数理解析研究所講究録 1383 巻 2004 年 43-47

Visual Approaches for Numerical Analysis Concerning Fuzzy Differential Equations

Norio SATOMI
Electrical and Electronic Engineering, College of Industrial Technology
Amagasaki, Hyogo, 661-0047, Japan
里見憲男
産業技術短期大学電気電子工学科

Seiji SAITO and Hiroaki ISHII
Graduate School of Information Science and Technology, Osaka University
Suita, Osaka, 565-0871, Japan
齋藤誠慈, 石井博昭
大阪大学大学院情報科学研究科情報数理学専攻

ABSTRACT

By a parametric representation of fuzzy numbers, normal strictly and upper semi-continuous fuzzy numbers with bounded supports are identified with bounded continuous curves in the two-dimensional metric space. We introduce fuzzified oil well equation and discuss the stability of solutions of the fuzzy differential equation (FDE) by the method of parametric representation of fuzzy numbers. We also show that the analysis of fuzzy differential inclusions plays an important role in discussing similar asymptotic behaviours to ordinary differential equations.

Keywords: fuzzy differential equation, fuzzy number, parametric representation, fuzzy differential inclusion.

1. PARAMETRIC REPRESENTATION

Let $I=[0,1]$. We define the following set of fuzzy numbers, where a fuzzy number x is characterized by a membership function μ_x as follows (cf. [1]):

Definition 1.
$F=\{\mu_x : R\rightarrow I$ satisfying (i) - (iv) below$\}$.

(i) There exists a unique $m\in\mathcal{R}$ such that $\mu_x(m)=1$;

(ii) $supp(\mu_x)=cl(\{\xi\in\mathcal{R}:\mu(\xi)>0\})$ is bounded in $\mathcal{R}$;

(iii) μ_x is strictly quasi-convex on the compact support $supp(\mu_x)$, i.e.,
$\mu_x(\lambda\xi_1+(1-\lambda)\xi_2)>\min[\mu_x(\xi_1),\mu_x(\xi_2)]$
for $0<\lambda<1$ and $\xi_1,\xi_2\in J$ such that $\xi_1\neq\xi_2$;

(iv) μ_x is upper semi-continuous on $\mathcal{R}$.

In usual case a fuzzy number x satisfies quasi-convex on $\mathcal{R}$, i.e.,
$\mu_x(\lambda\xi_1+(1-\lambda)\xi_2)>\min[\mu_x(\xi_1),\mu_x(\xi_2)]$
for $0<\lambda<1$ and $\xi_1,\xi_2\in\mathcal{R}$. Condition (iii) plays an important role in proving properties of membership function μ_x in Theorem 1, where we show significant properties concerning the end-points of the α-cut set

$L_\alpha(\mu_x)=\{\xi\in R:\mu_x(\xi)\geq\alpha\}$.

In the similar way as [1] we consider the following parametric representation of $\mu_x\in\mathcal{F}_b^{st}$ such that

$$x_1(\alpha)=\min L_\alpha(\mu_x), \tag{1.1}$$

$$x_2(\alpha)=\max L_u(\mu_x), \tag{1.2}$$

for $0<\alpha\leq 1$ and that $x_1(0)=\min \mathrm{cl}\,(supp\,(\mu_x))$, $x_1(0)=\max \mathrm{cl}\,(supp\,(\mu_x))$. In what follows we denote a fuzzy numbers by $x=(x_1,x_2)$.

By applying the above extension principle and the representation of fuzzy numbers we get the following results.

1) Addition. Let , $x=(x_1,x_2), y=(y_1,y_2)\in\mathcal{F}_b^{st}$. We get the addition

$$\begin{aligned}\mu_{x+y}(\xi)&=\sup_{\xi=\xi_1+\xi_2}\min[\mu_x(\xi_1),\mu_y(\xi_2)]\\&=\sup\{\alpha\in I:\xi=\xi_1+\xi_2,\ \ \xi_1\in x_\alpha,\xi_2\in y_\alpha\}\\&=\sup\{\alpha\in I:x_1(\alpha)+y_1(\alpha)\leq\xi\leq x_2(\alpha)+y_2(\alpha)\}\end{aligned} \tag{1.3}$$

which means that $x+y=(x_1+y_1,x_2+y_2)$. Here $x_\alpha=L_\alpha(\mu_x)$ etc.

