Theoretical Light Curves of Eclipsing Variables of the β Lyrae Type

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Introduction

§ 1. The problem of analysing a light curve of the β Lyrae type was undertaken by H. N. Russell[1] in his celebrated work on the determination of the orbital elements of eclipsing variable stars. According to his method, the light curve is first rectified by using the ellipticity constant determined from the slope of the curve near maximum and then the ordinary process of the determination for the Algol type is followed. A similar ellipsoidal form is hereby assumed for both components.

It is easily observed that the light elements thus obtained depend sensibly on this constant of ellipticity. Consider, for example, the case in which two minima are observed, primary minimum corresponding to the total eclipse. Let λ_1 and λ_2 be the brightness of these minima, and a and b be the lengths of the semi-axes. Then the ratio, r, of the radii of the two components is given by the relation,

$$r^2=\left(\frac{b}{a}-\lambda_2\right)/\lambda_1$$

when the stellar discs are uniformly brightened. Thus the dependency of r on b/a is clearly shown.

Moreover, each of the components will not in general be deformed similarly and this dissimilarity can not be found with any accuracy from a small part of the light curve near maximum. Rather, a weighted mean of the ellipticity constants of the two stars is determined thereby.

§ 2. The main purpose of the present paper is to examine, for Eddington's Model, whether the effect of the tidal deformation on the light curve conflicts with the observed data or not. It has been noted[2] that the uniform value of b/a agrees remarkably well with Darwin's theoretical value for a hypothetical homogeneous fluid; the darkened stars show considerably less ellipticity than the homogeneous bodies

1. Ap. J. **36** (1912) 60. 2. Shapley; Contr. Princeton Univ. Obs. **3** (1915) 115.

for given separation. The predicted distortion will be much smaller in a gaseous star than in a fluid one. It may be noted, however, that since the surface of the star in radiative equilibrium is not uniformly illuminated but is brighter nearer the centre, it tends to magnify the phase effect due to the gravitational distortion,—that is, to increase the apparent value of ellipticity and more so if we take the limb-darkening into account. On the other hand, it behaves as if there were a central darkening when we want to estimate the dimension of the smaller star from minimum of the light curve corresponding to the annular eclipse and it results in a greater value for r.

In a previous paper[1], I calculated, in the first order, the phase effect for the polytrope, $n=3$, by taking into account both the variation in surface brightness and the darkening to the limb. It was shown therein that the results for a semi-darkened solution satisfactorily accord statistically with observations. Here we intend to compare them for individual stars.

For this purpose we are not satisfied with Russell's method of rectification, but would like to find what is the effect of the dissimilarity of the deformation on the light curve. The model star assumed in the previous paper was used and the results applied to the light curve of β Lyrae.

I. Loss of Light by the Eclipse

§ 3. *Description of the model star.* Take the centre of gravity of the primary, M—the eclipsed star—as the origin of the coordinate system, the x-axis in the direction of the secondary, M'—the eclipsing star—and the z-axis parallel to that of the revolution. Now we assume that in our model star, the distance, η, of a point on the surface from the centre is given by

$$\eta=\eta_1\left[1-0.088\,v+\frac{1}{2}(1-\nu^2)v+\left\{P_2(\lambda)+\eta_1 P_3(\lambda)+\eta_1^2 P_4(\lambda)\right\}v_1\right] \quad (1),$$

and the net flux, πH of radiation through the surface by

$$\eta^2 H=H_0[1-(1-\nu^2)v-\{2P_2(\lambda)+3\eta_1 P_3(\lambda)+4\eta_1^2 P_4(\lambda)\}v_1] \quad (2),$$

where λ, μ and ν are the direction-cosines of the radius vector and η_1, H_0, v and v_1 are constants, the mean distance of the two stars being taken to be unity, and $P_i(\lambda)$ being the Legendre function of ith order. The shape of our model star is, therefore, distorted ellipsoid which is more elongated towards the other component, and its swollen portion

1. These Memoirs., A, **17** (1934) 197.

is less illuminated. As was shown in my previous paper (loc. cit. pp. 210 and 211), the relations, (1) and (2), represent to the first order the surface nature of a gaseous star in radiative equilibrium and the constants are connected by the equations,

$$\left.\begin{aligned} v_1 &= \eta_1^3 M'/M \\ v &= \eta_1^3 + v_1 \end{aligned}\right\} \qquad (3).$$

To the same order of accuracy, the total emission is given by

$$4\mathfrak{L} = 4\pi H_0\left(1 - \frac{2}{3}v\right) \qquad (4),$$

and the phase effects are

$$\mathfrak{L}_u(l_0, n_0) = \pi H_0[1 + (n_0^2 - 1)v - \{2P_2(l_0) - \eta_1^2 P_4(l_0)\}v_1] \qquad (5)_u$$

and

$$\mathfrak{L}_d(l_0, n_0) = \pi H_0\left[1 + \left(\frac{8}{5}n_0^2 - \frac{6}{5}\right)v - \left\{\frac{16}{5}P_2(l_0) + \frac{15}{8}\eta_1 P_3(l_0)\right\}v_1\right] \qquad (5)_d$$

respectively for the uniform and the darkened solution, where l_0, m_0 and n_0 are the direction-cosines of the line of sight.

§ 4. *Loss of Light by the Eclipse.* Let I_1 and I_2 be the flow per unit cross-section per second of the proper and reflected radiations travelling in the direction, α, to the surface normal whose direction-cosines are l, m and n. Then the loss of light by the eclipse is

$$\Delta\mathfrak{L} = \int (I_1 + I_2)\cos\alpha\ \eta^2 d\omega \qquad (6),$$

where $d\omega$ is the elementary solid-angle, and the integration is to be extended over the whole eclipsed area. Following the previous work, we may take for the flow of radiation,

$$\left.\begin{aligned} I_1 &= \frac{1}{1+s}H\left(1 + \frac{3}{2}s\cos\alpha\right) \\ I_2 &= \frac{\mathfrak{L}'}{\pi}\left\{\lambda + 2\eta_1 P_2(\lambda) + 3\eta_1^2 P_3(\lambda)\right\} \end{aligned}\right\}^* \qquad (7),$$

s being the coefficient of darkening. Finally we have

$$\begin{aligned} \cos\alpha &= ll_0 + mm_0 + nn_0 \\ &= L\left[1 + \left(\frac{n^2}{L}\nu - \nu^2\right)v + \left(\lambda - \frac{l_0}{L}\right)\left\{P_2'(\lambda) + \eta_1 P_3'(\lambda) + \eta_1^2 P_4'(\lambda)\right\}v_1\right] \end{aligned} \qquad (8),$$

where $P_i'(\lambda)$ is derivative of the Legendre function and

$$L = l_0\lambda + m_0\mu + n_0\nu \qquad (9).$$

We shall calculate, as usual, the loss in the proper radiation for

* The accent is used in this paper to indicate the corresponding entries for the secondary —the eclipsing star, unless otherwise mentioned.

two extreme cases corresponding to the uniform ($s=0$) and the darkened ($s=\infty$) solution. By combining (2) and (8), the complete forms of the integrands in (6) will be

$$H\eta^2 \cos\alpha = H_0 L\left[1 + v\left(\frac{n_0\nu}{L} - 1\right) + v_1 \sum_{i=2}^{4}\left\{P'_{i-1}(\lambda) - \frac{l_0}{L}P'_i(\lambda)\right\}\eta_1^{i-2}\right] \quad (10)_u$$

and

$$\frac{3}{2}H\eta^2 \cos^2\alpha = \frac{3}{2}H_0 L\left[L + v\left\{2n_0\nu - (1+\nu^2)\right\}\right.$$
$$\left. + v_1 \sum_{i=2}^{4}\left\{\overline{\lambda P'_i(\lambda) + P'_{i-1}(\lambda)}\cdot L - 2l_0 P'_i(\lambda)\right\}\eta_1^{i-2}\right] \quad (10)_d$$

respectively for these two solutions.

As for the reflected radiation, we neglect the deformation of the surface and reserve only the terms to the order of η^4. Accordingly it gives for the corresponding integrand

$$I_2\eta^2 \cos\alpha = H'_0\eta_1^2 L\{\lambda + 2\eta_1 P_2(\lambda) + 3\eta_1^2 P_3(\lambda)\} \quad (11).$$

Now if we write $\Delta\mathfrak{L}_u$, $\Delta\mathfrak{L}_d$ and $\Delta\mathfrak{L}_r$ for sub-integrals in (6) corresponding to these three, we have for the loss of total light

$$\Delta\mathfrak{L} = \frac{1}{1+s}(\Delta\mathfrak{L}_u + s\Delta\mathfrak{L}_d) + \Delta\mathfrak{L}_r \quad (12).$$

§ 5. *Transformation of the Coordinate.* To evaluate these integrals, it is more convenient to refer to another coordinate system (X, Y, Z), in which X-axis coincides with the line of sight (l_0, m_0, n_0) and Z-axis lies in xX-plane. To fix the idea we always take Z-coordinate of the secondary to be negative, so that if we write (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively for the direction-cosines of Y- and Z-axis, we have

$$\left.\begin{array}{l} l_1 = 0, \quad m_1 = -n_0/l_2, \quad n_1 = m_0/l_2\,; \\ l_2 = -\sqrt{1-l_0^2}, \quad m_2 = -m_0 l_0/l_2, \quad n_2 = -n_0 l_0/l_2 \end{array}\right\} \quad (13).$$

Now let (L, M, N) be the direction-cosines of any radius vector in this new system, then

$$\left.\begin{array}{l} \lambda = l_0 L + l_2 N \\ \nu = n_0 L + n_1 M + n_2 N \end{array}\right\} \quad (14),$$

and hence (10) and (11) can be expressed in M and N, which are now taken to be independent variables. L is always positive as far as we are concerned.

Further, since

$$L d\omega = dM dN \quad (15),$$

we find

$$\frac{1}{H_0}\Delta\mathfrak{L}_u = \int dM dN\left[1 + v\left\{n^2 - 1 + \frac{n_0}{l_2}\left(m_0\frac{M}{L} - n_0 l_0\frac{N}{L}\right)\right\}\right.$$

$$+v_1\left\{(1-3l_0^2)-3l_0l_2\frac{N}{L}\right\}$$

$$+\frac{3}{2}v_1\eta_1\left\{l_0(2-5l_0^2)L+2l_2(1-5l_0^2)N-l_0(5l_2^2N^2-1)\frac{1}{L}\right\}$$

$$+\frac{1}{2}v_1\eta_1^2\left\{5l_0^2(3-7l_0^2)L^2+15l_0l_2(2-7l_0^2)NL-15l_2^2(1-7l_0^2)N^2\right.$$

$$\left.\left.+3(5l_0^2-1)-5l_0l_2(7l_2^2N^2-3)\frac{N}{L}\right\}\right] \qquad (16)_u,$$

$$\frac{1}{H_0}\Delta\Omega_d=\frac{3}{2}\int dMdN\left[L+v\left\{2\frac{n_0}{l_2}(m_0M-n_0l_0N)\right.\right.$$

$$+(2\cdot\overline{n_0^2-1}+\left(\frac{n_0}{l_2}\right)^2+2\frac{l_0m_0n_0}{l_2^2}MN+\overline{1+n_0^2-2\left(\frac{n_0}{l_2}\right)^2}\cdot N^2)L$$

$$\left.-2\frac{n_0}{l_2}(m_0M-n_0l_0N)L^2+\overline{1-n_0^2-\left(\frac{n_0}{l_2}\right)^2}\cdot L^3\right\}$$

$$+3v_1\{l_0^2L^3+2l_0l_2NL^2+(l_2^2N^2+1-6l_0^2)L-2l_0l_2N\}$$

$$+\frac{3}{2}v_1\eta_1\left\{5l_0^3L^4+15l_2l_0^2NL^3+(15l_2^2N^2+1-10l_0^2)l_0L^2\right.$$

$$\left.+(5l_2^2N^2+1-20l_0^2)l_2NL-2l_0(5l_2^2N^2-1)\right\}$$

$$+v_1\eta_1^2\left\{\frac{35}{2}l_0^4L^5+70l_2l_0^3NL^4+l_0^2(105l_2^2N^2-35l_0^2)L^3\right.$$

$$+l_0l_2(70l_2^2N^2-105l_0^2)NL^2$$

$$\left.\left.+(\frac{35}{2}l_2^4N^4-\frac{3}{2}-105l_0^2l_2^2N^2+15l_0^2)L-5l_0(7l_2^2N^2-3)l_2N\right\}\right]$$

$$(16)_d$$

and

$$\Delta\Omega_r=H_0'\eta_1^2\int\left\{\lambda+2\eta_1P_2(\lambda)+3\eta_1^2P_3(\lambda)\right\}dMdN \qquad (16)_r.$$

§ 6. *Equation of the Shadow Cylinder.* The range of integration is to be determined by the section of the surface of the primary and the shadow cylinder of the secondary. The equation of the generating line of the latter will be

$$\frac{x-\bar{x}}{l_0}=\frac{y-\bar{y}}{m_0}=\frac{z-\bar{z}}{n_0} \qquad (17),$$

where parameters $\bar{x}$, $\bar{y}$ and $\bar{z}$ mean the coordinates of the tangential point on the secondary and we have

$$(\bar{x}-1)^2+\bar{y}^2+\bar{z}^2=\eta'^2 \qquad (18)$$

and

$$\cos\bar{\alpha}'=\bar{l}'l_0+\bar{m}'m_0+\bar{n}'n_0=0 \qquad (19).$$

Here by premise, the accent means the corresponding entries for the secondary, and the bar denotes the tangential point. Since $\cos\alpha$ differs from L only by a small amount of the first order,

$$\bar{L}' = \bar{\lambda}' l_0 + \bar{\mu}' m_0 + \bar{\nu}' n_0 = 0$$

in the zero order. Furthermore,

$$\bar{x} - 1 = \bar{\lambda}'\bar{\eta}', \quad \bar{y} = \bar{\mu}'\bar{\eta}' \quad \text{and} \quad \bar{z} = \bar{\nu}'\bar{\eta}',$$

so that each of (17) is equal to

$$X - l_0 - \bar{L}'\eta_1'$$

to the order of accuracy. Or by introducing our current coordinate and by using the properties of direction cosines, we have as the equation of the shadow cylinder

$$\left.\begin{aligned} \bar{x} - 1 &= \qquad\qquad l_2(Z - l_2) + l_0\bar{L}'\eta_1' \\ \bar{y} &= m_1 Y + m_2(Z - l_2) + m_0\bar{L}'\eta_1' \\ \bar{z} &= n_1 Y + n_2(Z - l_2) + n_0\bar{L}'\eta_1' \end{aligned}\right\} \qquad (20),$$

whence by squaring and summing up

$$Y^2 + (Z - l_2)^2 = \bar{\eta}'^2 = \eta_1'^2(1 + 2\Delta\bar{\eta}') \qquad (21)$$

where

$$\Delta\eta = -0.088v + \frac{1}{2}(1 - \nu^2)v + \left\{P_2(\lambda) + \eta_1 P_1(\lambda) + \eta_1^2 P_4(\lambda)\right\}v_1 \qquad (22)$$

by (1). $\bar{\lambda}'$ and $\bar{\nu}'$ in $\Delta\bar{\eta}'$ are also functions of Y and Z, as given in (20) but here it is sufficient to take only their zero order terms.

