On the Summation of an Expansion of Polynomials of Hermite

By

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(Received June 25, 1937)

Mr. N. Wiener has derived the following formula

$$\sum_{n=0}^{\infty} t^n \psi_n(x)\psi_n(y) = \frac{1}{\sqrt{\pi(1-t^2)}} \exp\left[\frac{4xyt-(x^2+y^2)(1+t^2)}{2(1-t^2)}\right],$$

where $\psi_n(x)$ represents the system of functions of Hermite. In regard to this formula, Mr. G. H. Hardy has made a note in his paper,[1] in which he said that he has never seen such formula. Recently I have read Mr. Wiener's book,[2] and found out that the method of proof of the formula as he gives it is rather complicated.

In this paper I treat this formula, simplifying it to some extent, and suggesting more general formula which seems to me better. But original idea is of course based on the suggestion in Mr. Wiener's method.

The function $K(x_1, x_2, \ldots\ldots, x_\nu; t)$ is defined over any finite complex domain of $x_1, x_2, \ldots\ldots, x_\nu$ and the complex plane of t. Even more it satisfies the four following hypotheses:

(I) each of K, $\frac{\partial K}{\partial x_i}$ and $\frac{\partial^2 K}{\partial x_i \partial x_j}$ $(i, j = 1, 2, 3, \ldots\ldots, \nu)$ has only two points $t = \pm 1$ as isolated singular points in the complex plane of t and is elsewhere regular,

(II) when expanded as a series in powers of t, each of $K(x_1, \ldots\ldots, x_\nu; t)$, $K(x, x, \ldots\ldots, x; t)$, $\frac{\partial K}{\partial t}$, $\frac{\partial^2 K}{\partial x_i^2}$ and $\frac{\partial K}{\partial x_i}$ is respectively dominated by the expansion of some corresponding function over $|x_i| < \mathfrak{A}_i$,[3] which is analytic in t for $|t| < 1$,

(III) K satisfies the partial differential equation

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1. "Summation of a series of polynomials of Laguerre," Jour. of the London Math. Soc., vol. VII. (1932).

2. "The Fourier integral and certain of its applications," Camb. pp. 51–62.

3. It is noticed that $\mathfrak{A}_i$ can be taken for any finite positive number.

(L) $\dfrac{\partial^2 K}{\partial x_i^2} - x_i^2 K = -2t\dfrac{\partial K}{\partial t} - K,\ (i=1, 2, 3, \ldots\ldots, \nu)$

(IV) if we put $\dfrac{1}{n!}\left[\dfrac{\partial^n K}{\partial t^n}\right]_{t=0} = K_n,\ (n=1, 2, 3, \ldots\ldots\ldots)$ then K_n and $\dfrac{\partial K_n}{\partial x_i}$ vanish together as $|x_i| \to \infty$.

To begin with, by Taylor's expansion,

$$(1)\quad K(x_1, \ldots\ldots, x_\nu; t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n K}{\partial t^n}\right]_{t=0} t^n$$

$$= \sum_{n=0}^{\infty} K_n(x_1, \ldots\ldots, x_\nu) t^n, \text{ for } |t| < 1.$$

From the hypothesis (II), if $|t| < 1$, the series of the above second member converges uniformly as to all ν-variables $x_1, x_2, \ldots\ldots, x_\nu$. Since the expression (1) satisfies the given partial differential equation (L), it results at once that

$$\frac{\partial^2 K_n}{\partial x_i^2} - x_i^2 K_n = -(2n+1) K_n.$$

Now if we put

$$(2)\quad K_n(x_1, x_2, \ldots\ldots x_\nu) = A_n(x_1, x_2, \ldots\ldots, x_\nu) e^{-\frac{x_1^2 + x_2^2 + \cdots + x_\nu^2}{2}},$$

we have

$$(3)\quad \frac{\partial^2 A_n}{\partial x_i^2} - 2x_i \frac{\partial A_n}{\partial x_i} + 2n A_n = 0.$$

Here we consider two differential equations as follows

$$(\mathrm{L}_1)\quad \frac{d^2 z}{dx^2} - x^2 z + (2n+1) z = 0,$$

$$(\mathrm{L}_2)\quad \frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + 2nz = 0.$$

The differential equation (L_1) has one and only one solution which satisfies such boundary conditions that z and $\dfrac{dz}{dx}$ vanish together as $|x| \to \infty$; for, let z_1 and z_2 be two such solutions of (L_2), then we have

$$z_1'' - x^2 z_1 = -(2n+1) z_1 \text{ and } z_2'' - x^2 z_2 = -(2n+1) z_2.$$

By multiplying the first equation by z_2, the second by z_1, and subtracting the one from the other, it may be obtained that $(z_1' z_2 - z_2' z_1)' = 0$, accordingly $z_1' z_2 - z_2' z_1 = const.$; here let $|x| \to \infty$, then we have $const. = 0$. This contradicts the independency of z_1, z_2.