2) Subtraction. It follows that

$$\begin{aligned}\mu_{x-y}(\xi)&=\sup_{\xi=\xi_1-\xi_2}\min[\mu_x(\xi_1),\mu_y(\xi_2)]\\&=\sup\{\alpha\in I:\xi=\xi_1-\xi_2,\ \ \xi_1\in x_\alpha,\xi_2\in y_\alpha\}\end{aligned} \tag{1.4}$$

means that $x-y=(x_1-y_2,x_2-y_1)$.

3) Product. It follows that

$$\begin{aligned}\mu_{xy}(\xi)&=\sup_{\xi=\xi_1\xi_2}\min[\mu_x(\xi_1),\mu_y(\xi_2)]\\&=\sup\{\alpha\in I:\xi=\xi_1\xi_2,\ \ \xi_1\in x_\alpha,\xi_2\in y_\alpha\}\end{aligned} \tag{1.5}$$

means that the following relation.

$$xy = \begin{cases} (x_1y_1, x_2y_2) & (0 \le x_1, 0 \le y_1) \\ (x_2y_1, x_2y_2) & (0 \le x_1, y_1 \le 0 \le y_2) \\ (x_2y_1, x_1y_2) & (0 \le x_1, y_2 \le 0) \\ (x_1y_2, x_2y_2) & (x_1 \le 0 \le x_2, 0 \le y_1) \\ (\min(x_2y_1, x_1y_2), & \\ \quad \min(x_1y_1, x_2y_2)) & (x_1 \le 0 \le x_2, y_1 \le 0 \le y_2) \\ (x_2y_1, x_1y_1) & (x_1 \le 0 \le x_2, y_2 \le 0) \\ (x_1y_2, x_2y_1) & (x_2 \le 0, 0 \le y_1) \\ (x_1y_2, x_1y_1) & (x_2 \le 0, y_1 \le 0 \le y_2) \\ (x_2y_2, x_1y_1) & (x_2 \le 0, y_2 \le 0) \end{cases}$$

By the above parametric representation of fuzzy numbers we get the following theorem concerning properties of end-points.

Theorem 1. Denote $x = (x_1, x_2) \in \mathcal{F}_b^{st}$, where x_1, x_2: $I \to R$. Then the following properties (i)-(ii) hold:

(i) $x_i \in C(I)$, $i = 1,2$. Here $C(I)$ is the set of all the continuous functions on I;

(ii) There exists a unique $m \in \mathcal{R}$ such that $x_1(1) = x_2(1) = m$ and $x_1(\alpha) \le m \le x_2(\alpha)$ for $\alpha \in I$;

(iii) One of the following statements (a) and (b) holds;

(a) Functions x_1, x_1 are non-decreasing, non-increasing on I, respectively, with $x_1(\alpha) \le m \le x_2(\alpha)$ for $0 < \alpha \le 1$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ for $0 < \alpha \le 1$.

Conversely, under the above conditions (i)-(iii), if we denote $\mu_x(\xi) = \sup\{\alpha \in I : \; x_1(\alpha) \le \xi \le x_2(\alpha)\}$ then μ_x is the membership function of x, i.e., $\mu_x \in \mathcal{F}_b^{st}$ Let a metric between $x = (x_1(\cdot), x_2(\cdot))$, $y = (y_1(\cdot), y_2(\cdot))$ be defined as follows.

$$d(x,y) = \sup_{\alpha \in I}(|x_1(\alpha) - y_1(\alpha)| + |x_2(\alpha) - y_2(\alpha)|) \quad (1.6)$$

Then we get following result immediately (see [2,3]).

Theorem 2. $(\mathcal{F}_b^{st}, d)$ is complete metric space.

2. CALCULUS OF FUZZY FUNCTIONS

Consider a function $x : R \to \mathcal{F}_b^{st}$. Then $x(t)$ is said to be a fuzzy function. In [7] we find the following definition of fuzzy functions.

$$\begin{aligned} x(t,\cdot) &= \{(x_1(t,\alpha), x(t,\alpha))^T \in R : \alpha \in I\} \\ &= (x_1(t,\cdot), x_2(t,\cdot)) \end{aligned} \quad (2.1)$$

for $t \in [t_1, t_2]$. Denote $x(t) = (x_1(t), x_2(t))$.