§ 7. *Section of the Shadow Cylinder on the Primary.* Since on the boundary surface of the primary

$$\left.\begin{aligned} Y &= M\eta_1(1 + \Delta\eta) \\ Z &= N\eta_1(1 + \Delta\eta) \end{aligned}\right\},$$

substitution of these relations into (21) follows

$$M^2 + \left(N - \frac{l_2}{\eta_1}\right)^2 = \left(\frac{\eta_1'}{\eta_1}\right)^2(1 + 2\Delta\bar{\eta}') - 2\Delta\eta\left\{M^2 + N\left(N - \frac{l_2}{\eta_1}\right)\right\} \quad (23),$$

the required equation which determines the range of integration.

It may be noted that we have always

$$M^2 + N^2 = 1 - L^2$$

and L^2 is of the second order on the limb of the visible hemisphere of the star, so that in order to find the total light, for example, the range of integration can be taken for a circle in MN-plane as was actually done in my previous paper. But it is not the case for the eclipsed area in general; in fact the deviation is of the first order as shown by (23).

Furthermore, if we write for simplicity

$$\left.\begin{aligned} \eta_1' &= r\eta_1 \\ -l_2 &= N_0\eta_1 \end{aligned}\right\} \qquad (24),$$

the equation defining the eclipsed portion becomes

$$M^2+(N+N_0)^2=r^2(1+2\Delta\bar{\eta}')-2\Delta\eta\{r^2-N_0(N+N_0)\} \qquad (23_a),$$

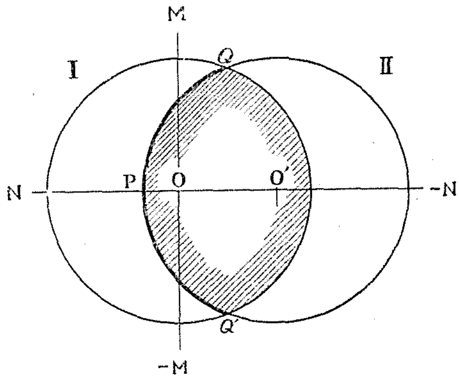
where $\Delta\bar{\eta}'$ and $\Delta\eta$ are of the form (22) and there

$$\left.\begin{aligned}\lambda&=l_0L+l_2N\\ \nu&=n_0L+\frac{m_0}{l_2}M-\frac{n_0l_0}{l_2}N\end{aligned}\right\} \qquad (14_a),$$

$$\left.\begin{aligned}r\bar{\lambda}'&=l_2(N+N_0)\\ r\bar{\nu}'&=\frac{m_0}{l_2}M-\frac{n_0l_0}{l_2}(N+N_0)\end{aligned}\right\} \qquad (25).$$

§ 8. *The Range of Integration.* Let I in Fig. 1 be the projection of the primary on MN-plane and II be that of the secondary. Then the former is a circle of unit radius and arc QPQ' is a distorted circle of radius, r, specified by (23), and their central distance $\overline{OO'}$ is N_0.

Fig. 1



It is our problem to calculate the light coming from the shaded portion. Now if we write

$$f(N,M)+\Delta f(N,M)$$

for the integrand in $\Delta\Omega$ at (16), Δf being of the first order, we can neglect the deviation of II from a circle for the integration of Δf. Hence the required integral is

$$\Delta\Omega=\left(\int_{-1}^{N_1}\int_{-M_*}^{M_*}+\int_{N_1}^{-N_0+r}\int_{-M_*'}^{M_*'}\right)(f+\Delta f)dNdM$$
$$+\int_{N_1}^{-N_0+r}\left[\Delta M\cdot f\right]_{-M_*'}^{M_*'}dN \qquad (26),$$

where

$$\left.\begin{aligned}M_*&=\sqrt{1-N^2}\\ M_*'&=\sqrt{r^2-(N+N_0)^2}\\ L_*&=\sqrt{M_*^2-M'^2_*}\end{aligned}\right\} \qquad (27)$$

and N_1 is the abscissa of the intersection, Q and Q', so that

$$-2N_0N_1=1-r^2+N_0^2 \qquad (28).$$

ΔM measures the deviation of arc QPQ' from a circle and from (23_a)

$$M\Delta M=r^2\Delta\bar{\eta}'-\{r^2-N_0(N+N_0)\}\Delta\eta \qquad (29).$$

Or referring to (22), (14_a), and (25), we have

$$\frac{M_*'}{f_*}\left[\Delta M\cdot f\right]_{-M_*'}^{M_*'}=\left[0.824r^2-M'^2_*+\left(\frac{n_0}{l_2}\right)^2\left\{M'^2_*-l_0^2(N+N_0)^2\right\}\right]\tau'$$

$$+2\{P_2(\bar\lambda')+\eta_1'P_3(\bar\lambda')+\eta_1'^2P_4(\bar\lambda')\}r^2v_1'$$
$$-\left[\left\{0.824-M_*'^2+\left(\frac{n_0}{l_2}\right)^2(M_*'^2-l_0^2N^2+2l_0l_2NL_*-l_2^2L_*^2)\right\}v\right.$$
$$\left.+2\{P_2(\lambda_*)+\eta_1P_3(\lambda_*)+\eta_1^2P_4(\lambda_*)\}v_1\right]\{r^2-N_0(N+N_0)\} \quad (30),$$

where

$$\left.\begin{aligned}\lambda_* &=l_0L_*+l_2N\\ r\bar\lambda' &=l_2(N+N_0)\end{aligned}\right\} \quad (31)$$

and $f_*=1$ and $\frac{3}{2}L_*$ respectively for the uniform and the darkened solution. The first two integrals in (26) do not depend on the distortion of the secondary. Hence, hereafter we shall call them "circular integrals" and the last integral will be referred to as "boundary correction".

In particular when the eclipse is annular, N_1 is to be replaced by $-(N_0+r)$ and naturally the first circular integral should be dropped off. Finally it should be added that the eclipse is

$$\left.\begin{array}{lll}\text{total when} & r-N_0>1 & (r>1),\\ \text{annular} \;'' & r+N_0<1 & (r<1)\\ \text{and partial} \;'' & r+N_0>1>|r-N_0| & \end{array}\right\} \quad (32).$$

§ 9. "*Circular Integrals.*" The integrand, $f+\Delta f$ is an algebraic function of N, M and L, its general terms being L^nN^m and L^nN^mM. But since

$$\int_{-M}^{M}L^nMdM=0,$$

only the former form need be considered.

Now put

$$\left.\begin{aligned}\beta_n^m &=\int_{-1}^{N_1}\int_{-M_*}^{M_*}L^nN^mdNdM,\\ \gamma_n^m &=\int_{N_1}^{r-N_0}\int_{-M_*'}^{M_*'}L^nN^mdNdM\\ \text{and}\quad \alpha_n^m &=\beta_n^m+\gamma_n^m\end{aligned}\right\} \quad (33).$$

Then we can express the circular integrals by means of these terms but in actual evaluations it is much more convenient to make some more reformations of terms with an even upper suffix such as

$$\left.\begin{aligned}\alpha_{2n}^{2m} &=2\phi_{2n+1}\cdot\phi_{2m,\,2n+2}\cdot\alpha_0+(\alpha_{2n}^{2m})\\ \alpha_{2n-1}^{2m} &=3\phi_{2n}\cdot\phi_{2m,\,2n+1}\cdot\alpha_1+(\alpha_{2n-1}^{2m})\end{aligned}\right\} \quad (34),$$

where

$$\phi_{m,n}=\frac{(m-1)(m-3)\ldots\times(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots} \quad (35)$$

whence**

$$\left.\begin{aligned}\phi_{2n+1}\cdot\phi_{2m,\,2n+2}&=\frac{(2m-1)(2m-3)\ldots1}{(2m+2n+2)(2m+2n)\ldots(2n+2)}\\ \phi_{2n}\cdot\phi_{2m,\,2n+1}&=\frac{(2m-1)(2m-3)\ldots1}{(2m+2n+1)(2m+2n-1)\ldots(2n+1)}\end{aligned}\right\}\qquad(34_a).$$

Thus α_0 and $\frac{3}{2}\alpha_1$ mean the losses of light by eclipse respectively for the uniformly brightened and the completely darkened star with no deformation. Their values have already been calculated and tabulated by Russell and Shapley.[1]

As will be seen later, it is most troublesome in numerical computations to find values of α_1, and it is the merit of such reformations to be able to pick up all the terms depending on it and bring them together into a simple form. In fact, performing the integration and picking up the coefficients of α_0 and α_1 in the expression of $\Delta\mathfrak{L}/H_0$, we get: for the uniform solution

$$\alpha_0\cdot[1+(n_0^2-1)v-\{2P_2(l_0)-\eta_1^2P_4(l_0)\}]\qquad(36)_u$$

and for the darkened solution

$$\frac{3}{2}\alpha_1\cdot\left\{1+\left(\frac{8}{5}n_0^2-\frac{6}{5}\right)v-\frac{16}{5}P_2(l_0)v_1\right\}-\alpha_0\cdot\frac{15}{8}P_3(l_0)\eta_1v_1\qquad(37)_d.$$

The forms in the bracket are quite the same as those found in the expression of the phase effect, (5), except for the asymmetrical term, P_3, in the darkened solution, which is proportional to the eclipsed area. Leaving this apart, Russell's method for the rectification of the curve is thus applicable in the present case so far as we consider only the terms depending on α_0 and α_1.

Other terms in the circular integrals are:
for the uniform solution

$$\begin{aligned}-n_0^2\frac{l_0}{l_2}\alpha'_{-1}\cdot v-\Big[&3l_0l_2\alpha'_{-1}+\Big\{3l_2(5l_0^2-1)\alpha'_0-\frac{3}{2}l_0(\alpha_{-1})+\frac{15}{2}l_0l_2^2(\alpha_{-1}^2)\Big\}\eta_1\\ &+\frac{5}{2}l_0l_2\Big\{3(7l_0^2-2)\alpha'_1+7l_2^2\alpha^3_{-1}-3\alpha'_{-1}\Big\}\eta_1^2\\ &+\frac{5}{2}\Big\{l_0^2(7l_0^2-3)(\alpha_2)+3l_2^2(7l_2^2-1)(\alpha_0^2)\Big\}\eta_1^2\Big]v_1\end{aligned}\qquad(38)_u$$

and for the darkened solution

$$\frac{3}{2}v\left[(\alpha_1^2)+(\alpha_3)+n_0^2\left\{\left(1-\frac{2}{l_2^2}\right)(\alpha_1^2)-\left(1+\frac{1}{l_2^2}\right)(\alpha_3)+2\frac{l_0}{l_2}(\alpha'_2-\alpha'_0)\right\}\right]+$$

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1. Ap. J. **35** (1912) 333 and **36** (1912) 243. ** In all subsequent notations with many suffixes, the suffix 0 will be omitted save where it is final: for example, $\alpha_n=\alpha_n^0$ and $\phi_n=\phi_{0,n}$.

$$+\frac{3}{2}v_1\Big[3\Big\{l_0^2(a_3)+l_2^2(a_1^2)\Big\}+6l_0l_2(a_2'-a_0')$$

$$+\frac{3}{2}\eta_1\Big\{5l_0^3(a_4)+15l_2^2l_0(a_2^2)+l_0(1-10l_0^2)(a_2)-10l_2^2l_0(a_0^2)$$

$$+15l_2l_0^2a_3'+5l_2^3a_1^3+l_2(1-20l_0^2)a'_1\Big\}$$

$$+\eta_1^2\Big\{\frac{35}{2}l_0^4(a_5)+105l_2^2l_0^2(a_3^2)+\frac{35}{2}l_2^4(a_1^4)-105l_0^2l_2^2(a_1^2)$$

$$-35l_0^4(a_3)+70l_2l_0(l_0^2a_4'+l_2^2a_2^3)$$

$$-35l_2l_0(3l_0^2a_2'+l_2^2a_0^3)+15l_0l_2a_0'\Big\}\Big] \qquad (39)_a.$$

§ 10. "*Boundary corrections*" are given by single integrations. If we write for simplicity

$$I_{\beta,\gamma}^m=\int_{N_1}^{-N_0+r}M_*^{\beta}L_*^{\gamma}N^m dN \qquad (40)$$

and

$$\left.\begin{aligned}\bar{I}_{\beta,\gamma}^m&=\int_{N_1}^{r-N_0}M_*^{\beta}L_*^{\gamma}(N+N_0)^m dN\\(I_{\beta,\gamma}^m)&=(r^2-N_0^2)I_{\beta,\gamma}^m-N_0I_{\beta,\gamma}^{m+1}\end{aligned}\right\} \qquad (41),$$

we get from (30)

$$\frac{1}{H_0}(\Delta\Omega_r)_b=v'\Big\{0.824r^2I_{-10}-I_{10}+\Big(\frac{n_0}{l_2}\Big)^2(I_{10}-l_0^2\bar{I}_{-10}^2)\Big\}$$

$$+v_1'\Big[3\Big(\frac{l_2}{r}\Big)^2\bar{I}_{-10}^2-I_{-10}+\eta_1'\Big\{5\Big(\frac{l_2}{r}\Big)^3\bar{I}_{-10}^3-3\frac{l_2}{r}\bar{I}'_{-10}\Big\}$$

$$+\frac{1}{4}\eta_1'^2\Big\{35\Big(\frac{l_2}{r}\Big)^4\bar{I}_{-10}^4-30\Big(\frac{l_2}{r}\Big)^2\bar{I}_{-10}^2+3I_{-10}\Big\}\Big]$$

$$-v'\Big[0.824(I_{-10})-(I_{10})+\Big(\frac{n_0}{l_2}\Big)^2\Big\{(I_{10})-l_0^2(I_{-10}^2)-l_2^2(I_{-12})+2l_0l_2(I'_{-11})\Big\}\Big]$$

$$-v_1[3\{l_0^2(I_{-12})+2l_0l_2(I'_{-11})+l_2^2(I_{-10}^2)\}-(I_{-10})]$$

$$+\eta_1v_1\Big[5\Big\{l_0^3(I_{-13})+3l_0^2l_2(I'_{-12})+3l_0l_2^2(I_{-11}^2)+l_2^3(I_{-10}^3)\Big\}$$

$$-3\Big\{l_0(I_{-11})+l_2(I'_{-10})\Big\}\Big]+\frac{1}{4}\eta_1^2v_1\Big[35\Big\{l_0^4(I_{-14})$$

$$+4l_0^3l_2(I'_{-13})+6l_0^2l_2^2(I_{-12}^2)+4l_0l_2^3(I_{-11}^3)+l_2^4(I_{-10}^4)\Big\}$$

$$-30\Big\{l_0^2(I_{-12})+2l_0l_2(I'_{-11})+l_2^2(I_{-10}^2)\Big\}+3(I_{-10})\Big] \qquad (42).$$

The first half depends on the distortion of the secondary and the second half on that of the primary and hence these terms may be combined with the circular integral when we calculate their numerical values as is actually done later.

