A Theory of the Annual Variation of Temperature of Ocean or Lake

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Abstract

It may be admitted that the chief controlling factor of the annual variation of temperature of ocean or lake is eddy convection, but it is very doubtful whether it will satisfy, as has been believed until recently, the simple diffusion-equation. The problem is elucidated when the transfer of heat is taken into consideration, i. e., when the occurrence of instability in the cooling stage, and the vertical transfer of heat at the time of its disappearance are taken into account. For this purpose, adopting L. F. Richardson's view that the equation varies according as an instantaneous value or a mean value of the temperature is taken, and regarding our observed temperatures as showing mean values, the writer has found a new form for the diffusion equation, that is as near to perfection as possible, and, comparing his solution with actual examples, he found that his new theory was justified. Thus the author believes that this theory has brought the physical explanation of this subject to completion, and has also thrown a light on the method of reduction of the records of the observed water-temperature.

I. General Consideration

Many scientists are of the opinion that eddy convection is the main factor controlling the actual distribution of water-temperature in the layer which is subject to annual variation. The researches of H. Jeffreys[1] and J. E. Fjeldstad[2] were made from this point of view. W. Schmidt,[3] though of the same opinion, noticed that, contrary to his expectation that better results would be obtained from annual variation than from daily variation, the results from the former were rather worse. These investigators carried on their work on the supposition that the eddy equation has the form $\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right)$ in the usual notation. A certain measure of agreement between the actual observation and the theory concerning the distribution of temperature may be arrived at by the application of the above equation with certain appropriate conditions. But, when the transfer of heat is considered,

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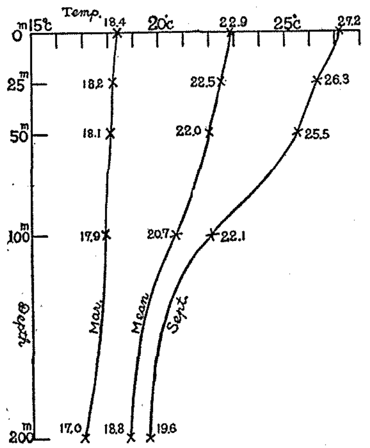
1. Phil. Mag. **39**, 578 (1920). 2. Geofys. pub. **10** No. 7.
3. Der Massenaustausch in freier Luft.........SS. 30–33.

Fig. 1



a contradiction will be met with. If the above diffusion equation is valid, the annual variation of the water-temperature ought to be of the same nature as that of the ground-temperature. Whereas, as shown in Fig. 1, the vertical distribution of the annual mean temperature of ocean or lake is not uniform as that of the ground-temperature is, but higher in the upper layer. Moreover, this annual mean temperature is to be regarded as the temperature in a steady state. If the state is steady, heat must be transmitted permanently from the upper to the lower part of the water. No evidence is found of this transmission, at least not in ocean or lake in the temperate zone. And, from a theoretical point of view, such a state, although not impossible, presupposes a somewhat extraordinary structure. In fact, it is most reasonable to suppose that the heat is carried down from the upper layer of water to the lower during the warm half-year and that the reverse process occurs during the cold half-year, the total exchange of heat flux, coming in the end to zero for the whole year. Thus, in the cooling stage, there must be an upward flux of heat. Then, according to the generally accepted theory, since the eddy flux of heat is $-k\dfrac{\partial\theta}{\partial z}$, the distribution of temperature in the winter season must generally show $\dfrac{\partial\theta}{\partial z}>0$ (z: positive downward); while actual observations give $\dfrac{\partial\theta}{\partial z}<0$ (though very near zero) in this stage. In the case of the ground-temperature, the theory and the actual state agree very well. Why, then, should the above mentioned contradiction be found in the case of water temperature? It is clear that the transfer of heat is not due to the eddy convection alone. Direct radiation has its effect, but this effect is limited to the surface layer, and to a very small portion of the whole layer subject to annual variation. Hence its influence may give a little correction only for the thin uppermost layer, but it cannot obliterate the doubt above stated in so far as the main body of water is concerned. Now, the reason why $\dfrac{\partial\theta}{\partial z}>0$ does not apply is clear, since $\dfrac{\partial\rho}{\partial z}<0$ (ρ:

