On the Pseudo-set

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Let X be a certain set and we shall call its element x, a *point* of set X. Taking X and an object M into consideration, we shall call M a *pseudo-set* with respect to X. F r all points x in X and M one of the following three fundamental relations holds:

$$(1)\quad \begin{cases} x \mathrel{\bar{\varepsilon}} M, \\ x \mathrel{\underline{\varepsilon}} M, \\ x \parallel M. \end{cases}$$

In the following pages, let us consider that these three fundamental relations and set X are fixed, and we shall call M, shortly, a pseudo-set. When we talk about a point, we mean a point of the set X.

Definition of equality and inequality of pseudo-sets: Two pseudo-sets L and M are called equal, when they satisfy the following conditions. We indicate them by $L=M$:

$$(2)\quad \begin{cases} \text{if } x \mathrel{\bar{\varepsilon}} L, \text{ then } x \mathrel{\bar{\varepsilon}} M, \\ \text{if } x \mathrel{\underline{\varepsilon}} L, \text{ then } x \mathrel{\underline{\varepsilon}} M, \\ \text{if } x \parallel L, \text{ then } x \parallel M, \text{ and vice versa.} \end{cases}$$

By this definition, it is clear that the equality is reflexive, symmetric and transitive. Let for example $L=M$, $M=P$ and $x \mathrel{\bar{\varepsilon}} L$, then $x \mathrel{\bar{\varepsilon}} M$ for $L=M$, and hence $x \mathrel{\bar{\varepsilon}} P$ for $M=P$.

Definition of the sum of two pseudo-sets: If there be an object S satisfying the following properties (3), then it is clear that S is a pseudo-set, and we shall calls S a sum of two pseudo-sets L and M, and denote it by $S=L+M$;

$$(3)\quad \begin{cases} x \mathrel{\underline{\varepsilon}} S \text{ when } x \mathrel{\bar{\varepsilon}} L,\ x \mathrel{\bar{\varepsilon}} M, \\ x \parallel S \text{ when } x \mathrel{\bar{\varepsilon}} L,\ x \mathrel{\underline{\varepsilon}} M, \\ x \mathrel{\bar{\varepsilon}} S \text{ when } x \mathrel{\bar{\varepsilon}} L,\ x \parallel M, \\ x \parallel S \text{ when } x \mathrel{\underline{\varepsilon}} L,\ x \mathrel{\bar{\varepsilon}} M, \\ x \mathrel{\bar{\varepsilon}} S \text{ when } x \mathrel{\underline{\varepsilon}} L,\ x \mathrel{\underline{\varepsilon}} M, \\ x \mathrel{\underline{\varepsilon}} S \text{ when } x \mathrel{\underline{\varepsilon}} L,\ x \parallel M, \\ x \mathrel{\bar{\varepsilon}} S \text{ when } x \parallel L,\ x \mathrel{\bar{\varepsilon}} M, \\ x \mathrel{\underline{\varepsilon}} S \text{ when } x \parallel L,\ x \mathrel{\underline{\varepsilon}} M, \\ x \parallel S \text{ when } x \parallel L,\ x \parallel M. \end{cases}$$

These relations are shown in the accompanying table.

		M		
		$\bar{\varepsilon}$	$\underline{\varepsilon}$	$\|$
L	$\bar{\varepsilon}$	$\underline{\varepsilon}$	$\|$	$\bar{\varepsilon}$
	$\underline{\varepsilon}$	$\|$	$\bar{\varepsilon}$	$\underline{\varepsilon}$
	$\|$	$\bar{\varepsilon}$	$\underline{\varepsilon}$	$\|$

From this definition we obtain the following theorem :

Among three pseudo-sets M_1, M_2 and M_3, we have

$$M_1+M_2=M_2+M_1,$$
$$(M_1+M_2)+M_3=M_1+(M_2+M_3),$$

i. e. the summation of pseudo-sets is commutative and associative.

For let $x \,\bar{\varepsilon}\, M_1$, $x \,\underline{\varepsilon}\, M_2$, $x \,\underline{\varepsilon}\, M_3$ for example, then $x \,\underline{\varepsilon}\, (M_1+M_2)+M_3$ for $x \,\|\, (M_1+M_2)$, $x \,\underline{\varepsilon}\, M_3$, and on the other hand $x \,\underline{\varepsilon}\, M_1+(M_2+M_3)$ for $x \,\bar{\varepsilon}\, M_1$, $x \,\bar{\varepsilon}\, (M_2+M_3)$.

