A Functional Equation with the Definite Kernel

By Tunezô Satô

(Received October 1, 1935)

I. We use the following linear operators

$$(\varphi, \psi)=\int_a^b \varphi(x)\psi(x)dx,$$

$$(A\varphi, \psi)=\int_a^b\int_a^b A(x, t)\varphi(t)\varphi(x)dtdx,$$

$$(\varphi, \psi)=(\psi, \varphi), \quad (\varphi, \varphi)=\int_a^b[\varphi(x)]^2dx.$$

Especially, let $A(x, y)$ be symmetric, then $(A\varphi, \psi)=(A\psi, \varphi)$.

Now we define the real function, $K(x, y)$, of two real variables x and y over the fundamental square

$$\Omega : \begin{bmatrix} a\leqq x\leqq b \\ a\leqq y\leqq b \end{bmatrix}$$

under the assumption that: they are continuous throughout Ω and symmetric as to x and y. Arrange the characteristic constants of $K(x, y)$ in the order of magnitude of their absolute values:

$$|\lambda_1|\leqq|\lambda_2 \leqq\ldots\ldots\ldots\leqq|\lambda_n|\leqq\lambda_{n+1}|\leqq\ldots\ldots\ldots,$$

where each λ_s is real. Put every corresponding characteristic function to each characteristic constant

$$\varphi_1(x), \varphi_2(x),\ldots\ldots\ldots, \varphi_n(x),\ldots\ldots\ldots\text{respectively},$$

then these functions $\varphi_n(x)$ will form a normalized orthogonal system, that is,

$$(\varphi_i, \varphi_j)=\begin{cases} 1 \\ 0 \end{cases} \text{for } \begin{matrix} i=j \\ i\neq j. \end{matrix}$$

Our present problem is to determine $\omega(x)$ so as to satisfy the following functional equation:

$$k=\int_a^b\int_a^b K(x, t)\omega(t)\omega(x)dtdx,$$

when k is a certain given constant. For convenience let $(A\varphi, \varphi)=A(\varphi)$, then we have

$$(1) \qquad K(\omega)=\int_a^b\int_a^b K(x, t)\omega(t)\omega(x)dtdx.$$

If we denote $\qquad (2) \quad u(x)=\int_a^b K(x, t)\omega(t)dt,$

the above equation (1) becomes $K(\omega)=(u, \omega)$. We now apply the Hilbert-Schmidt expansion theory to the right side of the equation (2). It becomes

(2′) $$u(x)=\sum_{\nu=1}^{\infty}\frac{C_\nu}{\lambda_\nu}\varphi_\nu(x),$$

where (3) $C_\nu=(\omega, \varphi_\nu)$, $\nu=1, 2, 3, \ldots\ldots\ldots$.

The series in (2′) is absolutely and uniformly convergent on (a, b). Therefore we have

$$(u, \omega)=\sum_{\nu=1}^{\infty}\frac{C_\nu}{\lambda_\nu}(\varphi_\nu, \omega)=\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_\nu},$$

namely by (1)

(1′) $$K(\omega)=\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_\nu}.$$

Apply the Bessel inequality to (3),

(4) $$\sum_{\nu=1}^{\infty}C_\nu^2\leqq\int_a^b[\omega(x)]^2dx.$$

On the other hand from (2′)

(5) $$\int_a^b[u(x)]^2dx=\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_\nu^2}.$$

Furthermore if we apply the Schwarz inequality to the right hand of (1), we obtain the following relation:

(6) $$[K(\omega)]^2\leqq\int_a^b[u(x)]^2dx\int_a^b[\omega(x)]^2dx,$$

while by computing the value of $K(\varphi_i)$,

$$\int_a^b\int_b^b K(x, t)\varphi_i(t)\varphi_i(x)dtdx=\frac{1}{\lambda_i}\int_a^b[\varphi_i(x)]^2dx.$$