A fuzzy function $x(t) = (x_1(t,\alpha), x(t,\alpha)) : R \to \mathcal{F}_b^{st}$ is H-differentiable at t in the sense of Hukuhara if there exists an $R \in \mathcal{F}_b^{st}$ such that (i) and (ii) hold as $h \to +0$.

(i) $x(t+h) = x(t) + h\eta + o(h)$;

(ii) $x(t) = x(t-h) + h\eta + o(h)$.

Here $o(h) = (o_1(h), o_2(h)) \in C[0,\varepsilon] \times C[0,\varepsilon]$ with $\varepsilon > 0$, which means that

$$\lim_{|h| \to 0} \frac{d(o(h), 0)}{|h|} = 0. \quad (2.2)$$

Then $x(t) = (x_1(t), x_2(t))$ is H-differentiable at t if and only if $x_1(t,\alpha), x(t,\alpha)$ are differentiable in t for each $\alpha \in I$ such that $\eta = (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}) \in \mathcal{F}_b^{st}$.

In [4] the author discuss the integration of fuzzy function $x(t)$.

Definition 2. A fuzzy function $x(t,\cdot) = (x_1(t,\cdot), x_2(t,\cdot))$ is called integrable over $[t_1, t_2]$ if $x_1(t,\alpha)$ and $x_2(t,\alpha)$ are integrable over $[t_1, t_2]$ for $\alpha \in I$. Define

$$\int_{t_1}^{t_2} x(s,\cdot)ds = \{(\int_{t_1}^{t_2} x_1(s,\cdot)ds, \int_{t_1}^{t_2} x_2(s,\cdot)ds)^T \in R^2 : \alpha \in I\}$$

3. FUZZY DIFFERENTIAL EQUATIONS

In [5] they discuss exponential decay problems, e.g., machine replacement and oil well extraction, etc. They analyze optimization problems for each oil well to determine its optimal replacement schedule. Denote the quality remaining in the well at time t by $x(t)$ and denote the rate of oil extraction by $D > 0$. Then they get the following rate of oil extraction $x'(t) = -Dx$ with $x(0) = v$. Then $x(t) = ve^{-Dt}$.

In what follows we consider the rate of oil extraction D as a constant fuzzy number $D = (D_1, D_2) \in \mathcal{F}_b^{st}$, where $D_1(\alpha)$ is the left end-point of the α-cut set and $D_1(\alpha) > 0$ for $\alpha \in I$. Then we assume that the oil quality $x(t) = (x_1(t), x_2(t)) \in \mathcal{F}_b^{st}$ is a fuzzy function which means the quality remaining in the well at time t and $v \in \mathcal{F}_b^{st}$. Consider an initial value problem of fuzzy differential equation

$$\frac{dx}{dt}(t) = -(Dx), \quad x(0) = v. \quad (3.1)$$

The above problem has a unique solution

$$x(t) = v + \int_0^t (-(Dx(s)))ds. \quad (3.2)$$

See [6].

It follows that as long as $x_1(t) \ge 0$, by the extension of principle

$$\frac{d}{dt}(x_1(t),x_2(t)) = -(D_1,D_2)(x_1,x_2)$$
$$= -(D_1x_1,D_2x_2) \qquad (3.3)$$
$$= (-D_2x_2,-D_1x_1).$$

Then we have two ordinary differential equations such as

$$x_1'(t) = -Dx_2, \quad x_2'(t) = -Dx_1 \qquad (3.4)$$

with $x(0)=(v_1,v_2)\in \mathcal{F}_b^{st}$. Therefore

$$x_1(t) = \frac{(v_1+v_2\sqrt{\frac{D_2}{D_1}})e^{-\sqrt{D_1D_2}t}}{2} + \frac{(v_1-v_2\sqrt{\frac{D_2}{D_1}})e^{\sqrt{D_1D_2}t}}{2}, \qquad (3.5)$$

$$x_2(t) = \frac{(v_1\sqrt{\frac{D_1}{D_2}}+v_2)e^{-\sqrt{D_1D_2}t}}{2} - \frac{(v_1\sqrt{\frac{D_1}{D_2}}-v_2)e^{\sqrt{D_1D_2}t}}{2}, \qquad (3.6)$$

for $t \geq 0$. These solutions, $x_1(t)$ and $x_2(t)$, decrease with an increase in time, and x_1 attain zero at the time

$$t_0 = \frac{1}{2\sqrt{D_1D_2}}\ln\left(\frac{\sqrt{D_2/D_1}\,v_2+v_1}{\sqrt{D_2/D_1}\,v_2-v_1}\right). \qquad (3.7)$$