Now as it is well known that the differential equation (L_1) has the solution $H_n(x) e^{-\frac{x^2}{2}}$ as the unique solution which satisfies the above stated boundary conditions, where $H_n(x)$ is a solution of the differential equation (L_2), we can conclude by the hypothesis (IV) that $K_n(x_1, x_2, \ldots\ldots, x_\nu) = H_n(x_i)e^{-\frac{x^2_i}{2}}\varphi(x_1, \ldots\ldots, x_{i-1}, x_{i+1}, \ldots\ldots x_\nu)$ necessarily, where the factor φ is independent of the variable x_i.

Thus we have

(4) $A_n(x_1, x_2, \ldots\ldots x_\nu) = d_n H_n(x_1)H_n(x_2)\ldots\ldots H_n(x_\nu)$,

where $H_n(x)$ means Hermite polynomial and d_n a pure constant to be determined in the sequel.

As previously shown, it is necessary that every function $A_n(x_1, x_2, \ldots\ldots, x_\nu)$ must be a polynomial of ν variables $x_1, x_2, \ldots\ldots, x_\nu$ and symmetric with respect to these variables.

Hence from (1), (2) and (4) we have

(5) $K(x_1, \ldots\ldots x_\nu;\ t) = \sum_{n=0}^{\infty} d_n\phi_n(x_1)\phi_n(x_2)\ldots\ldots\phi_n(x_\nu)t^n$,

where $\phi_n(x_i) = H_n(x_i)e^{-\frac{x^2_i}{2}}$. Here the system of functions $\phi_n(x)$ forms the orthogonal system of Hermite functions.

Now we wish to determine the values of coefficients of series (5), d_n. For the present purpose, let us consider the following series

(6) $\int_{-\infty}^{\infty} K(x, x, \ldots\ldots, x;\ t)dx = \sum_{n=0}^{\infty} c_n t^n$,

for $|t| < 1$. By means of (5) it results that

$$K(x, x, \ldots\ldots x;\ t) = \sum_{n=0}^{\infty} d_n[\phi_n(x)]^\nu t^n.$$

Since the hypothesis (II) shows the uniform convergence of series in the right hand of the above expression, if we combine this with the expansion of the second member of (6), then we obtain

(7) $c_n = d_n \int_{-\infty}^{\infty} [\phi_n(x)]^\nu dx$.

Now when we take as a value of $\left\{\int_{-\infty}^{\infty} [\phi_n(x)]^\nu dx\right\}^{\frac{1}{\nu}}$ a real one of ν roots of it, we have from (7) and (5)

(8) $K(x_1, \ldots\ldots, x_\nu;\ t) = \sum_{n=0}^{\infty} c_n\psi_n(x_1)\psi_n(x_2)\ldots\ldots\psi_n(x_\nu)t^n$,

where

(8′) $\psi_n(x_i) = \phi_n(x_i) \Big/ \left\{\int_{-\infty}^{\infty} [\phi_n(x_i)]^\nu dx_i\right\}^{\frac{1}{\nu}}$.

We now return to functions by which the previous hypotheses (I), (II), (III) and (IV) are all satisfied. Because of simplification of our treatment, let $\nu = 3$. Let us consider such a particular function as

(9) $K(x, y, z;\ t)$

$$= A.\ \exp\{-B(x^2+y^2+z^2) + 2Cxy + 2Dyz + 2Ezx\}.$$

It is easy to verify that this function may satisfy hypotheses (I), (II) and (IV). Thus we seek conditions that the function (9) may be a solution of the partial differential equation (L): in other words, relations to hold among the coefficients A, B, C, D and E, those which are functions of only t.

