Note on the Rate of Evaporation of small Water-Drops, such as Rain-Drops[1]

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Evaporation into still Air

1. Evaporation of water into still air is Stefan's classical problem.[2] Maxwell's treatment[3] of wet-bulb thermometer with a spherical bulb is applicable without modification in our problem. His result is

$$E = 4\pi RD/k.(p_1 - p_0) \quad \cdots\cdots (1)$$

Here, E : rate of evaporation,
R : radius of the water drops,
p_1 : saturation pressure at the temperature of evaporating surface,
p_0 : vapour pressure of the surrounding air,
D : diffusion constant,
$k = \frac{P}{\sigma\rho}$; P: Barometric pressure, $\sigma = 0.623$, ρ; density of air.

In our case, $E = -\frac{d}{dt}\left(\frac{4}{3}\pi R^3\right)$, therefore we have

$$\left.\begin{aligned} \frac{dR}{dt} &= -\frac{D}{Rk}(p_1 - p_0) \\ \text{or} \quad R^2 &= R_0^2 - 2D/k.\ (p_1 - p_0)t \end{aligned}\right\} \quad \cdots\cdots (2)$$

where R_0 is the radius of drops at $t = 0$.

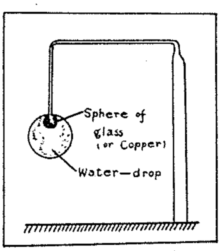
Taking $D = 0.24$ while $\rho = 0.00129$ gr/cm^3 and $P = 760$ mm. Hg.

$$\left.\begin{aligned} \frac{dR}{dt} &= -\frac{2.53 \times 10^{-7}}{R}(p_1 - p_0) \\ \text{or} \quad R^2 &= R_0^2 - 5.06 \times 10^{-7}(p_1 - p_0)t \end{aligned}\right\} \quad \cdots\cdots (2')$$

(R in cm, t in sec., p_1 and p_0 in mm. Hg.)

2. Some simple experimental verification of this result was made by T. Takahashi, one of the authors, in the following way. As shown in Fig. 1, a water drop was attached to the end of a fine stick of glass or copper, and the diminution of its diameter was recorded by a reading micrometer. In order to keep the temperature and humidity of the air uniform, and to prevent its motion, this experiment was conducted in a small box. The humidity was estimated by a hair hygrometer kept

Fig. 1



in comparison with Regnault's dew point apparatus, and also by a dry-and-wet bulb thermometer. For strict accuracy it is of course necessary to know the temperature of the water drop, but as no simple method of estimating it could be devised, the attempt had to be abandoned. Consequently, perfect experimental verification is impossible. Examples of the experimental data obtained are shown in Table 1.

Table 1

No. 1

Date of obs.: 19th Aug. 1931.

Hour	Obs. radius R	Cal. radius R
10 h. 17 m.	0.082 cm.	0.082 cm.
37	0.075	0.075
57	0.068	0.068
11 17	0.060	0.060
37	0.051	0.051

(Glass-suspender)

Dia. of axis=0.013 cm.
Rad. of sphere=0.024 cm.

Vapour tension=19.96 mm. Hg.

Dry-bulb temp.=26.7°C.
Wet-bulb temp.=23.9°C.

$$\frac{dR}{dt}=-\frac{4.21\times10^{-7}}{R}\qquad \begin{cases}R \text{ in cm.}\\ t \text{ in sec.}\end{cases}$$

$$R^2=6.67\times10^{-3}-8.42\times10^{-7}t$$

No. 2

Date of obs.: 19th Aug. 1931.

Hour	Obs. radius R	Cal. radius R
13 h. 39 m.	0.080 cm.	0.080 cm.
14 02	0.073	0.073
25	0.065	0.065
48	0.055	0.056
15 11	0.045	0.045

(Glass-suspender)

Dia. of axis=0.011 cm.
Rad. of sphere=0.024 cm.

Vapour tension=20.36 mm. Hg.

Dry-bulb temp.=26.8°C.
Wet-bulb temp.=24.2°C.

$$\frac{dR}{dt}=-\frac{4.03\times10^{-7}}{R}$$

$$R^2=6.43\times10^{-3}-8.05\times10^{-7}t$$

No. 3

Date of obs.: 1st Sep. 1931.