After this time t_0, we must use other system of the differential equations available for $x_1(t)\leq 0$. Then it follows that for $x_1(t)\leq 0$, by the extension of principle

$$\frac{d}{dt}(x_1(t),x_2(t)) = -(D_1,D_2)(x_1,x_2)$$
$$= -(D_2x_1,D_2x_2) \qquad (3.8)$$
$$= (-D_2x_2,-D_2x_1).$$

Then we have two ordinary differential equations such as

$$x_1'(t) = -D_2x_2, \quad x_2'(t) = -D_2x_1 \qquad (3.9)$$

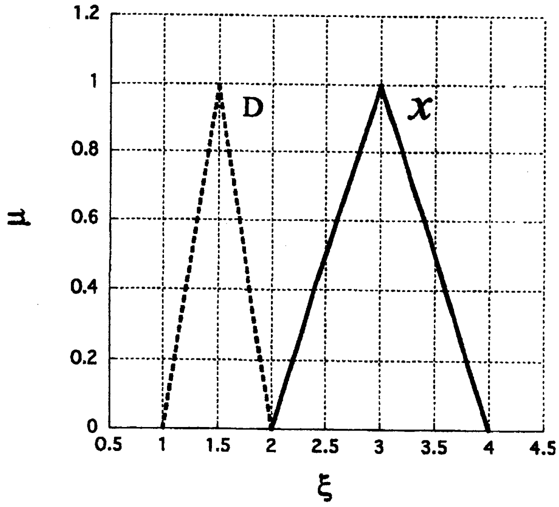

Fig. 1. Triangular membership functions for $x(0)$ and D used for the computations.

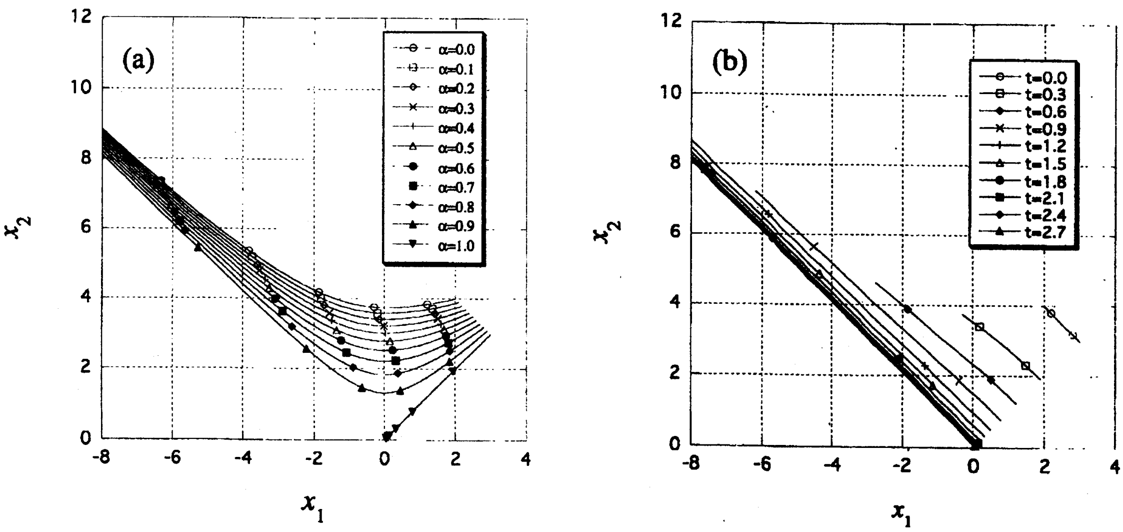

Fig. 2. Computed results of the fuzzy differential equation (3.1): (a) curves of solutions for each α, (b) temporal evolutions of the bounded continuous curves indicating the membership function.

