A Theoretical Note on the Diurnal Variation of Wind-Vector due to the Variation of Eddy Viscosity[1]

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I. Introduction

The diurnal variation of wind-vector in the lower atmosphere is a familiar phenomenon in meteorology and it is generally accepted that this phenomenon is a consequence of the daily convection. As for the variation in the atmospheric layer above the flat open land-area where no local orographic influence appears, the classical Espy-Köppen's theory is well known. Prof. G. I. Taylor[2] modified this theory and concluded that this phenomenon is to be explained as a result of the diurnal variation of the coefficient of eddy viscosity. Assuming a quasi-steady state, he got an excellent simple theory which gives a clear explanation of the chief characteristic nature of the diurnal variation of wind-speed; but which fails to give a satisfactory explanation of the diurnal characteristics of wind-direction. The diurnal variation of wind-direction is also a remarkable phenomenon, but there is no comprehensive dynamical theory which explains the diurnal characteristics of wind-direction as well as of wind-speed. Iswekov's report of his study,[3] dealing with cases where the eddy viscosity changes periodically, is full of instructive hints for the author's present study. It must be borne in mind, however, that he does not attempt to inquire into the question of why the diurnal variation of wind occurs as a result of the variation of the eddy viscosity; i. e., as he gives a periodic function for the wind at the earth's surface, his study must be considered to have been directed only to the propagative nature of diurnal wind-vector in an atmospheric layer which is subject to a periodic change of eddy viscosity. H. Futi[4] has also treated the propagating nature of the diurnal wind-vector in a case where the eddy viscosity varies with altitude. What the writers intend now is an attempt to construct a

1. The outline of this paper was read at the Annual Meeting of Physico-Mathematical Society of Japan in Apr. 1935. 2. Proc. Roy. Soc. London. Ser. A. XCIV, 137.
3. Met. Zs. **46**, 1 (1929). 4. Jap. J. Met. **10**, 6, 345.

dynamical theory concerning the origin of the diurnal wind-vector, under Prof. G. I. Taylor's ingenious suggestion, and not a general discussion of the diurnal variation of the wind.

II. The Problem and its Approximate Solution

Let us suppose a simplified case containing all the essential points of our problem, and by solving this idealized case, come to our final conclusion.

(i) *Dynamical equation*

With the usual notations, we have

$$\frac{\partial \bar{u}}{\partial t} = 2\omega\bar{v}\sin\varphi + \frac{1}{\rho}\frac{\partial}{\partial z}\left(K\frac{\bar{u}}{\partial z}\right)$$

$$\frac{\partial \bar{v}}{\partial t} = -2\omega\bar{u}\sin\varphi + \frac{1}{\rho}\frac{\partial}{\partial z}\left(K\frac{\partial \bar{u}}{\partial z}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y}.$$

Putting $\lambda = 2\omega\sin\varphi,\quad k = \frac{K}{\rho},\quad G = -\frac{1}{\rho\lambda}\frac{\partial p}{\partial y},\quad u = \bar{u} + G,$

$v = \bar{v},\quad V = u + iv,\quad \text{and}\quad i = \sqrt{-1},$

we get
$$\frac{\partial V}{\partial t} + i\lambda V = k\frac{\partial^2 V}{\partial z^2} \qquad \text{.................................(1)}$$

for the working dynamical equation.

(ii) *Coefficient of eddy viscosity* (k).

In order to simplify the mathematical treatment, we consider a most simple case which barely serves our purpose, i. e.,

$$k = k_0 - k_1\cos\sigma t \qquad \text{.......................................(2)}$$

($t = 0$ at midnight; $\sigma = {}^{2\pi}/_T$, $T = 24$ hours) where k_0 and k_1 are given constants independent of height (z) and time (t).

(iii) *Upper boundary condition.*

We assume that a given constant gradient wind velocity (G) prevails at a certain height (H), i. e., $\bar{u} = G$ or $u = 0$ at $z = H$, ...(3) where H is assumed independent of time.