Now let M be a pseudo-set and $\underline{M}$ be an object such as

$$x \,\underline{\varepsilon}\, \underline{M} \text{ when } x \,\bar{\varepsilon}\, M,$$
$$x \,\bar{\varepsilon}\, \underline{M} \text{ when } x \,\underline{\varepsilon}\, M,$$
$$x \,\|\, \underline{M} \text{ when } x \,\|\, M,$$

then, it is clear that $\underline{M}$ is a pseudo-set. We can call $\underline{M}$ a pseudo-anti-set of M. If all points x have the relation "$x \,\|\, M$" with respect to M, then we can call M a pseudo-zero-set, and we will in general denote it by N. By these definitions, we have

$$M+N=M,$$
$$M+\underline{M}=N.$$

Moreover, it is clear that M is a pseudo-anti-set of $\underline{M}$.

Definition of product of pseudo-sets : if there is an object P which has the following properties (4), then it is a pseudo-set and we shall call P a product of two pseudo-sets L and M, and denote it by $P=LM$:

$$(4)\quad \begin{cases} x \,\bar{\varepsilon}\, P \text{ when } x \,\bar{\varepsilon}\, L,\ x \,\bar{\varepsilon}\, M, \\ x \,\underline{\varepsilon}\, P \text{ when } x \,\bar{\varepsilon}\, L,\ x \,\underline{\varepsilon}\, M, \\ x \,\|\, P \text{ when } x \,\bar{\varepsilon}\, L,\ x \,\|\, M, \\ x \,\underline{\varepsilon}\, P \text{ when } x \,\underline{\varepsilon}\, L,\ x \,\bar{\varepsilon}\, M, \\ x \,\bar{\varepsilon}\, P \text{ when } x \,\underline{\varepsilon}\, L,\ x \,\underline{\varepsilon}\, M, \\ x \,\|\, P \text{ when } x \,\underline{\varepsilon}\, L,\ x \,\|\, M, \\ x \,\|\, P \text{ when } x \,\|\, L,\ x \,\bar{\varepsilon}\, M, \\ x \,\|\, P \text{ when } x \,\|\, L,\ x \,\underline{\varepsilon}\, M, \\ x \,\|\, P \text{ when } x \,\|\, L,\ x \,\|\, M. \end{cases}$$

These relations are shown by the accompanying table.

From this definition, we obtain the following theorem :

Between three pseudo-sets M_1, M_2 and M_3, we have

		M		
		$\bar{\varepsilon}$	$\underline{\varepsilon}$	$\|$
L	$\bar{\varepsilon}$	$\bar{\varepsilon}$	$\underline{\varepsilon}$	$\|$
	$\underline{\varepsilon}$	$\underline{\varepsilon}$	$\bar{\varepsilon}$	$\|$
	$\|$	$\|$	$\|$	$\|$

$$M_1M_2 = M_2M_1,$$
$$(M_1M_2)M_3 = M_1(M_2M_3),$$

i. e. the multiplication of the pseudo-sets is commutative and associative.

For let $x \mathrel{\bar{\varepsilon}} M_1$, $x \mathrel{\underline{\varepsilon}} M_2$, $x \mathrel{\underline{\varepsilon}} M_3$ for example, then $x \mathrel{\bar{\varepsilon}} (M_1M_2)M_3$ for $x \mathrel{\underline{\varepsilon}} (M_1M_2)$, $x \mathrel{\underline{\varepsilon}} M_3$, and on the other hand $x \mathrel{\bar{\varepsilon}} M_1(M_2M_3)$ for $x \mathrel{\bar{\varepsilon}} M_1$, $x \mathrel{\bar{\varepsilon}} (M_2M_3)$.