The algebraic expression of the corresponding integral for the darkened solution will be given if we increase γ-suffix for all I by 1 and multiply them by $\frac{3}{2}$.

§ 11. *Reflexion Effect* is given by (16), and there the integration should be extended over the same region as in the circular integral. Hence using the same notations, we have at once

$$\frac{\Delta\mathfrak{L}_r}{H_0'\gamma_1^2}=l_0a_1+l_2a'_0+\gamma_1\left\{3(l_0^2a_2+2l_0l_2a_1'+l_2^2a_0^2)-a_0\right\}$$
$$+\frac{3}{2}\gamma_1^2\left\{5(l_0^3a_3+3l_0^2l_2a_2'+3l_0l_2^2a_1^2+l_2^3a_0^3)-3(l_0a_1+l_2a_0')\right\} \quad (43).$$

The total reflexion in the direction of the line of sight, if uneclipsed, is also given by the same integral taken over the whole circle I in Fig. 1, thus:

$$\mathfrak{L}_r=H_0'\gamma_1^2\left[\frac{2}{3}\left\{(\pi-\varphi_0)l_0-l_2\right\}+\frac{\pi}{4}\left\{l_0+P_2(l_0)\right\}\gamma_1-l_2^3\gamma_1^2\right] \quad (44).$$

where

$$-l_2=\sin\varphi_0.$$

Our chief concern is in the net reflexion effect, $\mathfrak{L}_r-\Delta\mathfrak{L}_r$.

II. Construction of the Tables.

§ 12. *Reduction of the Integrals to Elementary Forms.* Put

$$\phi_{mn}(t)=\int_0^t t^m(\sqrt{1-t^2})^{n-1}dt \quad (45),$$

which is a kind of β-function[1] and in particular

$$\left.\begin{aligned}&\phi_{mn}(1)=\frac{\pi}{2}\phi_{mn} \quad \text{when both } m \text{ and } n \text{ are even,}\\ &\phi_{mn}(1)=\phi_{mn} \quad \text{otherwise}\end{aligned}\right\} \quad (46)$$

and

$$\left.\begin{aligned}&\phi_{2n+1}(t)=\phi_{2n+1}\cdot t\sum_{i=0}^{n}\phi_{2i}\cdot(1-t^2)^i\\ &\phi_{2n}(t)=\phi_{2n}\cdot\left[t\sum_{i=1}^{n}\phi_{2i-1}\cdot(1-t^2)^{i-\frac{1}{2}}+\sin^{-1}t\right]\end{aligned}\right\} \quad (47).$$

Then we have

$$\int_{-M}^{M}L^{n-1}dM=2M_*^n\phi_n\left(\frac{M}{M_*}\right) \quad (48).$$

In the present problem, required integrals (33) are those for which $M=M_*$ or M_*'; in the former case

1. K. Pearson, Tables of the Incomplete Beta-Function.

$$\beta_n^m = 2\phi_{n+1}(1)\int_{-1}^{N_1} M_*^{n+1} N^m dN = 2\phi_{n+1}(1)\Big\{\phi_{m,\,n+2}(N_1) + \phi_{m,\,n+2}(1)\Big\} \quad (49)$$

and in the latter case

$$\left.\begin{aligned} \gamma_{2n}^m &= 2\phi_{2n+1}\cdot\sum_{i=0}^{n}\phi_{2(n-i)}\cdot I_{2i,\,1,\,2(n-i)}^m \\ \gamma_{2n-1}^m &= 2\phi_{2n}\cdot\Big\{\sum_{i=1}^{n}\phi_{2i-1}\cdot I_{2(n-i),\,1,\,2i-1}^m + II_{2n}^m\Big\} \end{aligned}\right\} \quad (50),$$

where

$$\left.\begin{aligned} I_{\alpha,\beta,\gamma}^m &= \int_{N_1}^{r-N_0} M_*^\alpha M_*'^\beta L_*^\gamma N^m dN \\ II_{2n}^m &= \int_{N_1}^{r-N_0} M_*^{2n}\Big(\sin^{-1}\frac{M_*'}{M_*}\Big)N^m dN \end{aligned}\right\} \quad (51).$$

The last integral can be reduced a little more; since

$$d\sin^{-1}\frac{M_*'}{M_*} = -\frac{NL_*^2 + N_0 M_*^2}{M_*^2 M_*' L_*}dN \quad (52),$$

we have by integration by part

$$II_{2n}^m = -\theta_1\phi_{m,\,2n+1}(N_1) + \int_{N_1}^{r-N_0}\frac{NL_*^2 + N_0 M_*^2}{M_*^2 M_*' L_*}\phi_{m,\,2n+1}(N)dN \quad (53),$$

where

$$\theta_1 = \frac{\pi}{2} \quad \text{or} \quad 0$$

according to whether the eclipse is partial or annular; for in the latter case N_1 is to be replaced by $-(r+N_0)$. The second term in (53) can always be expressed by a combination of $I_{2\alpha,\beta,\gamma}^m$ in particular

$$\left.\begin{aligned} II_{2n} &= -\theta_1\phi_{2n+1}(N_1) + \phi_{2n+1}\cdot\sum_{i=0}^{n}\phi_{2i}\cdot(I_{2(i-1),\,-1,\,1}^2 + N_0 I'_{2i,\,-1,\,-1}) \\ II'_{2n} &= \frac{1}{2(n+1)}\Big\{(1-N_1^2)^{n+1}\theta_1 - I'_{2n,\,-1,\,1} - N_0 I_{2(n+1),\,-1,\,-1} \end{aligned}\right\} \quad (54).$$

Thus all the integrals can be expressed in the elementary form, $I_{2\alpha,\beta,\gamma}^m$, which, in turn, can be reduced to simpler forms when α is positive, the only one with negative α being $I_{-2,-1,1}$ in II_{2n}: namely since by premise

$$\left.\begin{aligned} & M_*^2 = 1 - N^2 = L_*^2 + M_*'^2 \\ \text{or}\quad & L_*^2 = 1 - r^2 + N_0^2 + 2N_0 N \equiv B^2 + 2N_0 N \end{aligned}\right\} \quad (55),$$

we have at once

$$\left.\begin{aligned} I_{2a}^{2m} &= \sum_{i=0}^{m}(-1)^i\binom{m}{i}I_{2\alpha+2i} \\ I_{2\alpha,\beta,\gamma}^{2m} &= \sum_{i=0}^{\alpha}\binom{\alpha}{i}I_{\beta+2i,\,\beta+2\alpha-2i}^m \\ 2N_0 I'_{2\alpha,\beta,\gamma} &= I_{2\alpha,\beta,\gamma+2} - B^2 I_{2\alpha,\beta,\gamma} \end{aligned}\right\} \quad (56)$$

and

$$\left.\begin{aligned} H_{2n}^{2m} &= \sum_{i=0}^{m}(-1)^i \binom{m}{i} H_{2(n+i)} \\ H_{2n}^{2m+1} &= \sum_{i=0}^{m}(-1)^i \binom{m}{i} H'_{2(n+i)} \end{aligned}\right\} \quad (57).$$

In short, our elementary integrals are $I_{\beta,\gamma}$ and $I_{-2,-1,1}$. Some of them can not be evaluated in a closed form and must be calculated by means of numerical integrations or series expansion. In this case the most troublesome is $I_{-2,-1,1}$; for it is easily seen that although these integrals depend on two parameters, N_0 and r, $I_{\beta,\gamma}$ can be transformed into an integral containing one parameter with a coefficient of a simple function of N_0 and r, while it is not the case for $I_{-2,-1,1}$, or more directly, this contains an elliptic integral of the third kind. But fortunately it is unnecessary to find it anew, since it appears only in α_1 of (38) eventually and its numerical values may be taken from Russell's table cited before.

Now we show the circumstances more closely. Comparing (50) and (54), we have at once

$$H_{2n} + \frac{1}{2\phi_{2n}}\beta_{2n-1} = \phi_{2n+1}\cdot\left\{\theta_1 + \sum_{i=0}^{n}\phi_{2i}\cdot(I^2_{2(i-1),-1,1} + N_0 I'_{2i,-1,-1})\right\} \quad (58)$$

and referring to (57) and (49)

$$H_{2n}^{2m} + \frac{1}{2\phi_{2n}}\beta_{2n-1}^{2m} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\left\{H_{2(n+i)} + \frac{1}{2\phi_{2(n+1)}}\beta_{2(n+i)-1}\right\} \quad (59).$$

Further, if we write

$$\left.\begin{aligned} (H_{2n}) &\equiv H_{2n} + \frac{1}{2\phi_{2n}}\beta_{2n-1} - \frac{3}{2}\phi_{2n+1}\cdot\alpha_1 \\ &= \phi_{2n+1}\cdot\left\{\sum_{i=2}^{n}\phi_{2i}\cdot(I^2_{2(i-1),-1,1} + N_0 I'_{2i,-1,-1}) - \frac{3}{2}I_{11}\right\} \\ (H_0) &= -\phi_2\cdot(3I_{11} + I^2_{-11} + N_0 I'_{2,-1,-1}) \end{aligned}\right\} \quad (60),$$

we have

$$(H_{2n}^{2m}) \equiv \sum_{i=0}^{m}(-1)^i\binom{m}{i}(H_{2n+2i}) = H_{2n}^{2m} + \frac{1}{2\phi_{2n}}\beta_{2n-1}^{2m} - \frac{3}{2}\phi_{2m,2n+1}\cdot\alpha_1 \quad (61)$$

and hence referring to (34) and (50)

$$(\alpha_{2n-1}^{2m}) = 2\phi_{2n}\cdot\left\{\sum_{i=0}^{n}\phi_{2i-1}\cdot I^{2m}_{2(n-i),1,2i-1} + (H_{2n}^{2m})\right\} \quad (62).$$

Quite similarly, we find

$$\alpha_{2n-1}^{2m+1} = 2\phi_{2n}\cdot\left\{\sum_{i=1}^{n}\phi_{2i-1}\cdot I^{2m+1}_{2(n-i),1,2i-1} + (H_{2n}^{2m+1})\right\} \quad (63),$$

where

$$\left.\begin{aligned}(H'_{2n}) &\equiv H'_{2n}+\frac{1}{2\phi_{2n}}\beta'_{2n-1}=-\frac{1}{2(n+1)}(I'_{2n,\,-1,\,1}+N_0I_{2(n+1),\,-1,\,-1})\\ (H^{2m+1}_{2n}) &=\sum_{i=0}^{m}(-1)\binom{m}{i}(H'_{2(n+i)})\end{aligned}\right\}\quad(64).$$

Thus $I_{-2,-1,1}$ is not involved in (a^{2m}_{2n-1}) and a^{2m+1}_{2n-1} and moreover we need not calculate γ and β separately for these circular integrals. Further it will be observed that for circular integrals with even lower suffix, β_{2n} comes out in place of (H_{2n}).

Finally it may be added that all integrals contained in boundary corrections and reflexion effect can be expressed by those obtained in this article, so that further considerations are unnecessary.

§ 13. *Normalization of* $I_{\beta,\gamma}$. Now introduce the following transformation :

$$N=-N_0+r(1-2t^2)\quad(65),$$

t being a new independent variable, then

$$\left.\begin{aligned}dN&=-4rtdt\\ M'^2_*&=4r^2t^2(1-t^2)\\ L^2_*&=A^2(1-k^2t^2)\end{aligned}\right\}\quad(66),$$

where

$$\left.\begin{aligned}A^2&=1-(r-N_0)^2\\ k^2&=4N_0r/A^2\end{aligned}\right\}\quad(67);$$

and $t=0,\ 1$ and k^{-1} (68)

respectively at $N=r-N_0$, $-(r+N_0)$ and N_1, so that the limits of integration for t become $(0,k_1)$, k_1 being unity when the eclipse is annular and k^{-1} when it is partial.