(iv) *Bottom-surface condition.*

At first, we consider a reasonable condition, viz.,

$$k\frac{\partial V}{\partial z}\bigg|_{z=0} = 0.002(V_0 + G)\,\big|\,V_0 + G\,\big|,$$

where suffix o indicates the surface-value of the wind-vector. But this condition is non-linear and causes some serious mathematical difficulties, therefore it is advantageous to make some simplification. Thus, in spite of unavoidable depreciation in the validity of the solution, we must perform the linearization of the above condition. In this way, we suppose a fundamental state and introduce a wind-vector, $\bar{V}$, which satisfies the equation of $i\lambda\bar{V}=k_0\frac{\partial^2\bar{V}}{\partial z^2}$; then we get $\bar{V}=C\sinh\beta_0(H-z)$ as a solution of the above equation which satisfies the boundary condition of $\bar{V}=0$ at $z=H$, where $\beta_0=\sqrt{\lambda/k_0}\,e^{i\pi/4}$ and the arbitrary constant C can be found by the aid of the method of numerical trials from the condition,

$$k_0\frac{\partial\bar{V}}{\partial z}\Big|_{z=0}=0.002(V_0+G)\,|\,V_0+G\,|$$

which is introduced also for the fundamental state.

Having simplified the non-linear surface condition by the elimination of the fundamental part above mentioned, we have for a deviating part $V'(=V-\bar{V})$,

$$\frac{\partial V'}{\partial z}\Big|_{z=0}-c\cos\sigma t\frac{\partial\bar{V}}{\partial z}\Big|_{z=0}=qV_0', \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

where $\quad q=\frac{0.002}{k_0}\,|\bar{V}_0+G|\quad$ and $\quad c=k_1/k_0$.

This adapted method of approximation which involves an assumption of $|\bar{V}_0+G+V_0'|\fallingdotseq|\bar{V}_0+G|$, may invite serious objection. Indeed, this is the most difficult point in the authors' present study; yet they believe that the essential character of the problem remains unaltered. It must be admitted, however, that the solution thus obtained is not exact and in fact this study degenerates into the framing of a semi-quantitative theory.

(v) *Stationary condition.*

Instead of an initial condition, we take $V(t)=V(t+T)$ ...(5) from the nature of the problem.

(vi) *The required solution.*

Now we consider a particular solution of eq. (1), which satisfies the condition (3),

$$V=A_n e^{(k_0\beta_n^2-i\lambda)t-\frac{k_1\beta_n^2}{\sigma}\sin\sigma t}\sinh\beta_n(H-z)$$

and β_n is determined by the condition (5), i. e.,

$$\beta_n^2=i(n\sigma+\lambda)/k_0 \qquad (n=0\ \pm1\ \pm2\ \ldots\ldots\ldots)$$

or

$$\beta_n=B_n e^{i\varepsilon\pi/4}\quad B_n=\sqrt{\frac{\varepsilon(n\sigma+\lambda)}{k_0}}\quad \begin{matrix}\varepsilon=1 & \text{for } n\sigma+\lambda\geqq0\\ \varepsilon=-1 & \text{for } n\sigma+\lambda<0\end{matrix}\Big\}\ \ldots(6)$$

Thus we have an appropriate solution,

$$V=\sum_{-\infty}^{+\infty}A_n e^{inx-i\gamma\sin\sigma t}\sinh\beta_n(H-z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$(x=\sigma t-c\sin\sigma t,\quad \gamma=\lambda k_1/\sigma k_0)$$

where constant-vectors A_n are determined by the remaining condition (4). Using the following two auxiliary expressions of Fourier's expansion,

$$e^{i\gamma \sin \sigma t} = \sum_{n=-\infty}^{n=+\infty} b_n e^{inx}, \text{ and } \cos \sigma t e^{i\gamma \sin \sigma t} = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

where $$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{i\gamma \sin \sigma t} e^{-inx} dx = \frac{\gamma}{nc+\gamma} J_n(nc+\gamma)$$

and

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \cos \sigma t e^{i\gamma \sin \sigma t} e^{-inx} dx = \frac{nJ_n(nc+\gamma)}{nc+\gamma}$$
$$-\frac{c}{2}\left\{\frac{(n-1)}{nc+\gamma} J_{n-1}(nc+\gamma) + \frac{n+1}{nc+\gamma} J_{n+1}(nc+\gamma)\right\} \quad \text{......(8)}$$

and noticing

$$V' = V - \bar{V} = \sum_{n=-\infty}^{+\infty} \left\{A_n \sinh \beta_n(H-z) - b_n C \sinh \beta_0(H-z)\right\} e^{-i\gamma \sin \sigma t + inx}$$

we get from the condition (4)

$$A_n = \frac{\beta_0 \cosh \beta_0 H(b_n + cC_n) + q b_n \sinh \beta_0 H}{\beta_n \cosh \beta_n H + q \sinh \beta_n H} C \quad \text{......(9)}$$