In the first place, put (9) into (L), then

$$-2t\{a-B'(x^2+y^2+z^2)+2C'xy+2D'yz+2E'zx\}-1$$
$$=-2B+4(Bx-Cy-Ez)^2-x^2,$$

where $a=A'/A$ and dashes mean the derivatives with respect to t. Since this equation ought to hold identically with respect to x, y, z, it follows that

(i) $2tB'-4B^2+1=0$,
(ii) $tC'-2BC=0$,
(iii) $tE'-2BE=0$,
(iv) $tB'-2C^2=0$,
(v) $tD'+2CE=0$,
(vi) $tB'-2E^2=0$,
(vii) $2ta+1-2B^2=0$.

Now if we replace (vii) by (i), we obtain

$$ta'-a-2ta^2=0,$$

hence $(t/a)'=-2t$. By integrating $a=t/(\rho-t^2)$. While $a=A'/A$. Again by integrating we deduce

$$A=\frac{a_0}{\sqrt{\rho-t^2}},$$

where ρ and a_0 are integral constants, but they are determined so as always $A(t)>0$ for all real t.

Thus from the remaining relations, it follows that

$$B=\frac{1}{2}\,\frac{\rho+t^2}{\rho-t^2},\quad C^2=E^2=\frac{t^2\rho}{(\rho-t^2)^2},\quad D^2=\frac{\rho^2}{(\rho-t^2)^2}.$$

However by means of the hypothesis (I), it needs that $\rho=1$. Therefore we have the following relations:

$$A=\frac{a_0}{\sqrt{1-t^2}},\quad B=\frac{1}{2}\,\frac{1+t^2}{1-t^2},\quad C^2=E^2=\frac{t^2}{(1-t^2)^2},$$
$$D^2=\frac{1}{(1-t^2)^2}.$$

Here we can give four pairs of values for C and E. Hence if we replace (9) by the above obtained results, we have in general four different functions as for K to be required. But to avoid repeating the same arguments we shall discuss only one of these functions; namely

$$(10)\quad K(x, y, z; t)=\frac{a_0}{\sqrt{1-t^2}}\exp\left\{\frac{-(1+t^2)(x^2+y^2+z^2)+4t(xy+xz)-4(1-t^2)yz}{2(1-t^2)}\right\}.$$

By evaluating the integral (6),

$$\int_{-\infty}^{\infty}K(x, x, x; t)dx=\frac{a_0}{\sqrt{1-t^2}}\int_{-\infty}^{\infty}e^{-\frac{7-4t-t^2}{2(1-t^2)}x^2}dx;$$

on the other hand, we see that the parabola, whose equation is $y=x^2+4x-7$, lies evidently under the X-axis in the interval $-1\leqq x\leqq 1$, hence for any real value of t such as $|t|<1$,

$$(7-4t-t^2)/2(1-t^2)>0.$$

Thus by the familiar integral formula,

$$\int_{-\infty}^{\infty}e^{-\frac{7-4t-t^2}{2(1-t^2)}x^2}dx=\sqrt{2\pi}\sqrt{\frac{1-t^2}{7-4t-t^2}},$$

therefore

$$\int_{-\infty}^{\infty}K(x, x, x; t)dx=a_0\sqrt{2\pi}(7-4t-t^2)^{-\frac{1}{2}}$$
$$=a_0\sqrt{2\pi}(e_0+e_1t+e_2t^2+\dots\dots\dots)$$

$e_0=7^{-\frac{1}{2}},\quad e_1=2.7^{-\frac{3}{2}},\quad e_2=5.7^{-\frac{5}{2}},\dots\dots\dots\dots\dots\dots\,.$

Consequently from (8) we obtain the following formula

$$(11)\quad \frac{a_0}{\sqrt{1-t^2}}\exp\left\{\frac{-(1+t^2)(x^2+y^2+z^2)+4t(xy+xz)-4(1-t^2)yz}{2(1-t^2)}\right\}$$
$$=\sum_{n=0}^{\infty}c_n\psi_n(x)\psi_n(y)\psi_n(z)t^n,\ \text{for}\ |t|<1,$$

where $\quad c_0=7^{-\frac{1}{2}}a_0\sqrt{2\pi},\quad c_1=a_0\sqrt{2\pi}.2.7^{-\frac{3}{2}},\quad c_2=a_0\sqrt{2\pi}.5.7^{-\frac{5}{2}},\dots\dots\dots$

and $\quad \psi_n(x)=\phi_n(x)/\left\{\int_{-\infty}^{\infty}[\phi_n(x)]^3dx\right\}^{\frac{1}{3}},\quad \phi_n(x)=H_n(x)e^{-\frac{x^2}{2}}.$

In conclusion the author wishes to express his hearty thanks to Professor Toshizô Matsumoto for his kind advice during the study.