On the Theory of Polynomials of Kernels

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Let

$$\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n, \ldots\ldots, \text{ where } |\lambda_i| \leqq |\lambda_{i+1}|,$$

and

$$\varphi_1, \varphi_2, \varphi_3, \ldots\ldots, \varphi_n, \ldots\ldots\ldots\ldots$$

be the system of characteristic constants and the corresponding normalized system of characteristic functions of the complete and symmetric kernel $K(x, y)$ respectively.

Corresponding to a polynomial of z

$$P(z) = c_1 z + c_2 z^2 + \ldots\ldots\ldots\ldots + c_m z^m,$$

where coefficients are all real, we consider the following polynomial of the kernel $K(x, y)$:

$$c_1 K + c_2 K^{(2)} + \ldots\ldots\ldots\ldots + c_m K^{(m)}.$$

Here every $K^{(i)}(x, y)$ means respectively the i-th iterated kernel of $K(x, y)$, and we denote this linear operator by $P(K)\binom{x}{y}$

We see easily from the property of iterated kernels that

$$\varphi_i(x) = \lambda_i^n \int_a^b K^{(n)}(x, t) \cdot \varphi_i(t) dt;$$

and moreover

$$\varphi_i(x) = \mu_i \int_a^b P(K)\binom{x}{t} \varphi_i(t) dt,$$

where $$\mu_i = \frac{1}{P\left(\dfrac{1}{\lambda_i}\right)} \qquad i = 1, 2, 3, \ldots\ldots\ldots\ldots ,$$

Now we begin with the following

Theorem 1. *When the polynomial of* z, $P(z)$, *is* $\geqq 0$ *for* $|z| \leqq \dfrac{1}{|\lambda_1|}$, *the polynomial of the kernel* $K(x, y)$, $P(K)\binom{x}{y}$, *is a symmetric and positive definite kernel.*

Proof. If we write

$$K^{(s)}(w) = \int_a^b \int_a^b K^{(s)}(x, t) \omega(t) \omega(x) dt dx,$$

then for any continuous function $\omega(x)$ we have

$$K^{(s)}(\omega) = \sum_{\nu=1}^{\infty} \frac{C_\nu^2}{\lambda_\nu^s}, \quad C_\nu = (\omega, \varphi_\nu),$$

since it is well known that for the symmetric kernel $K(x, y)$

$$K^{(s)}(x, y)=\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(x)\varphi_\nu(y)}{\lambda_\nu^s} \qquad (s\geqq 2),$$

where the series of the second member converges absolutely and uniformly over the square domain : $[a\leqq x\leqq b,\ a\leqq y\leqq b]$. Therefore

$$\begin{aligned}\int_a^b\int_a^b P(K)\binom{x}{t}\omega(t)\omega(x)dtdx &=\sum_{p=1}^{m} c_p K^{(p)}(\omega)\\ &=\sum_{p=1}^{m} c_p\sum_{\nu=1}^{\infty}\frac{C_\nu^2}{\lambda_\nu^p}\\ &=\sum_{\nu=1}^{\infty} C_\nu^2\sum_{p=1}^{m}\frac{c_p}{\lambda_\nu^p}\\ &=\sum_{\nu=1}^{\infty} C_\nu^2 P\left(\frac{1}{\lambda_\nu}\right).\end{aligned}$$

And since, by hypothesis, $P\left(\frac{1}{\lambda_\nu}\right)\geqq 0$ for all ν's, we have for any continuous function $\omega(x)$

$$\int_a^b\int_a^b P(K)\binom{x}{t}\omega(t)\omega(x)dtdx\geqq 0.$$

But if for a certain function $\omega(x)$, which is not identically zero,

$$\int_a^b\int_a^b P(K)\binom{x}{t}\omega(t)\omega(x)dtdx=0,$$

then for an arbitrary value of x

$$\int_a^b P(K)\binom{x}{t}\omega(t)dt=0,$$

thus we have the result that for all elements of the complete system $\{\varphi_\nu(x)\}$ $(\omega, \varphi_\nu)=0$; this contradicts the completeness of the system $\{\varphi_\nu\}$. Hence we have proved our theorem.

Another theorem resulting from the above viz.

Theorem 2. *None of the equalities*

$$P\left(\frac{1}{\lambda_i}\right)=0, \quad (i=1, 2, 3, \dots\dots\dots)$$

exist.