density of water) means instability, and generally $\frac{\partial\rho}{\partial z} \lesseqgtr 0$ according as $\frac{\partial\theta}{\partial z} \gtreqless 0$. To make matters simpler, our discussion will be confined to cases where $\frac{\partial\rho}{\partial z}$ and $\frac{\partial\theta}{\partial z}$ have the relation above given. The genrality of the problem may be preserved by making a slight alteration in our estimate according to the difference of relation. In the cooling stage, $\frac{\partial\theta_i}{\partial z} > 0$ for some time, but, this state, being unstable, cannot last long. Intense mixing must occur immediately, subsiding into the state $\frac{\partial\rho}{\partial z} = 0$, that is, $\frac{\partial\theta}{\partial z} = 0$. This is confirmed by actual observation (in regions near the water surface). It must be noticed, however, that heat is carried spontaneously upwards in this process. Since, during the intense mixing, k_i has a rather large value, and $\frac{\partial\theta_i}{\partial z} > 0$, a considerable quantity of heat is carried upwards. This process may also be considered as an eddy convection, if we use the instantaneous values of θ_i and k_i and express the eddy equation as $\frac{\partial\theta_i}{\partial t} = \frac{\partial}{\partial z}\cdot\left(k_i\frac{\partial\theta_i}{\partial z}\right)$. The data which the author is going to treat are, however, of a different nature. They are either a *monthly mean temperature* or a temperature observed on a particular day which may be adopted as the *mean* for the month concerned. Such being the case, if we use the ordinary equation $\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right)$ with the mean values of both θ and k, what has just been stated about the eddy convection due to instability is not allowed for anywhere in the equation.

L. F. Richardson[1] expresses a similar opinion in his study of the atmospheric eddy. Starting from $\frac{\partial\theta_i}{\partial t} = \frac{\partial}{\partial z}\left(k_i\frac{\partial\theta_i}{\partial z}\right)$, and putting $\theta_i = \theta + \theta'$, $k_i = k + k'$, he gets $\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z} + \overline{k'\frac{\partial\theta'}{\partial z}}\right)$. Here, θ and k are the *monthly mean values* of θ_i and k_i respectively, and $\overline{k'\frac{\partial\theta'}{\partial z}}$ is the *monthly mean value* of $k'\frac{\partial\theta'}{\partial z}$. The last term means the flux of heat occurring when instability disappears. There is no doubt, of course, of the existence of the process just described. It has, hitherto, been treated as a thermal transfer due to convection—whether regular convection or eddy convection has not yet been made clear. Nor has

1. Weather Prediction. pp. 87–88.

any scientist discussed its nature, so far as the author knows. What will be the value of this quantity, $\overline{k'\frac{\partial\theta'}{\partial z}}$? It is not easy to find. Here we are brought to an impasse, as L. F. Richardson says. To get out of this difficulty, the author is obliged to make a rather bold assumption, upon whose basis he frames up the problem in a practical form which leads to a solution. Comparing the solution with actual example, he wants to show that his theory can explain the real nature of the actual variation of temperature in ocean or lake.

2. The Problem and its Solution

(i) As the equation of eddy convection, we take $\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\cdot\left(k\frac{\partial\theta}{\partial z}+\overline{k'\frac{\partial\theta'}{\partial z}}\right)$ where θ is the monthly mean temperature, t the time, z the depth from the water surface and k the eddy diffusivity, and we consider the most simple case when $k=$const. (independent of t and z).

(ii) For the unknown convection term due to the rapid mixing process occurring when the instability of the layer happens in the cold season, we boldly assume as follows:

$\overline{k'\frac{\partial\theta'}{\partial z}}=qe^{-\alpha z}\varphi_1(\omega t)$ where q and α are constants, and

$$\begin{aligned}\varphi_1(\omega t)&=\sin\omega t \quad \text{when} \quad 0<\omega t<\pi\\ &=0 \quad \text{when} \quad -\pi<\omega t<0,\end{aligned}$$

the time origin being taken at the autumnal equinox.

$$\varphi_1(\omega t)=\frac{1}{\pi}\left\{1+\frac{\pi}{2}\sin\omega t-\frac{2}{2^2-1}\cos 2\,\omega t-\frac{2}{4^2-1}\cos 4\,\omega t-\ldots\ldots\right\}$$

$\omega=\frac{2\pi}{T}=\frac{2\pi}{365\times 24\times 60\times 60}\doteqdot 2\times 10^{-7}$ where T is the period (one year).