Hence we have

(7) $$K(\varphi_i)=\frac{1}{\lambda_i}, \quad i=1, 2, 3, \ldots\ldots\ldots .$$

Now when we take $\omega(x)$ so as to satisfy the condition

(α) $$\int_a^b[\omega(x)]^2dx=1,$$

from (5), (6) and (7) we have

$$[K(\omega)]^2-[K(\varphi_i)]^2\leqq\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}.$$

But by (4),

(4′) $$\sum_{\nu=1}^{\infty}C_\nu^2\leqq 1,$$

whence $\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_i^2}\leqq\frac{1}{\lambda_i^2}$. Put this relation into the above inequality, then

$$[K(\omega)]^2-[K(\varphi_i)]^2 \leqq \sum_{\nu=1}^{\infty}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)C_\nu^2.$$

For the further discussion of this inequality, we rewrite the right hand in the form

$$\sum_{\nu=1}^{i}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)C_\nu^2+\sum_{\nu=i+1}^{\infty}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)C_\nu^2.$$

Now $\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2} \gtreqless 0$, for $\nu \lesseqgtr i$, $i=1, 2, 3, \ldots\ldots\ldots$, by hypothesis.

Therefore

$$[K(\omega)]^2 \leqq [K(\varphi_i)]^2+\sum_{\nu=1}^{i-1}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)\cdot(\omega, \varphi_\nu)^2.$$

Specially, take $i=1$, then the above relation becomes

$$[K(\omega)]^2 \leqq [K(\varphi_1)]^2=\frac{1}{\lambda_1^2}, \text{ or } |K(\omega)| \leqq |K(\varphi_1)|.$$

Thus we have the following

Theorem 1.—*For any continuous normalized function $\omega(x)$, there exist always the following inequalities:*

(8) $$[K(\omega)]^2 \leqq [K(\pm\varphi_i)]^2+\sum_{\nu=1}^{i-1}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)\cdot(\omega, \varphi_\nu)^2,$$ [1]

(8′) $$|K(\omega)| \leqq |K(\pm\varphi_1)|=\frac{1}{|\lambda_1|}. \quad (i=1, 2, 3, \ldots\ldots\ldots)$$

Since the second member of the above inequality (8′) does not depend on $\omega(x)$, we have the following corollary to Theorem 1:

Corollary 1.—*For any continuous function $\omega(x)$ which satisfies the (a)-condition: $\int_a^b[\omega(x)]^2dx=1$, the maximum value of*

$$\left|\int_a^b\int_a^b K(x, t)\omega(t)\omega(x)dtdx\right| \textit{ is equal to } \frac{1}{|\lambda_1|}.$$

Remark. When we consider two appreciations (8) and (8′) for $|K(\omega)|$, we see that the one is relative, the other absolute; hence the following inequality must hold:

$$\frac{1}{\lambda_i^2}+\sum_{\nu=1}^{i-1}\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)\cdot(\omega, \varphi_\nu)^2 \leqq \frac{1}{\lambda_1^2} \quad (i=1, 2, 3, \ldots\ldots\ldots).$$

We can also show easily that the above inequality may give the following results:

$$\sum_{\nu=1}^{i-1}(\omega, \varphi_\nu)^2 \leqq 1, \text{ for any } i \text{ by } (4')$$

and $$C_\nu'=\left(\frac{1}{\lambda_\nu^2}-\frac{1}{\lambda_i^2}\right)\Big/\left(\frac{1}{\lambda_1^2}-\frac{1}{\lambda_i^2}\right)<1 \quad \text{by hypothesis;}$$

---

1. It is clear from the construction of $K(\omega)$ that $K(\omega)=K(-\omega)$ for any continuous function $\omega(x)$.

hence $\sum_{\nu=1}^{i-1} C'_\nu(\omega, \varphi_\nu)^2 \leqq 1$ for any i.

Next we take Green's function, instead of $K(x, y)$, which belongs to the linear differential equation of the second order. Schmidt has shown that the results of the Hilbert-Schmidt theory of continuous symmetric kernels still hold for a discontinuous kernel if

1) $\int_a^b K(x, t)f(t)dt$ for f continuous, is continuous in x on (a, b).