The required solution devised for the numerical computations is arranged as follows :—

$$V = \bar{V} + V' \qquad \bar{V} = C \sinh \beta_0(H-z)$$

$$V' = \bar{V} \sum_{n=-\infty}^{n=+\infty} w_n e^{-i\gamma \sin \sigma t + inx}$$

where

$$w_n = \left\{\frac{\beta_0(b_n + cc_n)\coth \beta_0 H + q b_n}{\beta_n \coth \beta_n H + q} \frac{\sinh \beta_n(H-z)}{\sinh \beta_n H} \frac{\sinh \beta_0 H}{\sinh \beta_0(H-z)} - b_n\right\}$$
$$\text{..................................(10)}$$

and the resultant wind vector is $V+G$ where G is the given gradient velocity.

III. Numerical Examples

By giving two specified examples, we now try to make clear some main characters contained in the formal solution of (10).

Ex. 1. Assume $k_0 = k_1 = 2\times 10^4$, $H = 10^5$, $G = 5\times 10^2$ and $\varphi = 45°$, where all the units adopted are C. G. S. system. Then we have $q = 3.3\times 10^{-5}$, $\gamma = 1.4$ and $c = 1$. The values of β_n, b_n and c_n are calculated in Table 1. Table 2 gives the values of $\bar{V}$ and V'(at $t=o$), and the final results which represent the diurnal characters of wind-vector are shown in Figs. 1 and 2.

Table 1.

$\beta_0 = 7.21 \times 10^{-5} e^{i\pi/4}$	$b_0 = +0.567$	$c_0 = -0.388$
$\beta_1 = 9.33$ ″ ″	$b_1 = +0.304$	$c_1 = +0.037$
$\beta_2 = 11.17$ ″ ″	$b_2 = +0.193$	$c_2 = +0.085$
$\beta_3 = 12.73$ ″ ″	$b_3 = +0.137$	$c_3 = +0.084$
$\beta_4 = 14.00$ ″ ″	$b_4 = +0.098$	$c_2 = +0.067$
$\beta_5 = 15.26$ ″ ″	$b_5 = +0.078$	$c_5 = +0.060$
$\beta_6 = 16.40$ ″ ″	$b_6 = +0.064$	$c_6 = +0.052$
$\beta_{-1} = 3.82$ ″ ″	$b_{-1} = -0.686$	$c_{-1} = +0.539$
$\beta_{-2} = 4.67 \times 10^{-5} e^{-i\pi/4}$	$b_{-2} = -0.102$	$c_{-2} = -0.104$
$\beta_{-3} = 7.64$ ″ ″	$b_{-3} = -0.064$	$c_{-3} = -0.043$
$\beta_{-4} = 9.76$ ″ ″	$c_{-4} = -0.045$	$c_{-4} = -0.029$
$\beta_{-5} = 11.45$ ″ ″	$b_{-5} = -0.035$	$c_{-5} = -0.026$
$\beta_{-6} = 12.87$ ″ ″	$b_{-6} = -0.030$	$c_{-6} = -0.014$

Table 2. V' (cm./sec., angle degree).