(iii) For the surface condition, we take $-k\left.\frac{\partial\theta}{\partial z}\right|_{z=0}=q\varphi_2(\omega t)$

$$\begin{aligned}\text{where} \quad \varphi_2(\omega t)&=0 \quad \text{when} \quad 0<\omega t<\pi\\ &=\sin\omega t \quad \text{when} \quad -\pi<\omega t<0\end{aligned}$$

$$\varphi_2(\omega t)=\frac{1}{\pi}\left\{1-\frac{\pi}{2}\sin\omega t-\frac{2}{2^2-1}\cos 2\,\omega t-\frac{2}{4^2-1}\cos 4\omega t-\ldots\ldots\right\}$$

(iv) $\theta=\theta_0$ at $z\rightarrow\infty$ (bottom condition).

(v) $\theta(t, z)=\theta(t+T, z)$ (stationary condition).

The solution of the above problem is very simple, and we can write as follows:

$\theta=\theta_1$(general solution)$+\theta_2$(particular solution)

and $\theta_2=e^{-\alpha z}\{C_0+\sum_{n=1}^{\infty}C_n\sin(n\omega t+\delta_n)\}$

where $C_0=\dfrac{q}{\pi k\alpha}$, $C_1=\dfrac{q\alpha}{2\omega}\dfrac{1}{\sqrt{1+\left(\dfrac{k\alpha^2}{\omega}\right)^2}}$, $\tan\delta_1=\dfrac{\omega}{k\alpha^2}$;

$$C_n=\frac{q\alpha}{\pi n\omega}\,\frac{2}{n^2-1}\,\frac{1}{\sqrt{1+\left(\dfrac{k\alpha^2}{n\omega}\right)^2}},\quad \tan\delta_n=\frac{-k\alpha^2}{n\omega}$$

when n=even integer, $C_n=0$ when n=odd integer.

$$\theta_1=\theta_0+\sum_{n=1}^{\infty}B_ne^{-\beta_n z}\sin(n\omega t-\beta_n z+\varepsilon_n)\text{ where }\beta_n=\sqrt{\frac{n\omega}{2k}}$$

and the constants B_n and ε_n adequate to the given conditions can be found by the aid of the following equations

$$\begin{cases}B_1\cos\varepsilon_1=-\dfrac{1}{2}\dfrac{\alpha}{\beta_1}\left[\dfrac{q}{2k\alpha}+C_1(\cos\delta_1+\sin\delta_1)\right]\\[2ex] B_1\sin\varepsilon_1=\dfrac{1}{2}\dfrac{\alpha}{\beta_1}\left[\dfrac{q}{2k\alpha}+C_1(\cos\delta_1-\sin\delta_1)\right]\end{cases}$$

$$\begin{cases}B_n\cos\varepsilon_n=-\dfrac{1}{2}\dfrac{\alpha}{\beta_n}\left[\dfrac{2q}{(n^2-1)k\pi\alpha}+C_n(\sin\delta_n+\cos\delta_n)\right]\\[2ex] B_n\sin\varepsilon_n=-\dfrac{1}{2}\dfrac{\alpha}{\beta_n}\left[\dfrac{2q}{(n^2-1)k\pi\alpha}+C_n(\sin\delta_n-\cos\delta_n)\right]\end{cases}$$

when n is an even integer, and $B_n=0$ when the integer is odd.

3. A Numerical Example (Verification of the Theory by Observed Data)

The method of treating the actual data will be illustrated by an example. Temperature records off the coast of Shionomisaki are employed for our present purpose.[1]

(i) The annual mean temperature $(\bar{\theta})$ has the theoretical form $\bar{\theta}=\bar{\theta}_0+C_0e^{-\alpha z}$. Though the value of $\bar{\theta}_0$ can not be found directly from our data, it may be estimated at about 17.0(C°) by reference to the value of the minimum temperature at 200 m. depth. The observed temperature values at different depths give the two constants C_0 and α, and we get $C_0\doteqdot 6.0$(C°), $\alpha\doteqdot 0.5\times 10^{-4}$.

1. Taken from K. Suda's paper. Jour. Ocea. II, 491.

(ii) The amount of annual exchange of thermal quantity Q can be found from the observed data. In fact

$$Q \doteqdot \left[\int_0^H \sigma\rho\theta dz\right]_{\text{Sept.}} - \left[\int_0^H \sigma\rho\theta dz\right]_{\text{Mar.}}$$

where $\sigma\rho$ is the thermal capacity of water and $\sigma\rho \doteqdot 1$ for rough calculation. H is the depth of invariable layer, and the suffixes *Sept.* and *Mar.* indicate the respective epochs. The value of $\int_0^H \theta dz$ for autumnal and vernal equinox can be simply found by measuring the area of the isochronous curve of the temperature-depth diagram for these epochs, and, in our present case, it is estimated as $Q \doteqdot 1.3 \times 10^5$ (cal.)