2) The second iterated kernel of $K(x, y)$ is continuous and does not vanish identically. These conditions are satisfied in the present instance and thus all of the Hilbert-Schmidt theory, as well as Theorem 1 and Corollary 1 remain true. Therefore by putting $i=3$ in (8) as a special case we have

Corollary 2.——*If $K(x, y)$ is the Green's function which belongs to the differential equation $y''+k^2y=0$ with the boundary conditions $y(0)=y(1)=0$, for a continuous function $\omega(x)$ such as satisfies the (a)-condition, then*

$$[K(\omega)]^2 \leqq \frac{1}{3^4\pi^4}\left\{1+80C_1^2+65C_2^2\right\},$$

where $$C_\nu=\sqrt{2}\int_0^1 \omega(x)\sin\nu\pi x dx.$$

Corollary 3.——*For the Green's function of Legendre's differential equation $[(1-x^2)y']'=0$ with the boundary conditions $y(-1)=y(+1)=0$, i. e. $K(x, y)=\log 2-\frac{1}{2}-\frac{1}{2}\log[(1\pm y)\cdot(1\mp x)]$ for $x\lesseqgtr y$,*

we have $$[K(\omega)]^2 \leqq \frac{1}{36}\left\{1+35C_1^2+3C_2^2\right\},$$

where $C_\nu=\sqrt{\frac{2\nu+1}{2}}\int_{-1}^{+1} P_\nu(x)\omega(x)dx$ *and $P_\nu(x)$ means a Legendre's polynomial of the ν-th degree.*

Generalizing the (a)-condition, let

$$(a') \qquad \int_a^b [\bar{\omega}(x)]^2 dx = \varkappa^2,$$

then for such $\bar{\omega}(x)$, we take $\bar{\omega}(x)/\varkappa$ for $\omega(x)$ in the above results. Hence we can conclude that

Theorem 2.——*For any given constant $\varkappa^2$ and any such continuous function $\omega(x)$ as satisfies the functional equation*

$$\varkappa^2=\int_a^b [\omega(x)]^2 dx,$$

we have always

(9) $$[K(\omega)]^2 \leqq \frac{\varkappa^2}{\lambda_i^2} + \sum_{\nu=1}^{i-1}\left(\frac{1}{\lambda_\nu^2} - \frac{1}{\lambda_i^2}\right) \cdot (\omega, \varphi_\nu)^2,$$

(9′) $$[K(\omega)]^2 \leqq \frac{\varkappa^2}{\lambda_1^2}. \quad (i = 1, 2, 3, \dots\dots\dots)$$

We return now to our original problem and will solve the following functional equation:

(10) $$k = \int_a^b \int_a^b K(x, t)\omega(t)\omega(x)dtdx$$

under the assumption

(a) $$1 = \int_a^b [\omega(x)]^2 dx.$$

For this purpose, it first becomes necessary, on account of (8′), that the value of k should be given as $k^2 \leqq 1/\lambda_1^2$. We consider the particular case where $k = 1/\lambda_1$, while since it has been shown from (7) that $K(\pm\varphi_1) = 1/\lambda_1$, two functions $\pm\varphi_1$ will be ω-solutions in the present instance. But it might be that another solution exists. We must examine this possibility.

If we put $i = 2$ into (8), by using $[K(\omega)]^2 = 1/\lambda_1^2$, we obtain

$$\frac{1}{\lambda_1^2} \leqq \frac{1}{\lambda_2^2} + \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right)C_1^2; \quad \text{that is,} \quad \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) \cdot (C_1^2 - 1) \geqq 0.$$

Now let $\lambda_1^2 \neq \lambda_2^2$, then $C_1^2 \geqq 1$.
On the other hand by (4′) $C_1^2 \leqq 1$. Therefore $C_1^2 = 1$. Then from (1′)

$$K(\omega) = \frac{C_1^2}{\lambda_1} + \sum_{\nu=2}^{\infty} \frac{C_\nu^2}{\lambda_\nu} = \frac{1}{\lambda_1} + \sum_{\nu=2}^{\infty} \frac{C_\nu^2}{\lambda_\nu},$$

which is, by hypothesis, equal to $1/\lambda_1$.