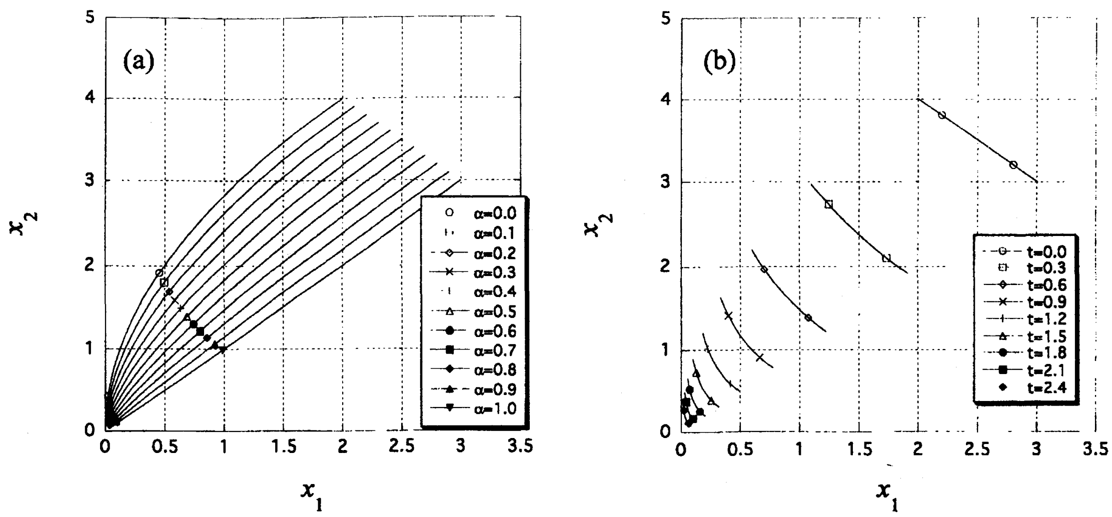

Fig. 3. Computed results of the fuzzy differential inclusion: (a) curves of solutions for each α, (b) temporal evolutions of the bounded continuous curves indicating the membership function.

with $x(t_0) = (0, v_{20}) \in \mathcal{F}_b^{st}$. Therefore

$$x_1(t-t_0) = \frac{v_{20}}{2}\left(e^{-D_2(t-t_0)} - e^{D_2(t-t_0)}\right), \quad (3.10)$$

$$x_2(t-t_0) = \frac{v_{20}}{2}\left(e^{-D_2(t-t_0)} + e^{D_2(t-t_0)}\right). \quad (3.11)$$

Then we get the unstable result of solution $x = (x_1, x_2)$ such that

$$\lim_{t \to +\infty} d(x(t), 0) = +\infty \quad (3.12)$$

where $0 \in R$, as well as it follows that

$$\lim_{t \to +\infty} \sup \left| \sqrt{D_1(\alpha)}\, x_1(t,\alpha) + \sqrt{D_2(\alpha)}\, x_2(t,\alpha) \right| = 0 \quad (3.13)$$

(see [3]).

To provide the graphs from these results, we consider membership functions for the initial value $x(0)$ and the constant D as shown in Fig. 1. The temporal evolutions calculated are shown in Fig.2. It is found that fuzzy differential equations enlarge its fuzziness of the system as time increases. Although the center of the fuzzy number for the decaying system described shows the exponential decay, other fuzzy numbers show asymptotic behaviors with $x_1(t,\alpha) \to -\infty$ and $x_2(t,\alpha) \to +\infty$ as $t \to +\infty$ for $\alpha \neq 1$.

4 FUZZY DIFFERENTIAL INCLUSIONS

In this section we introduce the idea of fuzzy differential inclusions in [6,7,8,9].

Example. Consider an initial value problem of fuzzy differential equation (3.2). According to the idea of fuzzy differential inclusions in which a family of differential inclusions plays an important role in finding some kind of fuzzy sets of (3.2) (See [10]). Let

$$F(\xi,\alpha) = [-D_2(\alpha)\xi, -D_1(\alpha)\xi] \subset R$$

defined on $R \times I$ to the set of compact and convex sets K_C^1 in R. Then one can solve the following differential inclusions

$$\xi'_\alpha(t) \in F(\xi,\alpha), \quad \xi_\alpha(0) \in L_\alpha(v) \quad (4.1)$$

where $L_\alpha(v) = [v_1(\alpha), v_2(\alpha)]$ for $\alpha \in I$, which means that differential inequalities