Hour	Obs. radius R	Cal. radius R
14 h. 23 m.	0.086 cm.	0.086 cm.
45	0.078	0.078
15 07	0.070	0.069
29	0.059	0.059

(Glass-suspender)

Dia of axis=0.013 cm.
Rad. of sphere=0.025 cm.

Vapour tension=18.29 mm. Hg.

Dry-bulb temp.=26.1°C.
Wet-bulb temp.=23.1°C.

$$\frac{dR}{dt}=-\frac{4.89\times10^{-7}}{R}$$

$$R^2=7.40\times10^{-3}-9.78\times10^{-7}t$$

No. 4

Date of obs.: 20th Aug. 1931.

Hour	Obs. radius R	Cal. radius R
13 h. 34 m.	0.066 cm.	0.066 cm.
44	0.061	0.061
54	0.054	0.054
14 04	0.047	0.047
14	0.039	0.039

(Copper-suspender)

Dia. of axis=0.020 cm.
Rad. of sphere=0.023 cm.

Vapour tension=21.50 mm. Hg.

Dry-bulb temp.=27.0°C.
Wet-bulb temp.=24.9°C.

$$\frac{dR}{dt}=-\frac{5.91\times10^{-7}}{R}$$

$$R^2=4.38\times10^{-3}-1.182\times10^{-6}t$$

In this table, it is clear that the results accord closely with those arrived at theoretically, namely $R\frac{dR}{dt}=\text{const.}$ In order to ascertain whether this constant agrees with $-D/k.(p_1-p_0)$, the value of p_1 must be estimated. As this has not been done otherwise we calculate the value of p_1 reasoning backward from the formula given above. Then we get what is shown in Table 2. The temperature of the water

Table 2

No.	$R\frac{dR}{dt}$	p_0 mm. Hg.	Dry-bulb Temp. °C	Wet-bulb Temp. °C	p_1-p_0* mm.Hg.	p_1 mm.Hg.	θ_1 °C	λ'
1	-4.21×10^{-7}	19.96	26.7	23.9	1.82	21.78	23.7	0.38
2	-4.03×10^{-7}	20.36	26.8	24.2	1.74	22.10	24.0	0.40
3	-4.89×10^{-7}	18.29	26.1	23.1	2.10	20.39	22.6	0.35
4	-5.91×10^{-7}	21.50	27.0	24.9	2.55	24.05	25.4	2.59

$\left(\frac{k}{D}=4.35\times10^{6}\text{ for this case of }P=752\text{ mm. Hg. and }\rho=0.001165\text{ gr/cm}^3\right)^*$

drop is found to be very close to that of the wet bulb. But in the case of a copper suspender, it is situated between the dry-bulb temperature and the wet-bulb temperature. The heat supplied by conduction, as in the case of the wet-bulb thermometer is $4\pi RK(\theta_1-\theta_0)$. Here $K(\fallingdotseq 0.6\times10^{-4})$, which is the thermal conductivity of air. θ_1 is the temperature of the water drops, and θ_0, that of the surrounding air. Assuming the supply of heat other than by conduction to be $\lambda(\theta_0-\theta_1)$, we get the equation of heat balance as follows:

$$p_1-p_0=Kk/LD.\{1+\lambda/4\pi KR\}(\theta_0-\theta_1)\ldots\ldots\ldots\ldots\ldots(3)$$

where L is the latent heat of the water. The value of $\lambda'(=\lambda/4\pi KR)$ is, when calculated by the help of this equation, *0.38* in the case of glass, and *2.6* in the case of copper. In view of this it is to be supposed that, in our case, the heat is supplied by conduction from the suspension rather than by radiation.

Evaporation into Moving Air

3. It is important from a meteorological point of view to ascertain the rate of evaporation into moving air as in the case of falling rain drops. A perfect experiment was impossible owing to lack of equipment, but we resorted to a makeshift by turning the set-up described above on the turn table of a gramophone and stopping it only when necessary to read the micrometer. We were able in this way to carry the experiment up to 2 m/sec. The humidity was measured by means of Assmann's Aspiration Psychrometer and Hair Hygrometer. Two examples of the results of the experiment are shown in Table 3, where the relation $R\frac{dR}{dt}=\text{const.}$ is found to hold. Assuming Assmann's Wet-bulb temperature to be that of the water drop, since this is unknown, we calculated p_1. In order to compare it with the condition in still air, we put

$$R\frac{dR}{dt}=-\beta'\frac{D}{k}(p_1-p_0) \qquad \text{.....................(4)}$$

and we get the value of β' shown in Table 4, where it is evident that it noticeably increases with the wind velocity.