Hence we have $$\sum_{\nu=2}^{\infty} \frac{C_\nu^2}{\lambda_\nu} = 0.$$

Now if we suppose that $\lambda_\nu > 0$ for all $\nu \geqq 2$, it will follow for all $\nu \geqq 2$ that $C_\nu^2 = 0$; in other words

$$(\omega, \varphi_\nu) = \begin{cases} 1 & \text{for } \nu = 1 \\ 0 & \phantom{\text{for }} \nu \geqq 2. \end{cases}$$

And besides, if we suppose that the system $\{\varphi_\nu\}$ is complete, we have the equality $\omega = \varphi_1$. Thus we can conclude that

Theorem 3.—*For the functional equation*

(10′) $$\frac{1}{\lambda_1} = \int_a^b \int_a^b K(x, t)\omega(t)\omega(x)dtdx$$

there exist two and only two continuous solutions $\pm\varphi_1(x)$ *under the assumptions*:

1) $\int_a^b [\omega(x)]^2 dx = 1.$

2) $\lambda_1^2 \neq \lambda_2^2.$

3) *$\lambda_\nu > 0$ for all $\nu \geqq 2$.*

4) *Functions $\varphi_\nu(x)$ form a complete system.*

It is clear that, in the above theorem, the positive definite and symmetric kernel satisfies assumptions 3) and 4). Accordingly the following corollary holds:

Corollary 1.—*There exist two and only two functions which maximize $|K(\omega)|$ with the (α)-condition, and the maximizing functions are $\pm\varphi_1$ under the following conditions:*

1) *$K(x, y)$ is symmetric and positive definite.*

2) *$\lambda_1 \neq \lambda_2$.*

We shall now assume that our kernel $K(x, y)$ is positive definite. Suppose $\lambda_1 = \lambda_2 \neq \lambda_3$. Then from (8) in Theorem 1

$$\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_3^2}\right) \cdot (C_1^2 + C_2^2 - 1) \geqq 0,$$

Hence $C_1^2 + C_2^2 \geqq 1$. On the other hand from (4′), we have $C_1^2 + C_2^2 = 1$. Thus in the proof of Theorem 3, we obtain

Corollary 2.—*When in our present problem $\lambda_1 = \lambda_2 \neq \lambda_3$, the required functions are given by $\omega = \cos\theta \cdot \varphi_1 + \sin\theta \cdot \varphi_2$.*

Analogically we can conclude that

Corollary 3. *When $\lambda_1 = \lambda_2 = \dots\dots = \lambda_s \neq \lambda_{s+1}$, the required solutions $\omega(x)$ are given by the following linear form*

$$\varepsilon_1\varphi_1 + \varepsilon_2\varphi_2 + \dots\dots\dots + \varepsilon_s\varphi_s,$$

where $$\varepsilon_1^2 + \varepsilon_2^2 + \dots\dots\dots + \varepsilon_s^2 = 1.$$

II. From results of the preceding section, we can show that

Theorem 4.—*The simultaneous functional equations*

$$\int_a^b [\omega(x)]^2 dx = 1, \quad \int_a^b [u(x)]^2 dx = 1,$$

where $u(x) = \int_a^b K(x, t)\omega(t)dt$, have the two and only two continuous solutions, and the two solutions are given by $\omega = \pm\varphi_1(x)$; moreover necessarily $\lambda_1 = 1$ follows.