The result of the transformation is

$$I_{\beta,\gamma}=2(2r)^{\beta+1}A^{\gamma}T_{\beta,\gamma}\quad(69),$$

where

$$T_{\beta,\gamma}=\int_0^{k_1}t^{\beta+1}(1-t^2)^{\frac{\beta}{2}}(1-k^2t^2)^{\frac{\gamma}{2}}dt\quad(70),$$

β being always odd in the present problem and $\beta+\gamma\leqslant 14$. $T_{\beta,\gamma}$ can not be evaluated in finite terms when γ is odd. It was found to be most convenient for numerical computations to expand some of them into a series in k or k^{-1} and to find others by recursion formulae.

a) Annular eclipse. $(k\leqslant 1)$.

$$T_{\beta,\gamma}=\sum_{i=0}^{s}(-1)^i\binom{\frac{\gamma}{2}}{i}\phi_{\beta+1+2i,\,\beta+1}(1)\cdot k^{2i}\quad(71),$$

where $s=\frac{\gamma}{2}$ when γ is even and ∞ when it is odd. In the latter case it will be seen from (35) that

$$(-1)^i\binom{\frac{2n-1}{2}}{i}=(-1)^n\phi_{2(i-n),\,2n} \quad \text{for} \quad n\leqslant i \tag{72}$$

On the other hand, there is a recursion formula,

$$T_{\beta,\gamma}=(k^2-1)T_{\beta,\,\gamma-4}+(2-k^2)T_{\beta,\,\gamma-2}-k^4T_{\beta+2,\,\gamma-4} \tag{73}$$

so that it is sufficient to evaluate $T_{\beta,\gamma}$ by (71) only for $\gamma\leqslant 2$ and $\beta\leqslant 5$.

b) Partial eclipse. $(k\geqslant 1)$.

If we take kt to be a new variable, its upper limit in (70) becomes unity and hence

$$T_{\beta,\,\gamma}=\left(\frac{1}{k}\right)^{\beta+2}\sum_{i=0}^{n}(-1)^i\binom{\frac{\beta}{2}}{i}\phi_{\beta+1+2i,\,\gamma+1}(1)\left(\frac{1}{k}\right)^{2i} \tag{74}$$

As for the recursion formula, changing the arrangement in (73), it follows

$$T_{\beta,\,\gamma}=\frac{1}{k^2}\left\{\left(\frac{1}{k^2}-1\right)T_{\beta-2,\,\gamma}+\left(\frac{2}{k^2}-1\right)T_{\beta-2,\,\gamma+2}-\frac{1}{k^2}T_{\beta-2,\,\gamma+4}\right\} \tag{75}$$

so that in this case it is only $T_{-1\gamma}$ to be calculated in series. Further $k^{-(\beta+2)}$ in $T_{\beta,\gamma}$ was actually kept apart and brought together with the constant factor in $I_{\beta,\gamma}$ (69).

§ 14. *Special Case—Central Eclipse.* Here we have

$$\left.\begin{aligned}&N_0=l_2=k=0,\ A^2=B^2=1-r^2\,;\\&I_{\beta,\,\gamma}=\phi_{\beta+1}\cdot\pi r^{\beta+1}A^{\gamma},\ I'_{\beta\gamma}=0,\ I_{-2,\,-1,\,1}=\pi\\&\frac{1}{r^2}(I^m_{\beta\gamma})=\bar{I}^m_{\beta,\gamma}=I^m_{\beta,\gamma}\end{aligned}\right\} \tag{76}$$

and can easily show

$$\left.\begin{aligned}&a_n=\frac{2}{n+2}\pi(1-A^{n+2})\\&(a_{2n})=\frac{\pi}{n+1}A^2(1-A^{2n})\\&(a_{2n+1})=\frac{2\pi}{2n+3}A^3(1-A^{2n})\\&(a_1^2)=-\frac{1}{5}\pi r^2A^3\end{aligned}\right\} \tag{77}$$

Hence inserting these values into equations (36)—(39), (42) and (43), we obtain

$$\frac{1}{r^2\pi H_0}(\Delta\mathfrak{L}_u)_c=\left\{1-v-(2-\eta_1^2)v_1\right\}-(3A\eta_1+5A^2\eta_1^2)v_1 \tag{78$_u$}$$

$$\frac{1}{\pi H_0}(\Delta\mathfrak{L}_d)_c=\left(1-\frac{6}{5}v-\frac{16}{5}v_1\right)(1-A^3)-\frac{15}{8}r^2\eta_1v_1$$
$$+\frac{3}{2}A^3r^2\left[\frac{1}{5}v+\left\{\frac{6}{5}-(9-5A^2)\eta_1^2\right\}\right]-\frac{3}{8}r^2A^2(17-10A^2)\eta_1v_1 \tag{78$_d$}$$

$$\frac{1}{r^2\pi H_0}(\Delta\mathfrak{L}_u)_b=0.324v'-\left(1-\frac{3}{4}\eta_1^2\right)v_1'$$
$$-\left(0.824-\frac{1}{2}r^2\right)v-2\left\{P_2(A)+\eta_1P_3(A)+\eta_1^2P_4(A)\right\}v_1 \quad (79)_u,$$

$$(\Delta\mathfrak{L}_d)_b=\frac{3}{2}A(\Delta\mathfrak{L}_u)_b \quad (79)_d$$

and

$$\Delta\mathfrak{L}_r=\pi H_0'\eta_1^2\left\{\frac{2}{3}(1-A^3)+\frac{1}{2}r^2(1+3A^2)\eta_1+3A^3r^2\eta_1^2\right\} \quad (80).$$

Similarly in the case of inner contact, these are found in explicit form, but are too long to be given here.

§ 15. *Explanation of the Tables.* Elementary integrals, $I_{\beta,\gamma}$ have been computed in unit of π for given values of N_0 and r, and then the circular integrals and the boundary corrections have been constructed for a few different values of η_{1l}, the suffix l meaning the larger star. So far N_0 and r were measured with in the unit of a mean radius, η_1 of the primary—the eclipsed star—(ref. (24)), but it is convenient hereafter to take that of the larger component to be unity throughout. Further in the following tables numerical corrections are given for residual light, and not directly for loss of light, separated into the rotational, ΔR (coefficient of $\pi H_0 v$) and the tidal terms, ΔT (coefficient of $\pi H_0 v_1$). In either case the parts of the boundary corrections depending on the deformation of the primary are combined with the circular integrals and the remainder (coefficients of $\pi H_0 v'$ and $\pi H_0 v_1'$) are given separately. We must further divide the rotational terms into two parts, those depending and those not depending on the inclination, but the former corrections are in general negligible. Finally, zero order terms in the circular integrals and constant part in the rotational trems are set aside.

Hence the predicted light curve of the binary can be expressed in the form,

$$\lambda=\pi H_0\Big[(1-a)(1+vf_n)+\Delta Tv_1+\Delta T'v_1'$$
$$+\left\{\Delta R+\Delta R_n\left(\frac{n_0}{\eta_{1l}}\right)^2\right\}v+\left\{\Delta R'+\Delta R_n'\left(\frac{n_0}{\eta_{1l}}\right)^2\right\}v'\Big]$$
$$+\pi H_0'(1+v'f_n+\Delta H'v_1') \quad (81),$$

but for the reflexion effect, which will be considered independently.

Here

$$\pi a=a_0, \qquad f_n=n^2-1$$

or $$\pi a=\frac{3}{2}a_1,\ f_n=\frac{8}{5}n_0^2-\frac{6}{5} \quad (82)$$

according to whether the star is uniformly illuminated or completely darkened towards the limb. It is, however, unnecessary to preserve f_n in this expression as it is constant for the same star, and hereafter we may neglect it by writing

$$\mathfrak{L}=\pi H_0(1+f_n v) \tag{83}$$

in place of πH_0 without disturbing the accuracy. ΔH means the tidal corrections in (5), and ΔT takes the same form outside of the eclipse. The usual method of rectification takes $v_1=v_1'$ and $\Delta T=\Delta H(1-\alpha)$ and neglects all other corrections. But it was found that the difference, $\Delta T-\Delta H(1-\alpha)$ is not insignificant.

Now ΔT and ΔR_n are functions of N_0, r and η_u, ΔR being independent of η_u, and we have actually computed them for different values of these three parameters. It is, however, too voluminous to print the whole result here, so that we give their values only for $\eta_u=0.45$ and their increments for $\Delta\eta=0.1$. Interesting cases are those in which the value of η_u is not far from this, for the deformation will not be great for its smaller values and our calculation fails for larger ones. Accordingly linear approximation for different η_u is sufficient for the present purpose.

For convenience' sake, $100\Delta T$ and $100\Delta R$ are tabulated, their increments for $\Delta\eta=0.1$ being printed in a smaller type. Steps of r and N_0 are self-evident; $N_0-r=-r$, $1-2r$ and 1.0 correspond respectively to the central eclipse, the inner contact and the outer. The upper half of each table corresponds to the case when the primary is larger, and the rest to the case when it is smaller. The uniform and the darkened values are given separately. For completeness we add the tables of α_0 and $\frac{3}{2}\alpha_1$ taken from Russell's paper.

§ 16. A direct method of the use of these tables is as follows: if we find approximate values for $\mathfrak{L}$, $\mathfrak{L}'$, r, η_u, i and the common ellipticity corresponding to v_1, we can rectify the observed light curve by means of these tables and then get better approximations successively.

In this connection, it is more convenient to rearrange the tidal terms. First we consider the case in which $n_0=0$. Let the maximum luminosity be unity and λ_0 be that of the minimum corresponding to the total eclipse. Then we have

$$\left.\begin{aligned}
1&=\mathfrak{L}+\mathfrak{L}'+\mathfrak{L}v_1\Delta H_{\frac{\pi}{2}}+\mathfrak{L}'v_1'\Delta H'_{\frac{\pi}{2}}\\
\mathfrak{L}_l&=\lambda_0-(\mathfrak{L}v_1\Delta H_0)_l\\
\mathfrak{L}_s&=1-\lambda_0-(\mathfrak{L}v_1\Delta H_{\frac{\pi}{2}}+\mathfrak{L}'v_1'\Delta H'_{\frac{\pi}{2}})+(\mathfrak{L}v_1\Delta H_0)_l
\end{aligned}\right\} \tag{84}$$

where suffixes l and s respectively signify the larger and the smaller component, and the accent the secondary as usual. Inserting these relations into (81) we obtain for net phase effect

$$\lambda = 1 - \lambda_0 \alpha + \Delta + \bar{T}\bar{v}\Delta\underline{T}\underline{v} + \mathfrak{L}v_1'\Delta T' + \text{ the rotational terms} \tag{85}$$

$$\left.\begin{aligned} \Delta\bar{T} &= -(\Delta H + \Delta H')_{\frac{\pi}{2}} + \alpha(\Delta H_l)_0 + \Delta T_l + \Delta H_s \\ \Delta\underline{T} &= \quad (\Delta H_s - \Delta H_l)_{\frac{\pi}{2}} + \alpha(\Delta H_l)_0 + \Delta T_l - \Delta H_s \end{aligned}\right\} \tag{86}$$

when the primary is larger and

$$\lambda = 1 - (1 - \lambda_0)\alpha + \Delta\bar{T}\bar{v} + \Delta\underline{T}\underline{v} + \mathfrak{L}v_1'\Delta T' + \text{ the rotational terms} \tag{87}$$

$$\left.\begin{aligned} \Delta\bar{T} &= -(\Delta H + \Delta H')_{\frac{\pi}{2}}(1-\alpha) - (\Delta H_l)_0\alpha + \Delta T_s + \Delta H_l \\ \Delta\underline{T} &= \quad (\Delta H_s - \Delta H_l)_{\frac{\pi}{2}}(1-\alpha) - (\Delta H_l)_0\alpha - \Delta T_s + \Delta H_l \end{aligned}\right\} \tag{88}$$

when it is smaller. In either case

$$\left.\begin{aligned} 2\bar{v} &= (\mathfrak{L}v_1)_l + (\mathfrak{L}v_1)_s \\ 2\underline{v} &= (\mathfrak{L}v_1)_l - (\mathfrak{L}v_1)_s \end{aligned}\right\} \tag{89}.$$

In an ideal case λ_0 corresponds to one of observed minima. When n_0 does not vanish, we must first find the value of λ_0, which, in turn, is calculated by (88), from minimum light. But so long as n_0 is small, the correction thus introduced is also small and may well be neglected.

Outside of the eclipse, it becomes simply

$$\lambda = 1 + \Delta\bar{H}\bar{v} + \Delta\underline{H}\underline{v} \tag{90},$$

a relation which gives total phase effect. Referring to (5) we find

$$\left.\begin{aligned} \Delta\bar{H} &= \Delta H_1 + \Delta H_2(1+r^2)\eta_{1l}^2 \\ \Delta\underline{H} &= \qquad\quad \Delta H_2(1-r^2)\eta_{1l}^2 \end{aligned}\right\} \tag{91$_u$}$$

$$\left.\begin{aligned} \Delta H_1 &= -2\{1 + 2P_2(l_0)\} \\ \Delta H_2 &= -\frac{3}{8} + P_4(l_0) \end{aligned}\right\} \tag{92$_u$}$$

for the uniform solution and

$$\left.\begin{aligned} \Delta\bar{H} &= \Delta H_1 \pm \Delta H_2(1-r)\eta_{1l} \\ \Delta\underline{H} &= \qquad \pm\Delta H_2(1+r)\eta_{1l} \end{aligned}\right\} \tag{91$_d$}$$

$$\left.\begin{aligned} \Delta H_1 &= -3.2\{1 + 2P_2(l_0)\} \\ \Delta H_2 &= -\frac{15}{8}P_3(l_0) \end{aligned}\right\} \tag{92$_d$}$$

for the darkened solution. In $(91)_d$ the upper sign is to be taken when the primary is larger and the lower sign when the primary is smaller.

$\Delta\bar{T}$ and $\Delta\underline{T}$ are tabulated in the same manner as the rest but as for $\Delta\bar{H}$ and $\Delta\underline{H}$ their components, ΔH_1 and ΔH_2 are given for different values of $-l_2 = N_0\eta_{1l}$. It will be seen that $\Delta\bar{T}$ or $\Delta\bar{H}$ is exclusively large.