	$z=0$	$z=50^m$	$z=100^m$	$z=200^m$	$z=300^m$
$\bar{V}$	210 ∠152°00′	164 ∠138°00′	127 ∠123°20′	78 ∠95°20′	48 ∠67°00′
$n=0$	59.6 ∠345°50′	46.6 ∠331°50′	36.1 ∠317°10′	22.2 ∠289°10′	13.6 ∠260°50′
+1	6.6 ∠343°30′	11.0 ∠345°40′	11.7 ∠332°40′	11.2 ∠304°20′	8.6 ∠274°20′
+2	2.3 ∠340°30′	7.2 ∠351°00′	9.4 ∠336°10′	9.4 ∠303°20′	7.5 ∠269°00′
+3	2.1 ∠340°40′	7.2 ∠349°50′	9.0 ∠332°50′	8.3 ∠298°30′	6.2 ∠262°40′
+4	2.3 ∠340°10′	6.1 ∠347°20′	7.4 ∠329°50′	6.7 ∠293°45′	4.8 ∠258°40′
+5	2.5 ∠339°10′	5.6 ∠344°30′	6.7 ∠326°30′	5.7 ∠290°30′	3.9 ∠255°30′
+6	2.3 ∠340°50′	4.9 ∠343°40′	6.0 ∠324°30′	4.9 ∠288°20′	3.2 ∠252°10′
−1	50.0 ∠172°00′	27.9 ∠150°10′	13.6 ∠112°30′	19.3 ∠352°30′	24.5 ∠319°20′
−2	44.5 ∠ 61°40′	43.8 ∠ 71°00′	47.0 ∠107°20′	32.2 ∠ 81°10′	23.3 ∠ 87°50′
−3	18.7 ∠ 81°20′	19.0 ∠ 89°10′	17.9 ∠101°20′	11.3 ∠ 99°20′	5.6 ∠104°00′
−4	10.7 ∠ 81°20′	11.8 ∠ 98°40′	10.5 ∠102°00′	6.1 ∠106°20′	2.5 ∠100°20′
−5	9.0 ∠ 95°40′	9.0 ∠103°30′	7.6 ∠107°50′	4.1 ∠109°50′	1.5 ∠ 91°50′
−6	7.6 ∠108°40′	6.9 ∠110°50′	5.7 ∠113°00′	3.0 ∠109°00′	1.2 ∠ 78°20′

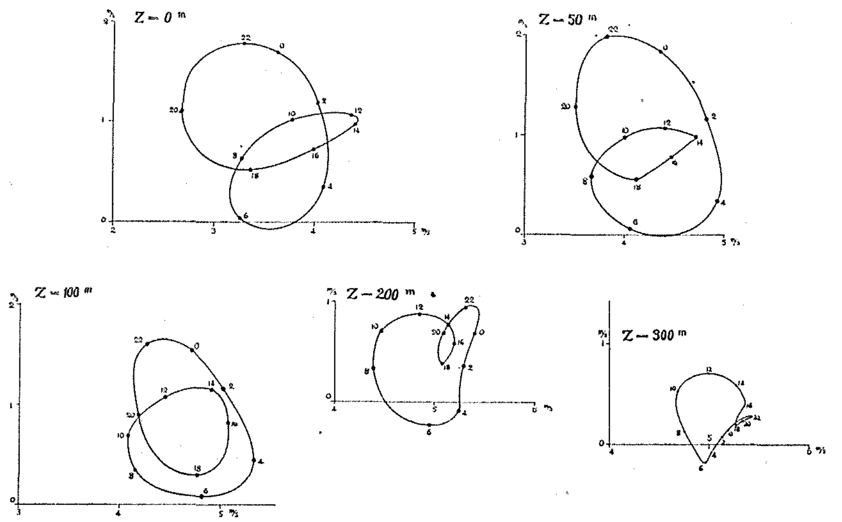
Table 3. $V+G$ (m./sec., angle degree.)

Height / Time	$z=0^m$	$z=50^m$	$z=100^m$	$z=200^m$	$z=300^m$
0^h	4.01 ∠25°10′	4.71 ∠23°00′	4.96 ∠18°10′	5.48 ∠7°00′	5.20 ∠1°20′
2	4.20 ∠16°30′	4.96 ∠13°30′	5.25 ∠12°40′	5.34 ∠3°50′	5.15 ∠0°30′
4	4.11 ∠ 5°00′	4.94 ∠ 3°50′	5.35 ∠ 4°50′	5.28 ∠−1°10′	5.04 ∠−1°10′
6	3.26 ∠ 0°30′	4.05 ∠ 0°40′	4.82 ∠ 0°50′	4.98 ∠−2°40′	4.98 ∠−2°30′
8	3.34 ∠11°00′	3.69 ∠ 9°00′	4.16 ∠ 4°40′	4.43 ∠4°00′	4.77 ∠1°20′

Table 3. Continued.