$$-D_2(\alpha)\xi_\alpha(t) \leq \xi'_\alpha(t) \leq -D_1(\alpha)\xi_\alpha(t) \quad (4.2)$$

$$v_1(\alpha) \leq \xi_\alpha(0) \leq v_2(\alpha) \quad (4.3)$$

for $\alpha \in I$. Then we emphasize that the function ξ_α is R-valued function defined on R without information on the grade of fuzzy number x, so $\xi_\alpha(t)$ is a real numbers but not fuzzy number. By basic calculation we get $\xi_\alpha(0)e^{-D_2(\alpha)t} \leq \xi_\alpha(t) \leq \xi_\alpha(0)e^{-D_1(\alpha)t}$ with $\xi_\alpha \in L_\alpha(v)$. Therefore we have $\xi_\alpha(t) \in [v_1 e^{-D_2(\alpha)t}, v_2 e^{-D_1(\alpha)t}]$ for $\alpha \in I$, $t \in R$, which is called a solution set denoted by $S_\alpha(L_\alpha(v),t) = [v_1(\alpha)e^{-D_2(\alpha)t}, v_2(\alpha)e^{-D_1(\alpha)t}]$. The solution set $S_\alpha(L_\alpha(v),t)$ is the α-cut set of the parametric representation of a fuzzy number $(v_1 e^{-D_2 t}, v_2 e^{-D_1 t})$. Thus we get a fuzzy solution of (3.1) as

$$x(t) = (v_1 e^{-D_2 t}, v_2 e^{-D_1 t}) \quad (4.4)$$

for $t \in R$. The temporal evolutions calculated by the fuzzy differential inclusion are shown in Fig.3.

5 CONCLUDING REMARKS

In classical analysis of the initial value problem (3.1) we observe the unstability of solutions by the method of parametric representation of fuzzy numbers. By applying differential inclusions to fuzzy differential equations (FDE) the same results of FDE as those in theory of ordinary differential equations. Much richer properties in fuzzy differential inclusions is significant but, in considering K_C^1-valued function $F(\xi,\alpha)$, one treats each fuzzy number $x(t) \in \mathcal{F}_b^{st}$x(t) as a real number $x(t) \in R$. Finally, we get solution sets which are the α-cut sets of a fuzzy set. By treating many practical modeling of real systems with uncertainty we can get better conclusions on comparison between fuzzy differential inclusions and the parametric representation of fuzzy numbers.

REFERENCES

[1] Jr. R. Goetschel, W. Voxman: Topological Properties of Fuzzy Numbers, Fuzzy Sets and Systems, Vol. 9, pp.87-99 (1983).

[2] P. Diamond and P. Kloeden: Metric Spaces of Fuzzy Sets -Theory and Applications-, World Scientific (1994).

[3] S. Saito: On Some Topics of Fuzzy Differential Equations and Fuzzy Optimization Problems via a Parametric Representation of Fuzzy Numbers, Differential Equations and Applications, Vol. 3, Nova Science Publishers, Inc., New York(in press).

[4] O. Kaleva: The Cauchy Problem for Fuzzy Differential Equation, Fuzzy Sets and Systems, Vol. 35, pp.389-396(1990).

[5] S. P. Sethi, G. L. Thopson and V. Udayabhanu: Profit Maximization Models for Exponential Decay Processes, European J. OR , Vol. 22, pp.101--115(1985).

[6] V. Lakshmikanthan and R.N. Mohapatra: Theory of Fuzzy Differential Equations and Inclusions, Taylor & Francis, London (2003).

[7] E. Hüllermeier: An Approach to Modeling and Simulation of Uncertain Dynamical Systems, Int. J. Uncertainty, Fuzziness, Knowledge-based Systems, Vol. 5, pp.117- 137(1997).

[8] P. Diamond: Time-Dependent Differential Inclusions, Cocycle Attractors and Fuzzy Differential Equations, IEEE Trans. on Fuzzy Systems, Vol. 7, No. 6, pp.734-740 (1999).

[9] P. Diamond: Stability and Periodicity in Fuzzy Differential Equations, IEEE Trans. on Fuzzy Systems, Vol. 8, No. 5, pp.583-590 (2000).

[10] J.P. Aubin and A. Celina: Differential Inclusions, Springer Verlag, New York (1984).