§ 17. As for the reflexion effect, we have computed only its terms of the lowest order (coefficients of $\mathfrak{L}'\eta_1^2$) in (43) and (44), and rearranged them similarly to those in the preceding article. Namely we divided the net reflexion effect, $\mathfrak{L}_r - \Delta\mathfrak{L}_r$ into the symmetrical, $\bar{w}\Delta\bar{\mathfrak{L}}_r$ and the antisymmetrical term, $\underline{w}\Delta\underline{\mathfrak{L}}_r$ and adjusted them to vanish at quadrature and mid-totality. Then

$$\left.\begin{aligned} \bar{w} &= \frac{1}{2}(\mathfrak{L}'\eta_1^2 + \mathfrak{L}\eta_1'^2) \\ \underline{w} &= \frac{1}{2}(\mathfrak{L}'\eta_1^2 - \mathfrak{L}\eta_1'^2) \end{aligned}\right\} \qquad (93)$$

$$\left.\begin{aligned} \pi\Delta\bar{\mathfrak{L}}_r &= \frac{2}{3}\Big\{(\pi - 2\varphi)l_0 - 2l_2 - 2(1-a)\Big\} - (l_0\alpha_1 + l_2\alpha_0') \\ \pi\Delta\underline{\mathfrak{L}}_r &= \frac{2}{3}\pi l_0 - (l_0\alpha_1 + l_2\alpha_0') \end{aligned}\right\} \qquad (94)$$

where $\pi a = \alpha_0$ or 0 according to whether the primary is larger or not, and in general $\underline{w}$ changes the sign before and after maximum.

These correction terms may be added simply to (86) or (87), if required.

§ 18. In actual applications, there are some difficulties concerning the terms defining the deformation. The usual method assumes a similar deformation but we prefer to differentiate it for each component. If we know the value of the mass-ratio, we can estimate them theoretically in connection with the dimensions of the two stars, but generally this is not the case. Moreover, our intention is rather to compare the results of our observation with such an estimate.

It is relatively simple to find the approximate value of $\bar{v}$ from the uneclipsed part of the light curve, but at present we can not hope to estimate $\underline{v}$ by actual observation for reasons that will be obvious later. Accordingly we turn into account more safely

$$v_1 v_1' = (\eta_1 \eta_1')^3 \qquad (95),$$

a theoretical relation which may be trusted to some degree. If this is taken for granted, we can find $\mathfrak{L}$, $\mathfrak{L}'$, v_1 and v_1' for given η_1 and η_1' in connection with the luminosities of maximum and one of minima which corresponds to the total eclipse.

In general there will be two solutions which satisfy this relation and give the observed value for $\bar{v}$. If there is only one solution, we may say that in a sense the stars are similarly deformed. Otherwise our theory fails ultimately.

In the first case it may be possible to see further which one follows a light curve in better conformity with the observation. At any rate when the solution is possible, we can find corresponding mass ratio by means of (3) and see whether it is plausible or not. Closer examinations of the circumstances will be given in the following section.

Phase Effect for Uniform Stars

Table I$_u$. $100\Delta T(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	83(−20)			84(−21)	85(−22)	92(−23)	106(−25)	103(−28)	106(−30)	100(−36)
5	48(−22)			49(−23)	57(−24)	76(−27)	74(−31)	83(−34)	89(−36)	85(−42)
6	20(−24)		20(−24)	27(−25)	47(−26)	43(−31)	53(−35)	65(−39)	73(−41)	69(−49)
7	8(−21)		10(−21)	27(−23)	15(−28)	23(−35)	34(−39)	47(−43)	56(−47)	52(−57)
8	16(−16)	16(−16)	25(−18)	0(−22)	−1(−30)	5(−38)	16(−43)	30(−47)	38(−53)	34(−66)
9	45(−9)	46(−10)	0(−15)	−12(−21)	−14(−30)	−11(−39)	−1(−46)	12(−50)	21(−59)	14(−74)
1.0	85(−7)	15(−7)	−7(−12)	−19(−20)	−25(−30)	−25(−38)	−17(−47)	−4(−54)	3(−64)	−6(−82)
9		0(0)	9(−6)	−10(−12)	−20(−22)	−22(−32)	−16(−41)	−2(−50)	12(−58)	12(−74)
8			0(0)	5(−5)	−12(−14)	−19(−24)	−16(−35)	−1(−46)	18(−53)	27(−65)
7				0(0)	2(−6)	−14(−14)	−16(−27)	−3(−40)	21(−48)	40(−58)
6					0(0)	−2(−6)	−15(−17)	−8(−32)	19(−44)	49(−52)
5						0(0)	−6(−8)	−13(−22)	12(−38)	54(−46)
4							0(0)	−10(−10)	−1(−30)	53(−42)

Table II$_u$. $100\Delta R$

r \ N_0-r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	12			12	12	12	12	−1	−4	−6
5	18			18	18	18	3	−2	−5	−7
6	23		23	23	23	6	1	−3	−6	−7
7	28		28	28	9	4	0	−4	−6	−7
8	32	32	32	11	7	3	−1	−4	−7	−8
9	34	34	12	9	6	2	−1	−4	−7	−8
1.0	14	14	11	8	5	2	−2	−5	−7	−8
9		0	14	10	7	3	0	−4	−7	−8
8		0	0	13	9	5	1	−3	−7	−9
7			0	0	12	8	3	−1	−6	−9
6			0	0	0	11	6	1	−5	−9
5				0	0	0	10	4	−3	−8
4				0	0	0	0	9	0	−7

These tables have been computed mainly by Dr. R. Kamiya.

Table III$_u$. $100\Delta R_n(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	0(0)			0(0)	− 8(5)	− 9(6)	− 11(5)	9(3)	7(1)	4(0)
5	0(0)			− 14(7)	− 15(7)	− 18(7)	16(4)	10(2)	7(1)	4(0)
6	0(0)		0(0)	− 25(10)	− 29(10)	28(6)	18(4)	11(3)	6(1)	3(0)
7	0(0)		− 39(12)	− 42(13)	52(8)	29(5)	17(3)	10(1)	6(0)	3(0)
8	0(0)	0(0)	− 64(14)	100(10)	47(6)	26(4)	16(3)	10(1)	5(1)	3(0)
9	0(0)	− 100(16)	185(8)	72(7)	39(5)	23(4)	14(3)	9(0)	5(0)	2(0)
1.0	0(0)	230(6)	95(6)	52(6)	32(4)	21(3)	13(2)	8(1)	4(0)	2(0)
9		0(0)	40(4)	58(6)	40(4)	26(4)	17(3)	10(1)	5(0)	3(0)
8		0(0)	0(0)	14(4)	42(4)	32(4)	22(4)	13(2)	7(0)	3(0)
7			0(0)	0(0)	11(4)	34(4)	26(4)	16(3)	9(2)	4(0)
6			0(0)	0(0)	0(0)	13(3)	29(4)	22(4)	12(2)	6(0)
5				0(0)	0(0)	0(0)	16(2)	25(3)	16(3)	9(2)
4				0(0)	0(0)	0(0)	0(0)	19(2)	22(4)	13(6)

Table IV$_u$. $100\Delta T'(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	16(0)			16(0)	15(0)	15(0)	14(0)	7(− 1)	3(− 1)	1(− 1)
5	24(0)			24(0)	23(− 1)	22(− 1)	12(− 1)	7(− 2)	3(− 2)	0(− 2)
6	34(0)		34(0)	34(− 1)	33(− 2)	18(− 2)	11(− 2)	6(− 2)	2(− 2)	− 1(− 3)
7	45(− 2)		45(− 2)	44(− 2)	24(− 3)	16(− 4)	10(− 4)	4(− 4)	0(− 4)	− 3(− 4)
8	57(− 3)	57(− 3)	57(− 3)	30(− 4)	21(− 4)	14(− 5)	7(− 5)	2(− 5)	− 2(− 5)	− 5(− 4)
9	71(− 4)	71(− 4)	35(− 4)	25(− 5)	18(− 6)	11(− 6)	5(− 6)	− 1(− 6)	− 5(− 6)	− 7(− 6)
1.0	85(− 7)	35(− 5)	28(− 6)	21(− 7)	14(− 8)	7(− 8)	1(− 8)	− 4(− 8)	− 8(− 7)	− 10(− 7)
9		0(0)	29(− 1)	25(− 6)	18(− 8)	11(− 8)	4(− 8)	− 3(− 8)	− 8(− 8)	− 10(− 7)
8		0(0)	0(0)	26(− 6)	22(− 8)	15(−10)	7(−10)	0(− 9)	− 7(− 8)	− 10(− 8)
7			0(0)	0(0)	24(− 8)	19(−10)	11(−11)	2(−11)	− 5(−10)	− 10(− 9)
6			0(0)	0(0)	0(0)	23(−10)	17(−14)	7(−13)	− 3(−12)	− 9(−10)
5				0(0)	0(0)	0(0)	22(−14)	13(−15)	1(−14)	− 8(−12)
4				0(0)	0(0)	0(0)	0(0)	21(−20)	8(−19)	− 6(−17)

Table V_u. $100\Delta R'$

$N_0 - r$ \ r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	− 5			− 5	− 5	− 5	− 5	− 2	− 3	− 3
5	− 8			− 8	− 8	− 8	− 4	− 4	− 4	− 4
6	−12		−12	−12	−12	− 5	− 6	− 6	− 6	− 5
7	−16		−16	−16	− 7	− 8	− 9	− 9	− 8	− 7
8	−21	−21	−21	− 9	−11	−11	−11	−11	−10	− 8
9	−26	−26	−12	−13	−14	−14	−14	−14	−12	−10
1.0	−32	−17	−17	−18	−18	−18	−17	−16	−14	−11
9		0	−22	−22	−22	−22	−21	−20	−17	−13
8		0	0	−26	−27	−27	−26	−24	−21	−17
7			0	0	−31	−33	−32	−30	−27	−20
6			0	0	0	−37	−40	−38	−35	−26
5				0	0	0	−46	−49	−45	−35
4				0	0	0	0	−60	−62	−50

Table VI_u. $100\Delta R'_u(\eta_u = 0.45)$ and its Increment for $\Delta\eta = 0.1$

$N_0 - r$ \ r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	− 2(−1)			− 2(−1)	− 2(−1)	− 2(−1)	− 2(−1)	− 3(0)	0(0)	1(0)
5	− 3(−1)			− 3(−1)	− 3(−1)	− 3(−1)	− 7(−1)	− 1(−1)	1(−1)	1(0)
6	− 4(−2)		− 4(−2)	− 4(−2)	− 4(−2)	−13(−1)	− 2(−1)	2(−1)	2(−1)	2(−1)
7	− 5(−2)		− 5(−2)	− 5(−2)	−24(−1)	− 3(−1)	2(−1)	3(−1)	2(−1)	2(−1)
8	− 7(−3)	− 7(−3)	− 7(−3)	−41(−2)	− 3(−1)	4(−1)	4(−1)	4(−1)	3(−1)	2(−1)
9	− 8(−4)	− 8(−4)	−56(−2)	3(−2)	8(−2)	7(−2)	6(−2)	4(−2)	3(−1)	1(−1)
1.0	−10(−1)	75(−2)	34(−2)	20(−2)	13(−2)	9(−2)	6(−2)	4(−2)	2(−2)	1(−1)
9		0(0)	140(−3)	45(−3)	20(−3)	12(−3)	8(−2)	4(−2)	2(−2)	1(−1)
8		0(0)	0(0)	92(−3)	32(−3)	17(−3)	9(−3)	5(−3)	2(−2)	1(−2)
7			0(0)	0(0)	49(−4)	24(−4)	11(−4)	5(−3)	2(−3)	0(−2)
6			0(0)	0(0)	0(0)	31(−4)	14(−4)	5(−4)	1(−4)	−1(−3)
5				0(0)	0(0)	0(0)	15(−5)	5(−6)	−1(−5)	−3(−4)
4				0(0)	0(0)	0(0)	0(0)	3(−7)	−4(−7)	−6(−6)

Table VII$_u$. $100\Delta\bar{T}(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

N_0-r / r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	−518(20)			−518(20)	−518(22)	−518(25)	−526(33)	−494(43)	−466(53)	−427(70)
5	−498(23)			−498(24)	−502(26)	−512(33)	−489(41)	−469(51)	−440(64)	−399(83)
6	−488(25)		−488(25)	−492(26)	−506(30)	−482(39)	−463(49)	−441(60)	−412(74)	−368(96)
7	−498(25)		−499(25)	−512(28)	−481(36)	−459(45)	−438(57)	−414(69)	−383(85)	−336(111)
8	−530(22)	−530(22)	−537(24)	−491(30)	−464(40)	−438(51)	−413(65)	−387(78)	−353(97)	−302(127)
9	−589(18)	−589(19)	−522(23)	−483(32)	−449(42)	−417(55)	−389(69)	−360(86)	−322(109)	−266(145)
1.0	−575(11)	−565(17)	−516(23)	−472(31)	−433(43)	−396(56)	−364(72)	−332(93)	−289(121)	−227(165)
1.0	0(0)	− 63(7)	− 87(16)	−116(25)	−143(40)	−171(54)	−201(71)	−224(91)	−230(120)	−206(165)
9		0(0)	− 51(7)	− 82(16)	−116(29)	−153(44)	−190(61)	−224(81)	−247(107)	−241(143)
8		0(0)	1(0)	− 46(8)	− 82(19)	−124(33)	−170(51)	−217(71)	−257(94)	−271(123)
7			0(0)	4(0)	− 42(9)	− 86(22)	−139(39)	−200(60)	−259(82)	−295(106)
6				1(0)	7(0)	− 41(12)	− 96(27)	−168(48)	−248(70)	−311(92)
5				0(0)	4(0)	10(4)	− 43(16)	−118(34)	−221(59)	−319(79)
4				0(0)	1(0)	7(0)	15(4)	− 51(19)	−166(45)	−309(68)

Table VIII$_u$. $100\Delta\underline{T}(\eta_u=0.45)$ for Increment $\Delta\eta=0.1$