Table 3

No. 3
Date of obs.: 6th Sep. 1931

Hour	Obs. radius R	Cal. radius R
13h. 45.5m.—14h. 05.5m.	0.085 cm.	0.085 cm.
14 07.5 — 27.5	0.075	0.075
29.5 — 49.5	0.063	0.063
	0.049	0.049

(Glass-suspender No. 2 in the still air exp.)
$v=0.58$ mps. Readings of Assmann's psychrometer
dry 26.32°C
wet 24.35°C
Vapour tension=21.64 mm. Hg.

$$\frac{dR}{dt}=-\frac{6.80\times10^{-7}}{R} \qquad \begin{cases}R \text{ in cm.}\\ t \text{ in sec.}\end{cases}$$

$$R^2=R_0^2-2A't=0.728\times10^{-2}-1.359\times10^{-6}t$$

No. 8
Date of obs.: 4th Sep. 1931

Hour	Obs. radius R	Cal. radius R
9 h. 28 m.— 9 h. 40 m.	0.083 cm.	0.083 cm.
42 — 54	0.073	0.073
56 —10 08	0.060	0.060
	0.045	0.045

(Glass-suspender No. 2 in the still air exp.)
$v=1.59$ mps. Readings of Assmann's psychrometer
dry 26.00°C
wet 23.52°C
Vapour tension=20.31 mm. Hg.

$$\frac{dR}{dt}=-\frac{1.139\times10^{-6}}{R}$$

$$R^2=0.690\times10^{-2}-2.278\times10^{-6}t$$

4. It is not easy task to ascertain theoretically the rate of evaporation of a moving water drop. To explain the above experimental fact in the face of various difficulties, we adopted the boundary layer hypothesis, and our discussion is carried on in reference to H. Jeffrey's estimation of the order of the terms of the diffusion equation.(4) He says "If the dimensions of the liquid surface are order of l, and the time needed for any considerable change of condition over it is of order of τ, we see that the terms like $\frac{\partial v}{\partial t}$, $u\frac{\partial v}{\partial x}$, and $D\frac{\partial^2 v}{\partial x^2}$ are relatively orders of $1/\tau$, U/l, and D/l^2." As, inside the boundary layer, U is to be considered extremely small, so

Table 4

No.	v mps.	$R\frac{dR}{dt}$ $\times 10^{-6}$	Assmann Dry °C.	Assmann Wet °C.	p_1-p_0 mm. Hg.	β'
1	0.48	−0.571	26.3	24.4	0.97	2.05
2	0.56	−0.766	26.4	24.3	1.04	2.48
3	0.58	−0.679	26.3	24.4	0.98	2.44
4	0.81	−0.792	26.5	24.4	1.03	2.57
5	0.99	−0.850	26.0	23.9	1.03	2.75
6	1.09	−0.889	26.1	24.0	1.07	2.88
7	1.26	−1.063	25.8	23.4	1.22	3.00
8	1.59	−1.139	26.0	23.5	1.21	3.21
9	1.76	−1.098	26.0	24.0	1.01	3.74
10	1.91	−1.244	26.0	23.7	1.15	3.66

$U/l \ll D/l^2$ seems to hold even when $D=0.24$. And on the outside, D is the 10^2-10^3 order and, while, as in our case, $l=10^{-1}$ cm. or thereabout, the relation $U/l \ll D/l^2$ seems to hold even when $U=2\times 10^2$ cm/sec. When $l=10^{-1}$ cm, $1/\tau \ll D/l^2$ is valid on both sides of the boundary layer. From these considerations, the relation of diffusion in this case as the first approximation, may be treated in the following way :

(a) $\frac{d}{dr}\left\{Dr^2\frac{dp}{dr}\right\}=0$ holds on both sides of the boundary layer.