We begin with this observation :—Corresponding to a polynomial of z with the constant term, e. g.

$$P_m(z)=c_0+c_1z+c_2z^2+\dots\dots\dots\dots+c_mz^m,$$

we cannot define the polynomial of kernel $K(x, y)$, being parallel to the above stated definition of one which corresponds to a polynomial of z without the constant term.

Now if we take $P_m^*(K)\binom{x}{y}$ as such a polynomial of a kernel which corresponds to $P_m(z)$, we must define the polynomial of the kernel, corresponding to $zP_m(z)$, as

$$KP_m^*(K)\binom{x}{y} \quad \text{or} \quad P_m^*(K)K\binom{x}{y}.$$

Thus we obtain

$$(1) \qquad \varphi_i(x) = \frac{1}{\frac{1}{\lambda_i}P_m\left(\frac{1}{\lambda_i}\right)}\int_a^b KP_m^*(K)\binom{x}{t}\varphi_i(t)dt, \qquad (i=1, 2, 3, \ldots\ldots\ldots\ldots).$$

Hence from this result characteristic properties of the kernel $P_m^*(K)\binom{x}{y}$ follow.

If we put

$$(2) \qquad \psi_i(u) = \int_a^b P_m^*(K)\binom{u}{t}\varphi_i(t)dt, \quad (i=1, 2, 3, \ldots\ldots\ldots)$$

from (1)

$$\mu_i\varphi_i(x) = \int_a^b K(x, u)\psi_i(u)du,$$

where

$$\mu_i = \frac{1}{\lambda_i}P_m\left(\frac{1}{\lambda_i}\right).$$

While by the expansion theory

$$\mu_i\varphi_i(x) = \sum_{\nu=1}^{\infty}\frac{d_{\nu i}}{\lambda_\nu}\varphi_\nu(x) \quad \text{and} \quad d_{\nu i} = (\psi_i, \varphi_\nu) \qquad (\nu=1, 2, 3, \ldots\ldots\ldots\ldots).$$

Since the system $\{\varphi_\nu\}$ is normalized and orthogonal,

$$d_{\nu i} = \begin{cases}\lambda_i\mu_i \\ 0\end{cases} \text{for} \begin{array}{l}\nu = i \\ \nu \neq i,\end{array}$$

that is, $d_{ii} = P_m\left(\frac{1}{\lambda_i}\right)$ and $d_{\nu i} = (\psi_i, \varphi_\nu) = 0.$

Therefore, when we use

$$\bar{\psi}_i = \psi_i / P_m\left(\frac{1}{\lambda_i}\right)$$

in place of ψ_i, then for one of the i's and for all ν's

$$(\bar{\psi}_i, \varphi_i) = 1, \quad (\bar{\psi}_i, \varphi_\nu) = 0,$$

and on account of the completeness

$$\bar{\psi}_i \equiv \varphi_i, \quad \text{for} \quad i = 1, 2, 3, \ldots\ldots\ldots\ldots .$$

Hence we have $\psi_i = P_m\left(\frac{1}{\lambda_i}\right)\varphi_i$. Put this result into (2).

Then we have

$$(3) \qquad P_m\left(\frac{1}{\lambda_i}\right)\cdot\varphi_i(x) = \int_a^b P_m^*(K)\binom{x}{t}\varphi_i(t)dt. \qquad (i=1, 2, 3, \ldots\ldots\ldots\ldots)$$

Now we consider the following symmetric function :

$$\chi(x, t) \equiv P_m^*(K)\binom{x}{t} - \sum_{p=1}^{m} c_p K^{(p)}(x, t).$$

By using (3)

$$\int_a^b \chi(x,\, t)\varphi_i(t)dt = \int_a^b P_m^*(K)\binom{x}{t}\varphi_i(t)dt - \sum_{p=1}^m c_p \int_a^b K^{(p)}(x,\, t)\varphi_i(t)dt$$

$$= \left\{ P_m\left(\frac{1}{\lambda_i}\right) - \sum_{p=1}^m c_p \left(\frac{1}{\lambda_i}\right)^p \right\} \varphi_i(x)$$

$$= c_0 \varphi_i(x),$$

that is,

$$\varphi_i(x) = \frac{1}{c_0} \int_a^b \chi(x,\, t)\varphi_i(t)dt$$

for all $i = 1, 2, 3, \ldots\ldots\ldots\ldots$

But in order that these equations may be true for the same constant c_0, the characteristic constant $1/c_0$ cannot have a finite value. Hence we have $c_0 = 0$.

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