N_0-r / r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	79(18)			79(18)	74(18)	60(16)	35(13)	28(9)	14(1)	2(− 5)
5	96(21)			96(20)	82(20)	54(19)	50(16)	34(11)	16(3)	2(− 3)
6	103(22)		103(22)	94(22)	67(21)	70(21)	58(18)	40(12)	20(3)	4(− 3)
7	90(20)		87(20)	66(20)	81(21)	77(21)	65(19)	45(12)	23(4)	5(− 2)
8	54(14)	54(14)	43(16)	76(18)	84(20)	82(21)	71(20)	50(13)	26(5)	7(− 1)
9	− 7(6)	−10(9)	50(12)	75(16)	87(19)	87(21)	76(19)	54(12)	30(4)	10(− 2)
1.0	0(0)	6(5)	46(11)	73(13)	87(17)	90(21)	80(17)	57(11)	32(1)	10(− 7)
1.0	0(0)	− 6(− 5)	−46(−11)	−73(−13)	−87(−17)	−90(−21)	−80(−17)	−57(−11)	−32(− 1)	−10(+ 7)
9		0(0)	−10(− 2)	−46(− 8)	−72(−13)	−84(−17)	−81(−19)	−63(−15)	−38(− 7)	−14(− 0)
8			1(0)	−11(− 4)	−48(− 8)	−72(−13)	−79(−17)	−68(−17)	−44(−10)	−19(− 4)
7			0(0)	4(0)	−11(− 3)	−50(− 6)	−71(−14)	−70(−17)	−50(−15)	−23(− 6)
6				1(0)	6(0)	−14(− 2)	−54(− 7)	−69(−14)	−55(−16)	−26(− 9)
5				0(0)	4(0)	10(3)	−18(0)	−58(− 9)	−59(−15)	−31(−11)
4				0(0)	1(0)	6(0)	15(4)	−25(− 2)	−58(−11)	−36(−14)

PHASE EFFECT FOR DARKENED STARS

Table IX_a. $100\Delta T(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	255(0)			255(0)	252(− 4)	247(−15)	246(−27)	243(− 40)	229(− 59)	190(− 84)
5	194(− 7)			194(− 8)	193(−15)	196(−26)	202(−39)	204(− 52)	196(− 71)	156(− 99)
6	139(−11)		139(−11)	140(−14)	144(−24)	152(−37)	167(−50)	174(− 64)	163(− 83)	120(−113)
7	95(−11)		101(−10)	97(−18)	102(−31)	117(−45)	133(−60)	141(− 76)	129(− 97)	84(−127)
8	65(−10)	65(−10)	63(−11)	63(−22)	73(−37)	86(−53)	100(−69)	108(− 86)	94(−109)	46(−139)
9	47(− 6)	42(− 7)	36(−12)	41(−25)	48(−42)	57(−60)	70(−77)	76(− 95)	60(−119)	8(−149)
1.0	0(0)	24(− 4)	28(−14)	29(−28)	29(−45)	33(−65)	41(−83)	44(−102)	26(−126)	−31(−155)
9		0(0)	21(− 4)	26(−14)	27(−30)	32(−50)	44(−71)	58(− 88)	53(−112)	12(−141)
8		0(0)	0(0)	19(− 4)	24(−15)	28(−34)	41(−56)	63(− 76)	75(− 97)	48(−129)
7			0(0)	0(0)	18(− 4)	23(−18)	34(−41)	60(− 63)	88(− 83)	81(−113)
6			0(0)	0(0)	0(0)	17(− 6)	25(−24)	49(− 49)	91(− 70)	108(− 96)
5				0(0)	0(0)	0(0)	17(− 8)	34(− 30)	81(− 56)	126(− 80)
4				0(0)	0(0)	0(0)	0(0)	16(− 12)	55(− 41)	132(− 65)

Table X_a. $100\Delta R$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	13			13	12	9	3	− 4	− 6	− 5
5	18			18	15	7	− 2	− 6	− 7	− 5
6	22		22	20	11	2	− 3	− 6	− 7	− 5
7	25		24	15	5	0	− 4	− 7	− 8	− 6
8	25	25	17	9	4	0	− 4	− 7	− 8	− 6
9	20	15	11	8	4	− 1	− 5	− 8	− 8	− 6
1.0	0	11	11	8	4	− 1	− 5	− 8	− 8	− 6
9		0	12	11	7	2	− 3	− 7	− 9	− 7
8		0	0	12	10	5	− 1	− 6	− 9	− 8
7			0	0	12	9	3	− 4	− 8	− 8
6			0	0	0	12	7	0	− 7	− 9
5				0	0	0	11	4	− 5	−10
4				0	0	0	0	10	0	−10

Table XI$_a$. $100\Delta R_n(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	0(0)			0(0)	8(6)	10(5)	17(4)	15(2)	8(0)	3(0)
5	0(0)			16(7)	18(7)	30(6)	25(3)	14(1)	7(0)	3(0)
6	0(0)		0(0)	31(10)	53(7)	40(4)	23(2)	14(2)	7(0)	3(0)
7	0(0)		57(10)	94(9)	61(6)	34(4)	21(2)	12(1)	6(0)	2(0)
8	0(0)	0(0)	192(9)	87(6)	46(5)	28(4)	18(2)	10(1)	5(0)	2(0)
9	0(0)	535(7)	102(4)	54(5)	35(4)	24(3)	15(1)	9(0)	5(0)	2(0)
1.0	0(0)	−7(4)	39(4)	36(4)	27(4)	20(2)	13(1)	8(0)	4(0)	2(0)
9		0(0)	−42(2)	23(4)	28(4)	23(3)	16(1)	10(1)	5(0)	2(0)
8		0(0)	0(0)	−27(2)	18(3)	24(3)	20(2)	13(1)	7(0)	3(0)
7			0(0)	0(0)	−16(2)	17(3)	21(3)	16(2)	9(0)	4(0)
6			0(0)	0(0)	0(0)	−8(2)	18(2)	19(2)	12(1)	5(0)
5				0(0)	0(0)	0(0)	−3(2)	18(3)	15(2)	7(0)
4				0(0)	0(0)	0(0)	0(0)	+ 5(2)	18(2)	9(0)

Table XII$_a$. $\Delta T'(\eta_u=0.45)$ and its Increment $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	22(0)			22(0)	21(0)	18(0)	12(−1)	7(−1)	3(−2)	0(−1)
5	31(0)			31(0)	28(−1)	20(−1)	12(−2)	7(−2)	2(−2)	0(−1)
6	40(−1)		40(−1)	38(−1)	29(−2)	18(−3)	11(−3)	5(−3)	1(−3)	−1(−2)
7	49(−1)		48(−2)	37(−2)	25(−3)	17(−4)	9(−5)	4(−4)	−1(−4)	−2(−3)
8	52(−2)	52(−2)	42(−3)	30(−4)	22(−5)	15(−6)	7(−6)	1(−5)	−3(−4)	−4(−3)
9	46(−3)	40(−2)	31(−3)	25(−5)	18(−7)	11(−8)	4(−8)	−2(−7)	−6(−6)	−5(−4)
1.0	0(0)	24(−4)	25(−6)	21(−8)	14(−9)	7(−9)	0(−9)	−5(−8)	−8(−6)	−7(−5)
9		0(0)	21(−4)	23(−7)	18(−9)	11(−10)	3(−10)	−4(−9)	−8(−7)	−8(−5)
8		0(0)	0(0)	20(−5)	21(−9)	15(−11)	7(−12)	−1(−10)	−7(−9)	−8(−6)
7			0(0)	0(0)	19(−7)	20(−11)	12(−14)	2(−12)	−6(−10)	−8(−7)
6			0(0)	0(0)	0(0)	19(−9)	18(−14)	7(−16)	−3(−14)	−8(−10)
5				0(0)	0(0)	0(0)	20(−12)	15(−18)	1(−18)	−8(−12)
4				0(0)	0(0)	0(0)	0(0)	21(−18)	9(−22)	−6(−17)

Table $XIII_d$. $100 \Delta R'$

$N_0 - r$ / r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	−7			−7	−7	−6	−4	−4	−3	−2
5	−10			−10	−10	−7	−6	−6	−5	−3
6	−14		−14	−13	−10	−8	−9	−8	−7	−4
7	−17		−17	−13	−11	−12	−12	−11	−8	−5
8	−19	−19	−14	−14	−15	−16	−15	−13	−10	−6
9	−17	−14	−16	−18	−20	−20	−18	−16	−12	−7
1.0	0	−15	−20	−22	−24	−23	−22	−19	−14	−8
9		0	−18	−24	−27	−28	−26	−23	−17	−10
8		0	0	−22	−30	−33	−32	−29	−22	−13
7			0	0	−27	−37	−40	−37	−29	−17
6			0	0	0	−34	−46	−47	−39	−23
5				0	0	0	−44	−58	−53	−34
4				0	0	0	0	−62	−74	−52

Table XIV_d. $100 \Delta R_n'(\eta_u = 0.45)$ and its Increment for $\Delta\eta = 0.1$

$N_0 - r$ / r	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	−2(−1)			−2(−1)	−2(−1)	−2(−1)	−4(0)	0(0)	1(0)	1(0)
5	−3(−1)			−3(−1)	−5(−1)	−9(−1)	−1(−1)	2(−1)	2(0)	1(0)
6	−4(−2)		−4(−2)	−11(−1)	−17(−1)	−2(−1)	4(−1)	4(−1)	3(−1)	1(0)
7	−5(−2)		−5(−2)	−36(−2)	1(−2)	7(−2)	7(−1)	5(−1)	3(−1)	1(0)
8	−6(−3)	−6(−3)	−87(−2)	6(−2)	12(−2)	13(−2)	9(−2)	5(−2)	3(−1)	1(0)
9	−5(−2)	−319(−2)	42(−2)	35(−2)	23(−2)	14(−2)	9(−2)	5(−2)	3(−1)	1(0)
1.0	0(0)	206(−2)	78(−2)	41(−3)	25(−3)	15(−3)	9(−3)	5(−2)	3(−1)	1(−1)
9		0(0)	141(−2)	63(−3)	34(−3)	19(−3)	11(−3)	6(−3)	3(−1)	1(−1)
8		0(0)	0(0)	84(−3)	45(−3)	24(−4)	13(−4)	7(−3)	3(−2)	1(−1)
7			0(0)	0(0)	50(−3)	30(−4)	16(−4)	7(−4)	2(−3)	0(−2)
6			0(0)	0(0)	0(0)	30(−4)	17(−5)	7(−5)	1(−4)	−1(−3)
5				0(0)	0(0)	0(0)	15(−5)	6(−6)	0(−6)	−2(−4)
4				0(0)	0(0)	0(0)	0(−0)	0(−7)	−4(−8)	−6(−6)

Table XV$_a$. $100\Delta\bar{T}(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	−915(12)			−915(12)	−908(18)	−890(31)	−865(48)	−826(70)	−770(99)	−686(138)
5	−874(22)			−873(22)	−862(33)	−844(48)	−814(65)	−777(87)	−719(117)	−631(159)
6	−843(32)		−843(32)	−838(34)	−823(47)	−796(63)	−767(81)	−728(104)	−668(135)	−574(182)
7	−825(36)		−830(36)	−811(44)	−781(58)	−752(77)	−720(96)	−679(120)	−615(153)	−515(203)
8	−823(40)	−823(40)	−810(42)	−778(52)	−746(67)	−711(88)	−675(109)	−629(135)	−561(171)	−455(225)
9	−827(38)	−818(40)	−786(46)	−751(58)	−712(75)	−671(97)	−630(120)	−580(149)	−507(189)	−394(247)
1.0	−791(38)	−797(42)	−769(48)	−729(62)	−681(82)	−633(105)	−587(130)	−532(162)	−452(205)	−331(269)
1.0	0(0)	−60(6)	−121(18)	−188(38)	−253(62)	−315(89)	−369(118)	−403(156)	−391(203)	−317(269)
9		0(0)	−54(6)	−124(20)	−199(42)	−276(70)	−351(100)	−412(133)	−430(176)	−380(238)
8		0(0)	2(0)	−52(8)	−128(24)	−218(50)	−312(81)	−400(112)	−454(151)	−438(207)
7			0(0)	4(0)	−52(8)	−143(30)	−252(60)	−370(93)	−463(128)	−486(178)
6			0(0)	2(0)	7(2)	−56(12)	−168(38)	−309(73)	−449(107)	−522(151)
5				0(0)	4(0)	12(3)	−66(16)	−215(49)	−402(88)	−542(127)
4				0(0)	2(0)	7(2)	17(4)	−84(22)	−303(64)	−535(103)

Table XVI$_a$. $100\Delta\underline{T}(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	−23(−3)			−23(−3)	−21(0)	−23(7)	−28(12)	−33(15)	−34(20)	−21(30)
5	2(4)			1(5)	0(9)	−4(16)	−10(20)	−20(23)	−23(28)	−9(38)
6	16(9)		16(9)	15(11)	14(16)	15(22)	5(27)	−7(31)	−11(36)	−5(46)
7	16(11)		12(8)	20(15)	30(22)	28(28)	19(33)	7(38)	3(42)	20(52)
8	2(9)	2(9)	11(10)	29(17)	38(26)	40(34)	31(39)	20(43)	17(47)	35(55)
9	−27(2)	−11(6)	13(10)	31(17)	44(28)	50(38)	44(43)	33(47)	31(50)	49(54)
1.0	0(0)	−9(2)	8(8)	29(17)	48(29)	58(40)	55(46)	46(49)	45(49)	66(43)
1.0	0(0)	9(−2)	−8(−8)	−29(−17)	−48(−29)	−58(−40)	−55(−46)	−47(−49)	−45(−49)	−65(−43)
9		0(0)	9(−2)	−9(−8)	−31(−17)	−48(−30)	−50(−39)	−41(−44)	−34(−47)	−47(−53)
8		0(0)	2(0)	9(−2)	−11(−8)	−34(−19)	−45(−31)	−37(−39)	−25(−43)	−33(−51)
7			0(0)	4(0)	10(0)	−14(−9)	−35(−21)	−36(−33)	−20(−38)	−19(−47)
6			0(0)	2(0)	7(2)	10(0)	−18(−10)	−33(−24)	−18(−32)	−7(−40)
5				0(0)	4(0)	12(2)	10(0)	−20(−12)	20(−26)	2(−33)
4				0(0)	2(0)	7(2)	17(4)	5(−1)	−21(−18)	4(−27)