On the other hand $Q = -\int_{\omega t=0}^{\omega t=\pi} q \sin \omega t dt$.

$$\therefore \quad q = \frac{\omega Q}{2} \doteqdot \frac{2 \times 10^{-7} \times 1.3 \times 10^5}{2} \doteqdot 1.3 \times 10^{-2}$$

(iii) From the expression $C_0 = \frac{q}{\pi k a}$, we can find the numerical value of k, $k = \frac{q}{\pi a C_0} \doteqdot \frac{1.3 \times 10^{-2}}{\pi \times 0.5 \times 10^{-5} \times 6.0} = 13.8$ (C. G. S.)

(iv) If we take the estimated values $k = 13.8$, $q = 1.3 \times 10^{-2}$, $a = 0.5 \times 10^{-4}$, $\theta_0 = 17.0(\text{C}^\circ)$, as the proper constants for the ocean considered, the annual variation of temperature can be found by the present theory. The constants of the solution become

$$\begin{cases} C_0 = 6.0(\text{C}^\circ). & \\ C_1 = 1.6(\text{C}^\circ). & \delta_1 = 80^\circ 20' \\ C_2 = 0.34(\text{C}^\circ). & \delta_2 = 355^\circ 05' \\ C_4 = 0.03(\text{C}^\circ). & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \end{cases} \qquad \begin{cases} B_1 = 4.07(\text{C}^\circ). & \varepsilon_1 = 144^\circ 20' \\ B_2 = 1.17(\text{C}^\circ). & \varepsilon_2 = 220^\circ 00' \\ \cdots\cdots & \cdots\cdots \end{cases}$$

Neglecting the higher harmonics, the calculated temperature approximately becomes

$$\theta \doteqdot 17.0 + 4.1 e^{-\beta_1 z} \sin(\omega t + 144^\circ - \beta_1 z) + 1.2 e^{-\beta_2 z} \sin(2\omega t + 220^\circ - \beta_2 z) + \{6.0 + 1.6 \sin(\omega t + 80^\circ) + 0.3 \sin(2\omega t + 355^\circ)\} e^{-\alpha z}$$

where $\alpha = 0.5 \times 10^{-4}$, $\beta_1 = 0.85 \times 10^{-4}$ and $\beta_2 = 1.20 \times 10^{-4}$ and $t = 0$ at the autumnal equinox (or in September).

To compare the actual observed data with these theoretical results, both are put into the harmonic series

$$\theta = \theta_0 + a_1 \sin(\omega t + \varphi_1) + a_2 \sin(2\omega t + \varphi_2) + \ldots\ldots$$

The comparison of the harmonic components is listed in Table 1.

Table 1

	θ_0 (C°.)		a_1 (C°.)		φ_1		a_2 (C°.)		φ_2	
Depth.	cal.	obs.	cal.	obs.	cal.	obs.	cal.	obs.	cal.	obs.
0^m	23.0	22.9	4.9	4.5	128°	107°	0.9	0.5	235°	145°
50^m	21.7	22.0	3.6	3.6	107°	97°	0.4	0.3	194°	50°
100^m	20.7	20.7	2.6	2.2	90°	86°	0.2	0.4	126°	321°

The coincidence for the constants of the main term a_1 and φ_1 is very good, though some discrepancies may be seen for the constants a_2 and φ_2. Remembering the bold assumption adopted in this theory, and seeing the fair agreement of the observed variations with those calculated, the writer believes that the suggested mechanisms of the variations of temperature are correct, at least qualitatively or semi-quantitatively.

4. Conclusion

(i) Applying L. F. Richardson's equation of eddy transfer of heat for mean state, the author adds, to the ordinary eddy diffusion equation, a new term of the upward flux of heat due to the spontaneous mixing occurring only in the cooling stage of water.

(ii) Though he adopts a bold assumption for the unknown terms of the upward flux of heat above mentioned, he finds comparatively good agreement between the theory and the observations.

(iii) In this paper, readers may find a physical explanation of the vertical distribution of the annual mean temperature; also a new method of finding the mean value of eddy diffusivity from the data of the vertical distribution of the annual mean temperature and the amount of annual heat-exchange of lake or ocean. He believes that his study will throw some light upon the method of reduction of the accumulated observed data of the annual variation of temperature of lake or ocean.

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