Time \ Height	$z=0^m$	$z=50^m$	$z=100^m$	$z=200^m$	$z=300^m$
10^h	3.91 ∠15°10′	4.11 ∠13°40′	4.13 ∠ 9°40′	4.55 ∠ 9°20′	4.73 ∠6°20′
12	4.50 ∠13°50′	4.53 ∠13°40′	4.57 ∠13°30′	4.95 ∠10°00′	5.04 ∠7°40′
14	4.52 ∠12°30′	4.82 ∠11°40′	5.04 ∠13°20′	5.23 ∠ 8°30′	5.31 ∠6°00′
16	4.06 ∠10°10′	4.53 ∠ 9°50′	5.13 ∠ 9°20′	5.26 ∠ 6°10′	5.38 ∠3°50′
18	3.36 ∠ 8°50′	4.15 ∠ 7°30′	4.78 ∠ 3°40′	5.11 ∠ 4°00′	5.27 ∠1°40′
20	2.92 ∠22°40′	3.74 ∠20°10′	4.28 ∠12°00′	5.17 ∠ 7°40′	5.35 ∠2°10′
22	3.50 ∠30°40′	4.29 ∠27°30′	4.56 ∠20°40′	5.42 ∠10°00′	5.44 ∠2°30′

Fig 1.



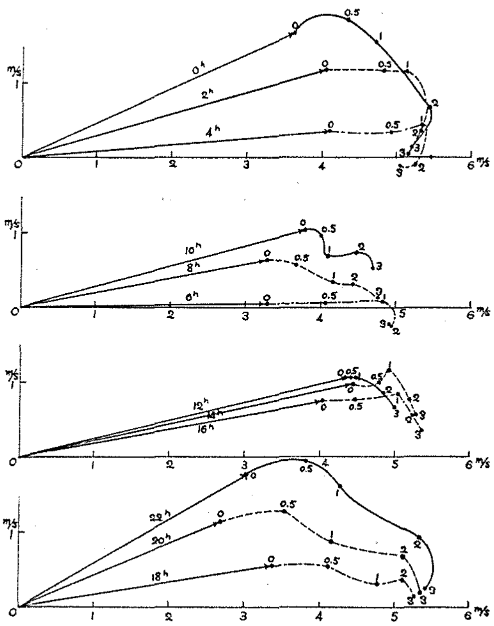
Ex. 2. All the assumed data are the same as in *Ex.* 1, except $k_0 = 2k_1 = 2\times 10^4$. Results in this case are indicated by Tables 4 and 5.

IV. Conclusions and Remarks

(i) This paper is a short study of the diurnal variation of wind-vector caused by the variation of eddy viscosity. If we compare the proposed theory with the existing theories, we have to make the following remarks :

(a) Retaining the terms which contain $\frac{\partial}{\partial t}$, G. I. Taylor's quasi-

Fig. 2.



steady theory is modified by the writers, and a hydrodynamical theory is obtained which may explain some of the chief diurnal characters of wind direction as well as wind speed.

(b) But as the present theory included the linearization of the non-linear surface condition, it is only valid for semiquantitative purposes. Moreover as the solution is given by the infinite series, inconvenience and inaccuracy unavoidably accompany the numerical computation.

(c) Iswekov's study does not touch our theory; indeed, it is an essentially different problem.

(ii) It is clear that the observed diurnal variation at a given locality is not entirely due to the variation of eddy viscosity, i.e., several other causes are associated; therefore a direct identification of the theory with observed facts is very difficult. Moreover the existing statistical records of the diurnal wind-vectors are invalid, because the usual method of the reduction of wind-vector is not suited for testing our theory; although we find that the theoretical results agree to some extent with the well-known facts observed at the

Table 4. (*Ex.* 2) V' (cm./sec., angle degree.)