Reflexion Effect

Table XVII. $100\Delta\bar{\mathfrak{L}}_r(\eta_u=0.45)$ and its Increment for $\Delta\eta_1=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	9(0)			9(0)	9(0)	8(0)	8(−1)	10(−2)	12(−3)	14(−4)
5	1(0)			1(0)	1(0)	1(−1)	4(−2)	7(−3)	10(−3)	12(−4)
6	− 8(0)		− 8(0)	− 8(0)	− 8(−1)	− 4(−2)	1(−2)	5(−3)	9(−4)	11(−5)
7	−18(0)		−18(0)	−17(−1)	−13(−1)	− 7(−2)	− 2(−3)	3(−4)	7(−4)	9(−5)
8	−28(0)	−28(0)	−27(0)	−22(−1)	−16(−2)	−10(−2)	− 4(−3)	1(−4)	5(−5)	8(−6)
9	−37(0)	−36(0)	−32(−1)	−26(−1)	−19(−2)	−13(−3)	− 7(−3)	− 1(−4)	3(−5)	6(−6)
1.0	−42(0)	−39(0)	−33(−1)	−27(−1)	−20(−2)	−14(−3)	− 9(−4)	− 3(−5)	1(−6)	4(−7)
1.0	0(0)	− 2(0)	− 2(−1)	− 1(−1)	1(−2)	2(−3)	4(−4)	5(−5)	5(−6)	6(−7)
9		0(0)	− 2(0)	− 2(−1)	0(−2)	1(−3)	3(−3)	5(−4)	6(−5)	7(−6)
8		0(0)	0(0)	− 2(0)	− 2(−1)	0(−2)	2(−3)	4(−4)	6(−5)	8(−6)
7			0(0)	0(0)	− 2(−1)	− 2(−1)	0(−2)	3(−4)	6(−5)	9(−5)
6			0(0)	0(0)	0(0)	− 3(−1)	− 2(−2)	2(−3)	6(−4)	10(−5)
5				0(0)	0(0)	0(0)	− 3(−2)	− 1(−2)	4(−4)	10(−4)
4				0(0)	0(0)	0(0)	0(−1)	− 3(−1)	1(−3)	9(−4)

Table XVIII. $100\Delta\underline{\mathfrak{L}}_r(\eta_u=0.45)$ and its Increment for $\Delta\eta=0.1$

r \ N_0-r	$-r$	−0.8	−0.6	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8
0.4	51(0)			51(0)	51(0)	51(− 1)	51(− 2)	52(− 3)	54(− 4)	54(− 5)
5	43(0)			43(0)	43(0)	43(− 1)	46(− 2)	49(− 3)	51(− 5)	51(− 7)
6	34(0)		34(0)	34(0)	35(− 1)	39(− 2)	43(− 3)	46(− 4)	48(− 6)	49(− 8)
7	24(0)		25(0)	26(− 1)	30(− 1)	35(− 2)	40(− 3)	43(− 5)	46(− 7)	46(− 9)
8	14(0)	14(0)	16(− 1)	20(− 1)	26(− 2)	32(− 3)	37(− 4)	41(− 6)	43(− 8)	43(−11)
9	6(0)	6(0)	11(− 1)	17(− 1)	23(− 2)	29(− 3)	34(− 5)	38(− 7)	40(− 9)	40(−14)
1.0	0(0)	4(0)	9(− 1)	15(− 2)	21(− 3)	27(− 4)	32(− 6)	35(− 8)	37(−12)	36(−17)
1.0	0(0)	4(0)	9(− 1)	15(− 2)	21(− 3)	27(− 4)	32(− 6)	35(− 8)	37(−12)	36(−17)
9		0(0)	3(0)	9(− 1)	16(− 2)	23(− 3)	29(− 5)	34(− 7)	38(−10)	39(−14)
8		0(0)	0(0)	3(0)	10(− 1)	17(− 3)	25(− 4)	32(− 6)	38(− 8)	41(−12)
7			0(0)	0(0)	3(− 1)	10(− 2)	19(− 3)	29(− 5)	37(− 7)	43(−10)
6			0(0)	0(0)	0(0)	3(− 1)	13(− 2)	24(− 4)	35(− 6)	44(− 8)
5				0(0)	0(0)	0(0)	4(− 1)	16(− 3)	31(− 5)	44(− 7)
4				0(0)	0(0)	0(0)	0(0)	6(− 2)	23(− 4)	42(− 6)

Loss of Light by the Eclipse for Spherical Stars

Table XIX. $a_u = \frac{1}{\pi} a_0$

r \ $N_0 - r$	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	0.160			0.160	0.160	0.160	0.160	0.124	0.073	0.027
5	250			250	250	250	206	144	81	30
6	360		360	360	360	307	232	157	88	32
7	490		490	490	427	339	251	167	93	33
8	640	640	640	562	462	362	265	175	97	35
9	810	810	716	600	487	378	275	183	101	36
1.0	1.000	872	745	620	504	392	283	188	103	37
9		1.000	883	741	601	467	340	226	125	44
8			1.000	879	722	565	414	274	152	55
7				1.000	871	692	512	341	190	68
6					1.000	851	644	436	244	89
5						1.000	822	573	325	120
4							1.000	772	458	171

Table XX. $a_d = \frac{3}{2} \frac{1}{\pi} a_1$

r \ $N_0 - r$	$-r$	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
0.4	0.230			0.230	0.224	0.208	0.174	0.113	0.056	0.014
5	351			347	330	288	210	129	63	16
6	488		488	473	427	333	230	140	67	17
7	636		627	584	481	361	247	149	71	18
8	784	784	749	643	513	382	261	154	74	19
9	917	902	806	678	536	398	270	160	76	19
1.0	1.000	932	819	683	541	402	276	163	78	19
9		1.000	932	803	650	489	337	202	97	25
8			1.000	927	782	603	422	258	125	33
7				1.000	917	743	537	332	163	45
6					1.000	900	687	442	220	62
5						1.000	870	602	311	90
4							1.000	822	464	140

Phase and Reflexion Effect Outside of the Eclipse

Table XXI.

$-l_2$	$-\sin^{-1} l_2$		ΔH_{1u}	ΔH_{2u}		ΔH_{1d}	ΔH_{2d}		$\Delta\overline{\mathfrak{g}}_r$	$\Delta\underline{\mathfrak{g}}_r$
1.00	90.°0		0	0		− 0	0		0	0
98	78. 5		−0.24	−0.14		−0.38	0.52		0.01	0.13
95	71. 8		− 59	− 33		− 94	73		2	21
90	64. 2		−1.14	− 55		−1.82	84		4	29
85	58. 2		−1.67	− 70		−2.67	80		6	35
80	53. 1		−2.16	− 78		−3.46	68		8	40
75	48. 6		−2.62	− 80		−4.20	51		9	44
70	44. 4		−3.06	− 77		−4.90	+ 30		11	48
60	36. 9		−3.84	− 61		−6.14	− 15		14	53
0	0		−6.00	+ 63		−9.60	−1.88		0.67	0.67

III. Application to β Lyrae

§ 19. β Lyrae is, as is well known, one of the most interesting stars because of its peculiar behaviour with reference to both light-variation and spectral appearance. It is generally accepted that this star is a close binary, although the system will not be composed of such simple models as considered above. The shape and height of successive two maxima are not the same, and during different cycles of variation, the star does not exhibit exactly the same changes. It may be enclosed in a non-static gaseous envelope, and the shape and surface brightness of either component may not be symmetrical about the line joining their centres.

In our model star, asymmetry about this line is not considered. Such peculiarities, however, might be atmospheric, and we may assume that within the photosphere each component more or less resembles our model-star even in this extreme pair whose adjacent vertices are practically in contact. We will accordingly apply the results obtained in the preceding sections to the light curve of β Lyrae, by taking its appropriate mean.

§ 20. The light curve of β Lyrae shows two unequal minima and it is generally accepted that the $B8$-component is smaller and is hidden totally at principal minimum. The spectral class of the other component is usually assigned to $B5$.

Guthnick[1] has calculated the following elements of β Lyrae as derived from the photometric light curve;

Ratio of the axes of an equatorial section, $b:a$		0.8
Inclination of the orbit, i		90°
Semi-axes major of $B5$ and $B8$, respectively		0.5 and 0.3
Maximum brightness	,,	0.55 and 0.45
Densities in unit of the sun	,,	0.0022 and 0.0060

Since the ratio of maximum and minimum brightness predicted for our uniform models which are similarly distorted is approximately

$$\frac{b}{a}=\frac{1-2v_1}{1+v_1}=1-3v_1 \qquad (96),$$

we get at once $v_1=1/15$ by using the above data.

On the other hand relation (3) follows for similar deformation

$$v_1^2=(\eta_1\eta_1')^3 \text{ and } \left(\frac{M_s}{M_l}\right)^2=r^3 \qquad (97),$$

1. Handbuch der Astrophysik, VI 2, p. 431.

r being the ratio of the radii of the two stars, so that by inserting the above values, we find $v_1=0.06$ and $M_s/M_l=0.5$ which are quite in agreement with the observed values.

This result justifies the assumption of similar distortion and at the same time implies that our mode of attack is promising. Moreover we may say that the predicted value of v_1 is a little smaller than that observed, suggesting that the deformations of the two components are not quite similar, and this in turn requires a somewhat different value in mass-ratio.

In fact, the observed value of the mass-ratio is very flexible: for example, it is 0.407 by Curtiss[1] and 0.07 by Miss Maury.[2] This discrepancy comes from the indistinctness of the spectral lines of the larger component. Its spectral class is usually labelled as $B5$ but according to Miss Maury it seems to be $B2c$, and Struve[3] considers it as belonging rather to a later class, $B9$ or $A0$ from the study of the nature of its radiation, reminding us of the spectrum of P Cygni.[4] Curtiss deduced his result from the displacement of the bright lines and Miss Maury's mass ratio is based upon a comparison of her own determination of $K_l+K_s=196$ km/sec with Rossitor's determination of $K_s=0.184$ km/sec. It is very doubtful whether the bright lines of β Lyrae really originate in the atmosphere of $B5$-component, although it can not be believed that the mass ratio is so remarkable (cf. Struve, loc. cit. p. 267).

On the other hand Guthnick elements are the results of a uniform solution. The circumstances will differ when we consider the darkened solution. For example, according to Shapley (loc. cit. p. 86), the eclipse is partial throughout and $B5$-component was found to be less luminous: namely

$b/a=0.76$, $i=62^\circ$

Semi-axes major $=0.68$ and 0.27

and Maximum brightness $=0.4$ and 0.6 respectively.

§ 21. Now we will try to solve the problem quite anew from the present point of view by using Stebbins' adopted light curve[5] determined with his photo-electric photometer. Secondary minimum is not placed at the middle but comes $0^d.17$ later, the period being $12^d.92$. Hence

$$e\cos\omega(1+\operatorname{cosec}^2 i)=\frac{\pi}{P}\left(t_0-t_1-\frac{1}{2}P\right)=0.041,$$

1. Publ. Allegheny Obs., **2** (1911) 113. 2. H. A. **84** (1933) 219.
3. Obs. **57** (1934) 268. 4. Araki and Kurihara, Jap. Journ. Astron. and Geophys., **14** (1937) 305. 5. L. O. B., **16** (1916) 191.

which agrees with Miss Maury's elements of relative orbit, $e=0.02$ and $\omega=0$ provided the inclination is nearly 90°. In Table XXII anomaly, θ is calculated by using these constants and Δm is the difference in magnitude, β Lyrae $-\gamma$ Lyrae.

Table XXII. Light-Curve of β Lyrae (Stebbins)

Phase	θ	Δm	Phase	θ	Δm	Phase	θ	Δm	Phase	θ	Δm
d	°		d	°			°		d	°	
—1.0	—28.1	0.550	3.0	81.7	0.174	6.5	176.5	607	9.97	275.9	140
—0.5	—14.0	1.015	3.45	93.7	160	6.63	180.0	610	10.0	276.8	140
0.0	0.0	1.120	3.5	95.1	161	7.0	190.5	560	10.5	291.2	157
1.5	13.8	1.028	4.0	108.8	178	7.5	204.6	438	11.0	305.6	207
1.0	27.6	675	4.5	121.9	230	8.0	219.0	336	11.5	320.0	312
1.5	41.3	445	5.0	135.4	300	8.5	233.4	255	12.0	334.1	610
2.0	54.8	308	5.5	149.0	386	9.0	247.7	190	12.5	348.2	1.065
2.5	68.3	222	6.0	162.8	506	9.5	262.2	155	12.92	360.0	1.120

There is still a conspicuous dissimilarity about midway from primary minimum to maximum, although there is little divergence among the descending branches of the curve to secondary minimum. But since the asymmetry of the light curve coming from the orbital eccentricity is always very small compared with the displacement of secondary minimum,[1] such an irregularity observed in β Lyrae should be attributed to the real unevenness of the atmospheric structure, and it is outside of the scope of the present paper to trace its origin. As a preliminary attempt we shall take the simple mean of the two halves of the curve before and after minimum and compare it with the theoretical one. The adopted mean curve is given in the following table, the brightness of maximum being taken to be unity.

Table XXIII. Mean Light Curve

θ	0.0°	12.0	15.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
λ_θ	0.410	431	461	532	686	804	881	933	972	994	1.000
$\lambda_{\pi-\theta}$	0.656	686	710	741	801	852	898	939	974	993	1.000

§ 22. First we shall treat the uniform solution, the reflexion effect being put aside. At the beginning it may be noticed that the flatness of principal minimum suggests the deep totality of the eclipse while the shape of secondary minimum seems to deny the uniformity of the surface brightness. Further, the slope near maximum requires that the total distortion be greater than that found by Guthnick.