Proof. Since we see that the integral equation of the first kind

$$\varphi_i(x) = \int_a^b K(x, t)\omega(t)dt$$

has a continuous solution $\omega(x) = \lambda_i\varphi_i(x)$ as the unique solution, it follows, if we denote

$$D(u) = \int_a^b\int_a^b K(x, t)\omega(t)\omega(x)dtdx = K(\omega), \quad \text{where} \quad u(x) = \int_a^b K(x, t)\omega(t)dt,$$

that

$$D(\varphi_i) = (\varphi_i, \lambda_i\varphi_i) = \lambda_i(\varphi_i, \varphi_i) = \lambda_i$$

will exist. Hence we have

$$0 < D(\pm\varphi_1) \leqq D(\pm\varphi_2) \leqq D(\pm\varphi_3) \leqq \cdots\cdots$$

by hypothesis $0 < \lambda_i \leqq \lambda_{i+1}$, $i = 1, 2, 3, \cdots\cdots\cdots$.

While, from the theory of the boundary problem on the linear differential equation of the second order, it is well known that if we give the condition

$$(\beta) \qquad \int_a^b [u(x)]^2 dx = 1,$$

then by (1′) and (5) the following inequality will be proved:

$$(11) \qquad D(u) \geqq D(\pm\varphi_1) = \lambda_1 \quad \text{for any continuous function } u.$$

Therefore it follows from Theorem 1, (6) and (11) that

$$1 \geqq \frac{1}{\lambda_1} \geqq D(u) = K(\omega) \geqq D(\pm\varphi_1) = \lambda_1 \geqq 1.$$

Consequently we have $\lambda_1 = 1$ and $K(\omega) = D(\pm\varphi_1) = K(\pm\varphi_1) = 1$.

Thus by reason of Theorem 3, we can conclude that our theorem is true. Q. E. D.

Next we consider the following functional equation in the place of (10′):

$$(10'') \qquad \frac{1}{\lambda_i} = K(\omega), \qquad \text{where } (\alpha): \int_a^b [\omega(x)]^2 dx = 1.$$

We have seen by (7) that $\omega = \pm\varphi_i$ are solutions of (10″); but further we wish now to see whether equation (10″) has another solution or not. For this purpose from (1′) we take

$$K(\omega) = \sum_{\nu=1}^{\infty} \frac{C_\nu^2}{\lambda_\nu};$$

while by (10″) $= \frac{1}{\lambda_i}$.

Now let us choose $\omega(x)$ so as to satisfy $C_i^2 = 1$, then there would be $C_\nu^2 = 0$ $(\nu \neq i)$. Since the system $\{\varphi_\nu\}$ is complete, we must have $\omega = \pm\varphi_i$. Hence we have the following

Theorem 5.—*The functional equation*

$$\frac{1}{\lambda_i} = \int_a^b \int_a^b K(x, t)\omega(t)\omega(x)dtdx$$

with conditions: 1) $K(x, y)$ *is symmetric and positive definite.* 2) $(\omega, \omega) = 1$. 3) $(\omega, \varphi_i) = 1$: *has two and only two continuous solutions given by* $\omega = \pm\varphi_i$.

For the case where λ_i has the index s, we can obtain

Corollary.—*In Theorem 5, if* $\lambda_i = \lambda_{i+1} = \cdots\cdots = \lambda_{i+s-1}$, *the required solutions will be*

$$\omega = \varepsilon_1\varphi_1 + \varepsilon_1\varphi_2 + \cdots\cdots + \varepsilon_s\varphi_{i+s-1},$$

where $\qquad \varepsilon_1^2 + \varepsilon_2^2 + \cdots\cdots + \varepsilon_s^2 = 1.$

III. We shall now discuss the most general case where k is any given constant, namely we consider the functional equation as follows:

$$(1) \qquad K(\omega)=\int_a^b\int_a^b K(x, t)\omega(t)\omega(x)dtdx,$$

which satisfies the (a)-condition $1=(\omega, \omega)$.