	$z=0$	$z=50^{m}$	$z=100^{m}$
$\bar{V}$	210 ∠152°00′	164 ∠138°00′	127 ∠123°20′
$n=0$	18.5 ∠345°50′	14.4 ∠331°50′	11.2 ∠317°10′
+1	9.7 ∠165°40′	4.1 ∠ 76°40′	5.5 ∠ 17°00′
+2	4.4 ∠162°50′	2.3 ∠ 56°10′	3.9 ∠ 0°10′
+3	1.5 ∠160°10′	1.6 ∠ 19°00′	2.8 ∠346°30′
+4	0.2 ∠152°00′	1.3 ∠357°50′	1.8 ∠337°00′
+5	0.2 ∠152°00′	0.7 ∠ 3°00′	0.9 ∠337°00′
+6	0.2 ∠152°00′	0.3 ∠ 21°30′	0.6 ∠340°10′
−1	21.4 ∠172°00′	11.0 ∠145°50′	5.2 ∠ 89°20′
−2	11.1 ∠ 60°50′	11.0 ∠ 70°00′	10.4 ∠ 71°50′
−3	2.7 ∠ 80°30′	2.8 ∠ 88°10′	1.3 ∠ 94°00′
−4	1.0 ∠ 98°50′	1.0 ∠ 99°20′	1.0 ∠100°10′
−5	0.4 ∠ 88°30′	0.7 ∠104°20′	0.4 ∠105°00′
−6	0.2 ∠152°00′	0.2 ∠138°00′	0.1 ∠123°20′

Table 5. (*Ex.* 2) $V+G$ (m./sec., angle degree.)

Time \ Height	$z=0^m$	$z=50^m$	$z=100^m$
0^h	3.22 ∠21°00′	4.12 ∠18°30′	4.72 ∠14°10′
2	3.34 ∠19°10′	4.21 ∠17°00′	4.78 ∠13°10′
4	3.31 ∠16°20′	4.04 ∠15°20′	4.68 ∠12°00′
6	3.28 ∠15°10′	3.95 ∠14°10′	4.44 ∠11°50′
8	3.41 ∠15°10′	3.93 ∠14°20′	4.35 ∠12°20′
10	3.62 ∠16°10′	4.04 ∠14°50′	4.40 ∠13°10′
12	3.78 ∠15°00′	4.16 ∠14°10′	4.34 ∠13°30′
14	3.78 ∠14°10′	4.20 ∠13°40′	4.57 ∠12°00′
16	3.60 ∠14°00′	4.14 ∠12°50′	4·53 ∠12°10′
18	3.32 ∠15°00′	3.98 ∠13°50′	4.45 ∠12°00′
20	3.12 ∠17°20′	3.89 ∠15°30′	4.43 ∠12°50′
22	3.12 ∠20°10′	3.96 ∠17°50′	4.56 ∠14°10′

Eiffel Tower and at Lindenberg.

(iii) The resultant diurnal wind-vector is represented in our theory by a composition of the infinite number of the harmonic components. We see in the numerical examples that the components of the clockwise rotating wind-vectors are predominant in general for the localities in the northern hemisphere. The diurnal and semidiurnal clockwise components which are the most prominent among them, have the same order in magnitude; and the total amount of the less-prominent components—higher harmonics of the clockwise rotating vectors, and all the components with counterclockwise rotation, —is not negligible. These circumstances combine to give the resultant vector a complex form. Yet, there remains some tendency toward clockwise rotation. This fact has been already found by Sprung and others and has been explained as the result of the deflecting force of the earth's rotation and the frictional force acting near the ground. Our theory explains this physical mechanism more minutely.

(iv) It is worth while to notice the theoretical results of the vertical distribution of wind vectors for every two hours shown in Fig. 2. We shall find that this differs somewhat from the commonly accepted idea. A few meteorologists, assuming a steady condition, consider that any difference from the ordinary spiral shape of the vertical wind vector is due to the eddy viscosity varying as height varies. While our theoretical results also show that even under the assumptions of non-steady condition and constant eddy viscosity, some remarkable differences from the spiral form of the vertical distribution of wind-vector appear on some occasions. The writers of this paper believe that this information must be of some value to those meteorologists who conduct observations by a pilot-balloon or engage in the reduction of those observations.