1. Russell, Ap. J. **36** (1912) 59.

In fact, our preliminary determination referring to $(91)_u$ follows $\bar{v}=0.036$, although we can not hope at this time to estimate $\underline{v}$ with any accuracy. We are accordingly compelled to search for its value so as to give theoretically the observed luminosities at maximum and primary minimum by using relations (81) and (95) and by taking $\eta_{1l}=0.5$, $r=0.6$ and $i=90^\circ$ as a first approximation. Since $\lambda_{max}=1.0$ and $\lambda_0=0.410$, (81) follows

$$\left.\begin{aligned} \mathfrak{L}_l &= 0.410+(2-\eta_{1l}^2)(\mathfrak{L}v_1)_l \\ \mathfrak{L}_s &= 1-\mathfrak{L}_l-2\bar{v}-\frac{3}{8}\eta_{1l}^2\left\{(\mathfrak{L}v_1)_l+r^2(\mathfrak{L}v_1)_s\right\} \end{aligned}\right\} \qquad (98),$$

whence we find two solutions, $\underline{v}=\mp 0.024$. Other constants will be given as follows:

$\underline{v}$	η_1	$\mathfrak{L}v_1$	$\mathfrak{L}$	v_1	v	$\mathfrak{L}v_1'$	$\mathfrak{L}v$	$\mathfrak{L}v'$	ΔM
(I_u) —0.024	0.5 0.3	0.012 60	0.431 494	0.028 121	0.153 148	0.052 14	0.067 73	0.064 76	0.22
(II_u') 0.024	0.5 0.3	60 12	515 407	116 29	241 56	15 48	124 23	29 98	0.92

ΔM in the 10th column means mass-ratio, M_s/M_l.

It may be a regular course to use these constants to rectify the observed curve, by (85) or (87), and to seek a second approximation for η_1, r and i by the usual method. Here, however, we shall compute λ directly by means of these elements and compare it with the observation. It is convenient here to take N_0-r as argument and not θ, leaving free η_1 and i, which can then be redetermined by the relation

$$(\operatorname{cosec} i\,\eta_{1l})^2 N_0^2=\cot^2 i-\sin^2\theta \qquad (99)$$

so as to reconcile the curve to the observation as well as possible. Once they are known, we can find θ for each N_0-r by the same equation and write down $\lambda(O-C)$. The following example is shown for $\underline{v}=-0.024$, where each pair of lines gives corresponding entries for the larger and the smaller primary and $(\lambda)=1-\lambda_0\alpha$ and $1-(1-\lambda_0)\alpha$ respectively.

(I_u) Computation of Theoretical Light Curve

N_0-r	$\bar{v}\Delta\bar{T}$	$\underline{v}\Delta\underline{T}$	$\mathfrak{L}v_1'\Delta T''$	$\mathfrak{L}v\Delta R$	$\mathfrak{L}v'\Delta R'$	(λ)	λ_c	θ	$O-C$
—0.2	—0.171 0.001	—0.019 —0.001	0.017	0.016	—0.007	0.852 410	0.685 410	7.°5	—0.015 0
0.0	— 166 — 13	— 19 4	9 2	4 8	— 3 — 28	874 498	699 471	15.0	11 — 10
0.2	— 158 — 30	— 17 14	5 1	1 4	— 4 — 30	905 620	732 579	21.8	20 — 19

Table Continued

N_0-r	$\bar{v}\Delta\bar{T}$	$\underline{v}\Delta\underline{T}$	$\Omega v_1'\Delta T'$	$\Omega v\Delta R$	$\Omega v'\Delta R'$	(λ)	λ_c	θ	$O-C$
0.4	−148	−11	2	−2	−4	936	773	28.5	19
	−52	18	0	1	−29	743	681		−16
0.6	−135	−5	0	−4	−4	964	816	35.5	12
	−78	15	−1	−3	−26	856	763		−7
0.8	−114	0	−1	−5	−3	987	864	43.2	1
	−95	7	−2	−6	−20	948	832		0
0.9	−102	1	−1	−4	−2	995	887	47.4	−1
	−92	5	−2	−5	−13	982	875		−11
$-l_2=$ 0.8	−88	3				1000	915	52.7	−5
									−17
0.9	−48	2				1000	954	63.8	−1
									−4

θ in the ninth column is computed for $\cot^2 i=0.023$ and $(\eta_u \operatorname{cosec} i)^2 =0.251$, whence $i=81.^\circ 4$ and $\eta_u=0.495$. A little change in r will follow smaller residuals on the whole but it would be superfluous further to revise the elements in this example.

The case is somewhat different for positive $\underline{v}$: it apparently makes secondary minimum too luminous. We shall accordingly take as an alternative approximation $r=0.65$, $\eta_u=0.45$ and $i=90^\circ$ for which we find

	$\underline{v}$	η_1	Ωv_1	Ω	v_1	v	$\Omega v_1'$	Ωv	$\Omega v'$	ΔM
(II_u)	0.028	0.45	0.064	0.525	0.122	0.213	0.011	0.113	0.024	1.3
		$0.45r$	8	398	20	45	49	18	85	

	N_0-r	−0.4	−0.2	0.0	0.2	0.4	0.6	0.8	0.9	$-l_2=0.8$	0.9
(II_u)	θ	6.°5	11.7	17.0	22.5	28.2	34.2	40.7	44.2	53.1	64.2
	$O-C$	−0.029	−7	−4	0	4	5	5	2	0	−4
		2	3	−5	−8	0	0	−3	−11	−12	0

Here θ is given for $i=90^\circ$ and $\eta_u=0.45$.

Thus the observed light curve can be predicted by the two sets of elements but for secondary minimum. It seems difficult, however, to explain at a time the flatness of principal minimum and the constant decrease in luminosity at secondary minimum so long as the stars are taken to be uniformly illuminated. The asymmetry of the light curve around maximum may also be left unexplained. Such asymmetry seems to continue to the eclipsed part of the curve: in fact, except close to secondary minimum, the corresponding residuals are algebraically greater in the upper lines where the primary is larger.

§ 23. Leaving this aside for a while, there are thus two solutions which are consistent with our theory if one of the components can be taken to be much more strongly distorted. Though either will do, it is required that the $B5$-component is about five times more massive if it is the $B8$-component that is more deformed, while they are comparable in the other case.

The observed light curve is equally explainable by these two solutions and it could not be decided from this point of view alone which one should be taken. The spectroscopic mass ratio differs so much among authors that we can not attach much importance to it. Nevertheless, there are, I think, good reasons to believe that solution (II) is more probable.

First, the $B8$-spectrum is that of a normal supergiant and its lines are almost constant in intensity, while the $B5$-spectrum is completely abnormal. This fact leads us to the conclusion that it is the $B5$-component that is violently distorted; otherwise we should expect to see an abnormal $B8$-spectrum.

Secondly, the total emissions of the two stars are comparable so that the mass-luminosity relation would not admit of such a great difference in masses found in solution (I), for it requires that $\mathfrak{L}$ vary nearly proportionally to M^3.

On the other hand it has been found[1] that at primary minimum the color of β Lyrae is redder than at maximum and at secondary minimum it is slightly bluer than at maximum. According to solution (II), at principal minimum we see only the $B5$-component edgewise and at secondary minimum the light comes mainly from the undistorted $B8$-component. This may be taken as a further illustration of the argument, for the surface of the $B5$-component is generally less luminous than that of the $B8$-component and darkest at the end of the tidal tip.

It seems to be some difficulty to the above argument that oscillating lines belonging to $B5$-component have not been observed. This might, however, be attributed to the impurity of the spectrum due to the strong distortion of the surface structure of this component. The same problem has been fully discussed by Struve (loc. cit., p. 271) from a somewhat different point of view.

§ 24. Now we shall turn to the darkened solution. Take as a first approximation

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1. Elvey, Ap. J. **81** (1935) 171.

$\bar{v}=0.027$, $r=0.5$, $\eta_u=0.5$ and $i=90°$.

Then by the relations

$$\left.\begin{aligned}\mathfrak{L}_l&=0.410+\left(3.2-\frac{15}{8}\eta_u\right)(\mathfrak{L}v_1)_l\\ \mathfrak{L}_s&=1-\mathfrak{L}_l-3.2\bar{v}\end{aligned}\right\}\qquad(100)$$

we find in consistence with (95)

	v	η_1	$\mathfrak{L}v_1$	$\mathfrak{L}$	v_1	v	$\mathfrak{L}v_1'$	$\mathfrak{L}v$	$\mathfrak{L}v'$	ΔM
(I_d)	−0.018	0.5 0.5r	0.009 45	0.430 483	0.021 93	0.146 109	0.040 10	0.063 53	0.042 71	0.17
(II_d)	0.018	0.5 0.5r	45 9	512 402	88 22	214 38	11 35	109 15	19 85	0.70

In the case of the darkened star, the contribution from ΔT is not so great during the eclipse that different sets of first approximations are required for two solutions. In fact, after calculations similar to those made for the uniform star, we find as follows:

$\lambda(O-C)$ for the Darkened Star

N_0-r	0.1	0.2	0.4	0.6	0.8	0.9	$-l_2=0.8$	0.9
θ	7.°9	13.5	22.2	30.2	38.5	42.6	51.5	63.1
(I_d)	−0.013 3	− 4 0	8 5	9 2	2 − 2	4 −15	3 −21	6 −13
(II_d)	− 13 4	− 1 3	10 24	15 21	2 8	6 − 4	−15 − 3	−16 9

Here θ is given for revised constants, $(\eta_u \operatorname{cosec} i)^2=0.275$ and $\cot^2 i=0.080$ or $i=74.°3$ and $\eta_u=0.504$. For solution (II_d), a closer agreement will result if a little smaller inclination is taken.

§ 25. Thus two sets of the elements can be determined also in this case and the general discussion made in § 23 remains unchanged, so that solution (II_d), I think, should be adopted.

There are naturally minor differences among the elements for the uniform and the darkened stars. The shape of secondary minimum seems to favour the darkened solution. We are, therefore, likely to admit that the star is darkened in some degree towards the limb, in spite of the indication of Pannekoek[1] who has shown that for *B*-type stars the limb darkening is small. It would be too much to say that in solution (II_d) the less massive *B*8-component is less luminous totally than the other component as the mass-luminosity relation leads one to

1. M. N., **95** (1935) 734.

expect. It seems, however, to be difficult for the present to determine the degree of darkening with any accuracy by an analysis of the observed light curve.

It may be of some interest to note that according to the darkened solution, the stars are just outside of the totality at principal minimum. In this case, the flatness of the light curve at this portion is due partly to the darkening of the limb of the eclipsed star and partly to the rotational term resulting from the eclipsing star, while in the uniform solution it is ascribed to the deep totality.

The asymmetry before and after maximum remains unexplained. In solution (I_d) predicted asymmetry is opposite in sense and in solution (II_d) its maximum effect occurs near $-l_2=0.9$ and results in over-correction.

§ 26. Finally we shall therefore consider the reflexion effect briefly. The coefficient of reflexion defined by (93) will be as follows:

	(I_u)	(I_d)	(II_u)	(II_d)
ϖ	0.081	0.075	0.063	0.066
$\underline{w}$	42	45	18	34

On the other hand, if the whole asymmetry observed is taken to be due to the reflexion effect, it will be found, referring to Table XXIII that the nearer to maximum, the smaller $\underline{w}$ is; for example, it is, approximately, 0.04 at $\theta=50°$ and 0.02 at $\theta=60°$.

Thus the coefficient is of the value expected theoretically, although it fails in details. However, there is so remarkable a difference in luminosity before and after principal minimum that our adopted mean curve depends sensibly on the mode of taking the mean and we can not insist too much on whether it is plausible or not.

Nevertheless it will be worth while to see theoretically what degree the reflexion effect amount to and whether our results obtained in the preceding articles would be affected or not if it should be taken into account.

In the case of solution (II_u) we find:

Reflexion Effect (II_u)

N_0-r	—0.4	—0.2	0.0	0.2	0.4	0.6	0.8	$-l_2=0.8$	0.9	1.0
$\varpi\Delta\overline{\mathfrak{L}}_r$	—0.008 0	—0.006 — 1	—0.003 — 2	0 — 1	0.003 + 2	0.005 4	0.006 6	0.005	0.003	0
$\underline{w}\Delta\underline{\mathfrak{L}}_r$	5 0	6 0	7 — 1	7 — 3	8 — 5	9 — 6	9 — 8	± 7	± 4	0
Sum	— 3 0	0 — 1	4 — 3	7 — 4	11 — 3	14 — 2	15 — 2	12 — 2	7 — 1	0

On the whole the reflexion effect is too small to affect the determined elements. It rather cancels the systematic deviations noted at the end of § 22, except in solution II_a, which it may be necessary to reconsider if the reflexion is taken into account, though the general results would remain unchanged.

Summary

1) Theoretical light curves are computed for the model stars described in § 1, which are equivalent to the first order to the tidally distorted polytrope, $n=3$.

2) The shape of the model star is a distorted ellipsoid which is more elongated towards the other component, its swollen portion being less luminous.

3) In the first section, the boundary of the eclipsed portion is first determined (eq. 24) and then loss of light by the eclipse is formulated.

4) For convenience' sake, it is divided into two parts: "circular integrals" (eqs. 37 and 38), which are independent of the distortion of the secondary—the eclipsing star—and "boundary corrections" (eq. 42).

5) In the second section, the integrals are reduced to an elementary form (eq. 69) convenient for evaluation.

6) Numerical results are given in tables. Their meaning and direction for use are explained in articles 15–18.

7) The predicted light curve is given in the form of equations (81), (85) and (87). Corrections due to the distortion are divided into the tidal and the rotational terms.

8) The reflexion effect is also considered and its result is given in § 17.

9) In the third section the complete results are applied to Stebbins' light curve of β Lyrae.

10) Both for the uniform (§ 22) and the darkened star (§ 24), there are two alternative solutions which are consistent with our theory if one of components can be taken to be much more strongly distorted. If it is the smaller component, *B*8 that is more deformed, then the other component will be five or six times more massive, while they are comparable in the other case.

11) The observed light curve is equally explainable by these two sets of elements but other reasons discussed in § 23 seem to favor the latter solution.

12) The star seems to be darkened towards the limb in some degree (§ 25), though we can not say it quantitatively.

13) The reflexion effect is found to be insignificant (§ 26), so that the determined elements would remain unchanged if it is taken into account.

In conclusion I take this opportunity of expressing my deep indebtedness to Dr. R. Kamiya for his valuable assistance during the preparation of this paper and the patience he has shown in the laborious construction of the tables. It is also my pleasant duty to offer my warmest thanks to Nippon Gakujutu Sinkokai; without its help the present work could not have been undertaken.