These two processes, the summation and the multiplication, are also distributive:

$$M_1(M_2+M_3) = M_1M_2 + M_1M_3.$$

For, let $x \mathrel{\bar{\varepsilon}} M_1$, $x \mathrel{\underline{\varepsilon}} M_2$, $x \mathrel{\bar{\varepsilon}} M_3$ for example, then $x \mathrel{\|} M_1(M_2+M_3)$ for $x \mathrel{\bar{\varepsilon}} M_1$, $x \mathrel{\|} (M_2+M_3)$; on the other hand $x \mathrel{\|} M_1M_2 + M_1M_3$ for $x \mathrel{\underline{\varepsilon}} M_1M_2$, $x \mathrel{\bar{\varepsilon}} M_1M_3$.

If all points x have the relation "$x \mathrel{\bar{\varepsilon}} M$" with respect to M, we call M a pseudo-unit-set, and denote it by E. From this definition we have

$$\underline{E}\,\underline{E} = E,$$
$$M\underline{E} = \underline{M}.$$

Because, all x have the relation "$x \mathrel{\underline{\varepsilon}} \underline{E}$" with respect to $\underline{E}$; therefore $x \mathrel{\underline{\varepsilon}} M\underline{E}$, $x \mathrel{\bar{\varepsilon}} M\underline{E}$, $x \mathrel{\|} M\underline{E}$ provided $x \mathrel{\bar{\varepsilon}} M$, $x \mathrel{\underline{\varepsilon}} M$, $x \mathrel{\|} M$ respectively.

From the results which we have obtained, we may conclude the following proposition with respect to a system of pseudo-sets $\mathfrak{M}$:

If there exists a pseudo-zero-set in $\mathfrak{M}$, and if the sum and the product of any two pseudo-sets of $\mathfrak{M}$ and the pseudo-anti-set of any pseudo-sets of $\mathfrak{M}$ exist, then the system $\mathfrak{M}$ forms a ring, and if this $\mathfrak{M}$ contains also pseudo-unit-set, then $\mathfrak{M}$ forms a proper ring.

Example of pseudo-set: Let $\mathfrak{F}$ be a system of all one-valued functions $f(x)$ which are defined in the interval $0 < x < 1$, such as

$$f(x_i) = a_i \text{ (integer), for } i = 1, 2, 3, \dots\dots\dots, n,$$

where $0 < x_1, x_2, \dots\dots\dots, x_n < 1$, and let X be the set of these points $x_1, x_2, \dots\dots, x_n$. When we define $x_i \mathrel{\bar{\varepsilon}} f(x)$, $x_i \mathrel{\underline{\varepsilon}} f(x)$ and $x_i \mathrel{\|} f(x)$, so that $a_i \equiv 1 \pmod{3}$, $a_i \equiv 2 \pmod{3}$ and $a_i \equiv 0 \pmod{3}$ respectively, then it is clear, by the definition of $f(x)$, that these $f(x)$ form pseudo-sets with respect to these three fundamental relations and to the set X. Therefore when we consider $\mathfrak{F}$ as a system of these pseudo-sets, two functions $f_1(x)$ and $f_2(x)$ of $\mathfrak{F}$ such as

$$f_1(x_i) = a_i^{(1)},\ f_2(x_i) = a_i^{(2)} \quad (i = 1, 2, 3, \dots\dots, n),$$
$$a_i^{(1)} \equiv a_i^{(2)} \pmod{3} \text{ for all } i = 1, 2, 3, \dots\dots, n,$$

are considered equal: $f_1(x) = f_2(x)$.

Let f_1 and f_2 be any two functions of $\mathfrak{F}$. Then it is clear that $\mathfrak{F}$ contains functions $f_3(x)$, $f_4(x)$ as follows:

$$f_3(x_i) \equiv a_i^{(1)} + a_i^{(2)} \pmod{3}, \text{ for all } i = 1, 2, 3, \ldots\ldots n,$$

$$f_4(x_i) \equiv a_i^{(1)} \times a_i^{(2)} \pmod{3}, \text{ for all } i = 1, 2, 3, \ldots\ldots n,$$

and these functions $f_3(x)$, $f_4(x)$ are respectively the sum and the product of two pseudo-sets $f_1(x)$ and $f_2(x)$. By this and the definition of $\mathfrak{F}$, it forms a ring and a proper ring, for it contains a pseudo-zero-set, a pseudo-anti-set and the pseudo-unit-set.

At the conclusion we notice that we may give another definition to the sum and the product of pseudo-sets, although they are not fertile.

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