But since it has been shown that even in the particular case where the kernel $K(x, y)$ has only a finite number of characteristic constants, there will exist an infinite number of solutions, we demand here such a special solution as the following form:

$$(12) \qquad \omega(x)=\xi\varphi_{h-1}(x)+\eta\varphi_h(x),$$

where ξ and η are certain constants to be determined. Put (12) into the (a)-condition, then

$$1=(\xi\varphi_{h-1}+\eta\varphi_h, \xi\varphi_{h-1}+\eta\varphi_h)=\xi^2(\varphi_{h-1}, \varphi_{h-1})+2\xi\eta(\varphi_{h-1}, \varphi_h)+\eta^2(\varphi_h, \varphi_h),$$

hence for ξ and η, we must have $1=\xi^2+\eta^2$. On account of

$$\int_a^b\int_a^b K(x, t)\varphi_i(t)\varphi_j(x)dtdx=\frac{1}{\lambda_i}(\varphi_i, \varphi_j)=\begin{cases}\dfrac{1}{\lambda_i} & \text{for } i=j\\ 0 & \phantom{\text{for }} i\neq j,\end{cases}$$

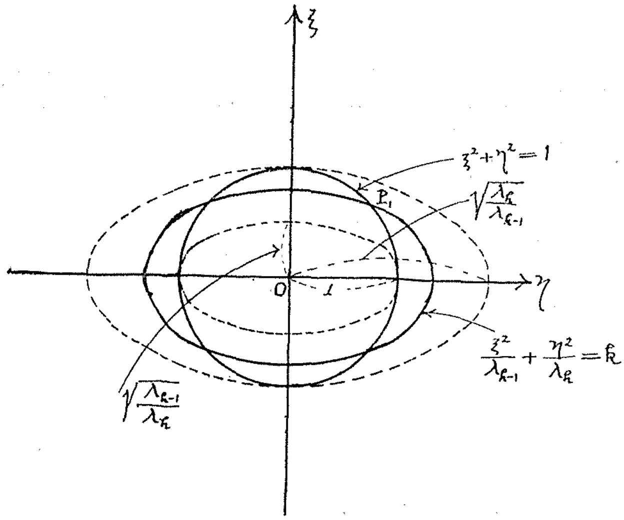
from (7) we obtain

$$(13) \qquad K(\omega)=\frac{\xi^2}{\lambda_{h-1}}+\frac{\eta^2}{\lambda_h}.$$

Now from our functional equation

$$(10) \qquad k=\int_a^b\int_a^b K(x, t)\omega(x)\omega(t)dtdx \quad \left(k\leqq\frac{1}{\lambda_1}\right),$$

we can show that for a given constant there exists such a number as satisfies the following inequality: Hence by (13) we must have



$$\frac{1}{\lambda_h} \leqq \frac{\xi^2}{\lambda_{h-1}} + \frac{\eta^2}{\lambda_h} \leqq \frac{1}{\lambda_{h-1}} \quad \text{and} \quad \xi^2 + \eta^2 = 1.$$

On the other hand we see that in general four pairs of values of ξ, η may satisfy above these relations (see Fig.). Consequently we have the following

Theorem 6.—*For the functional equation*

$$k = \int_a^b \int_a^b K(x, t)\omega(t)\omega(x)dtdx \quad \left(k < \frac{1}{\lambda_1}\right)$$

with the condition $(\omega, \omega) = 1$, *there exist such four pairs of solutions as* $\omega_1 = \xi_1\varphi_{h-1} + \eta_1\varphi_h$, $\omega_2 = -\xi_1\varphi_{h-1} + \eta_1\varphi_h$, $\omega_3 = -\omega_1$, $\omega_4 = -\omega_2$, *where* ξ_1, η_1 *gives the coordinates of one of the points at which the unit circle* $\xi^2 + \eta^2 = 1$ *and the ellipse* $\xi^2/\lambda_{h-1} + \eta^2/\lambda_h = k$ *intersect each other on the* (ξ, η)-*plane*; *and the number* h *is determined from the constant* k.

In conclusion the author wishes to express his hearty thanks to Professor Toshizô Matsumoto for his kind encouragement and for observations made.