(b) Inside the boundary layer

$p=p_1$ at $r=R$; R is the radius of water drop.

$p=p_b$ at $r=R(1+\delta)$; $R\delta$ is the effective thickness of the boundary layer.

$D=D_0$; D_0 is the coefficient of molecular diffusion.

(c) Outside the boundary layer

$p=p_b$ at $r=R(1+\delta)$

$p=p_0$ at $r\rightarrow\infty$

$D=D'$; D' is the eddy diffusion constant, considered as a constant. This assumption is not true, but the influence of this zone on the final result is very small, therefore for simplicity we make the assumption.

The distribution of the vapour pressure satisfying these relations can instantly be obtained, and the rate of evaporation is calculated, after elimination p_b and putting $r=R$:—

$$E=4\pi R\beta D_0/k.(p_1-p_0) \qquad (5)$$

Here $$\beta\equiv(1+\delta)/(\delta+D_0/D') \qquad (6)$$

If we assume $D_0/D' \ll \delta$, we have

$$\beta=(1+1/\delta) \qquad (6')$$

This β depends on δ or $R\delta$, which is the effective thickness of the boundary layer, which again depends on the motion of the air. Therefore, according as the velocity increases, δ decreases and the rate of evaporation increases.

From (5) we have

$$R\frac{dR}{dt}=\frac{-D_0\beta}{k}(p_1-p_0) \qquad (7)$$

which agrees with the experimental equation shown in (4).

5. Next we consider the thermal relations of evaporating water drops. Heat quantity supplied by conduction is

$$Q=4\pi RK(\theta_1-\theta_0)\beta$$

And the heat supplied by other means can be expressed $h=\lambda_m(\theta_1-\theta_0)$. Then we have from the equation of heat balance $EL+Q+h=0$,

$$p_1-p_0=\frac{kK}{LD_0}\left(1+\frac{\lambda_m}{4\pi RK\beta}\right)(\theta_0-\theta_1) \qquad (8)$$

But as λ_m is unknown, p_1 can not be obtained from this equation. As λ_m seems to be mainly due to the conduction from the suspension stem, we may assume $\lambda_m \doteqdot \lambda\beta$ and then $\lambda_m/4\pi RK\beta \doteqdot \lambda'$, where we can make use of 0.38, which is the value of λ' in still air in the case of glass suspension. In this way we obtain p_0 and θ_0 from the equation (8) and then we can calculate p_1 and θ_1. Accordingly we get the value of p_1-p_0 and that of β from the equation (7). The values of β thus obtained are shown in Table 5.

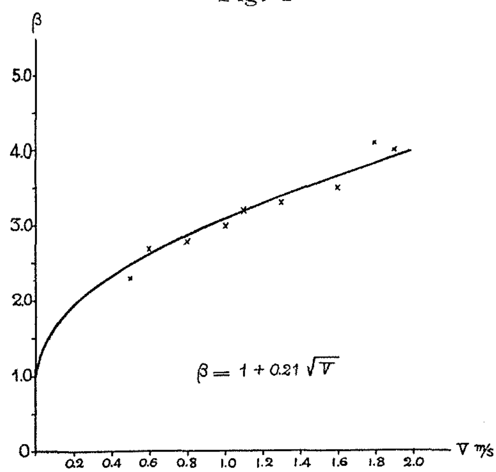
Table 5

No.	v mps.	$R\frac{dR}{dt}$ $\times 10^{-6}$	θ_0 °C.	p_0 mm. Hg.	p_1 mm. Hg.	θ_1 °C.	p_1-p_0 mm. Hg.	β
1	0.48	−0.571	26.3	21.65	22.75	24.5	1.10	2.25
2	0.56	−0.766	26.4	21.51	22.73	24.4	1.22	2.71
3	0.58	−0.679	26.3	21.64	22.74	24.5	1.10	2.66
4	0.81	−0.792	26.5	21.66	22.88	24.5	1.22	2.80
5	0.99	−0.850	26.0	20.99	22.21	24.0	1.22	3.01
6	1.09	−0.889	26.1	21.07	22.29	24.1	1.22	3.15
7	1.26	−1.063	25.8	20.15	21.55	23.5	1.40	3.28
8	1.59	−1.139	26.0	20.31	21.71	23.7	1.40	3.51
9	1.76	−1.098	26.0	21.11	22.27	24.1	1.16	4.09
10	1.91	−1.244	26.0	20.61	21.95	23.8	1.34	4.01

They show no great difference from those reached by way of

Assmann's wet-bulb thermometer. But assumptions remain in both of them. The temperature of the water drop must be measured before the exact value of β is found. The latter depends on the effective thickness of the boundary layer and it would be interesting to investigate their relations. However, it is not allowed to go so deep in discussions from experiments of this kind. More precise experiments are required for such work. The values of β so far ascertained are shown in Fig. 2.

Fig. 2



This can be expressed in the following experimental equation.

$$\beta = 1 + 0.21\sqrt{v} \quad \ldots\ldots\ldots (9)$$

(v in m/s, $v < 2.0$)

Summary

We have made a short theoretical and experimental study concerning the evaporation of a small water drop with a radius under 1 mm.. The decrease of the radius of such a drop due to evaporation can be put in the following equation.

$$\frac{dR}{dt} = \frac{-D\beta}{Rk} \cdot (p_0 - p_0)$$

where

R : radius of water drops,

D : molecular diffusion constant of water vapour into air,

$k = \frac{P}{\sigma\rho}$; P barometric pressure, $\sigma = 0.623$, ρ : density of air,

p_1 : saturation pressure for temperature of water drop,

p_0 : vapour pressure of surrounding air,

β : a constant depending on v, relative velocity of water drop to the air.

Within the limits of experiments, this can be expressed

$$\frac{dR}{dt} = -\frac{2.53 \times 10^{-7}}{R}(1 + 2.1\sqrt{v})(p_1 - p_0).$$

(R in cm., t in sec., v in m/s, $R < 0.1$ cm, $v < 2$ m/s, under ordinary pressure and temperature).

Additional Note

(i) The data of Prof. T. Okada's experiments for the rate of evaporation in still air are found in the "Glazed Frost," a paper published in 1914.(5) His result may be expressed as $R\frac{dR}{dt} = -0.351 \times 10^{-6}$ (C. G. S.) without any deviation.

(ii) The experimental fact that in the evaporation of small liquid drops in moving air the relation $R\frac{dR}{dt}$=const. holds, was noticed in 1931 by Prof. S. Majima(6) and others.

(iii) Mr. Y. Takahasi at the Central Meteorological Observatory of Japan published, in 1935, results of experiments similar to those of the present authors.(7) His discussion showed that $R\frac{dR}{dt}$=const. holds concerning moving air, that the change of shape of water drops in moving air has very little effect on the rate of evaporation, and that the effect of saturation pressure due to surface tension does not appear unless $R=10^{-6}$ cm or less than this. These results are the same as those reached by the present authors, but there are other points on which they have opinions very different from his. It must be added here that, at the first publication of his results, the present authors called his attention to the existence of their unpublished paper and its already published abstract.(8)

(iv) When the authors published the abstract of this paper they adopted $D=0.198$ (from Landolt and Bornsteen's table), but as $D=0.24$ has since been found better, they adopted the latter in the present paper. Hence the difference in coefficient used in the abstract and in the present paper.

(v) Although one of the authors noticed, in a recent number of the Meteorological Magazine, that H. G. Houghten has lately published a paper entitled "A Study of the Evaporation of Small Water Drops," they are not sure what kind of study it is, as they cannot as yet get a copy of the paper.

Literature Cited

(1) The outline of this note was read at the Annual Meeting of the Physico-Mathematical Society of Japan in 1932. Readers may find the abstract of this note in the Appendix of the Proceedings of the Society of that year.

(2) Sitzungsberichte der K. Akad. der Wis. Wien **68** (1873) **73** (1881).

(3) "Diffusion" Encyclopaedia Britannica 9th. ed.

(4) Phil. Mag. **35** (1918) 270.

(5) Jour. of the Met. Soc. of Japan, May 1914 or Monthly Weather Review. (1914) 284.

(6) Riken-iho, **9** 339.

(7) Jour. of the Met. Soc. of Japan July (1935).

(8) loc. cit. (1).