On the Proof of Carleman's Extension Theorem of Liouville Theorem

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As an extension theorem of Liouville theorem, Torsten Carleman proved two theorems[1] as follows:

I. *Let* $\omega(\varphi)\geqq 0$, *and*

$$\int_0^{2\pi}\omega(\varphi)d\varphi<K \qquad (\textit{finite}),$$

then an integral function $f(z)$ *which satisfies the inequality*

$$|f(z)|<e^{e^{\omega(\varphi)}} \qquad (z=re^{i\varphi})$$

should be constant;

II. *Let* $\omega(\varphi)\geqq 0$ *and* $\omega(a)$, $\omega(-a)$ *be finite. Let*

$$\int_{-\alpha}^{\alpha}\omega(\varphi)d\varphi<K.$$

Then a function $f(z)$, *which is regular and analytic in the angular domain* $-a<\varphi<a$ *and satisfies the inequality*

$$|f(z)|<e^{e^{\omega(\varphi)}},$$

should be bounded throughout the angular region $-a\leqq\varphi\leqq a$.

In this paper we wish to prove these theorems most briefly by a different method from the one by which Carleman followed.

Let $f(z)$ be regular and analytic for $|z|<R$, then the integral of F and R. Nevanlinna

$$M(r)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|f(re^{i\theta})|\,d\theta \qquad (r<R) \tag{1}$$

is a function which is convex and non-decreasing with respect to $\log r$.[2] And besides if, for any two numbers r_1, r_2 such as $r_1<r_2<R$, $M(r_1)=M(r_2)$, the function $M(r)$ will be constant by virtue of its convexity. Now first let $M(r)=0$. Then for every $r(<R)$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|f(re^{i\theta})|\,d\theta=\frac{1}{2\pi}\int_{|f|\geqq 1}\log|f(re^{i\theta})|\,d\theta=0.$$

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1. Acta Math., **48** (1926), pp. 363-366.
2. P. Montel; Jordan Jour., IX(1928), pp. 29-60.

Since the integrand is not negative, unless for all r the Lebesgue measure of the set $E(|f|\geqq 1)$ is zero,[1] almost all values of θ in $E(|f|\geqq 1)$

$$\log|f(re^{i\theta})|=0 \quad i.\ e. \quad |f(re^{i\theta})|=1.$$

Hence from the regularity of $f(z)$, we have $f(z)=const.$ throughout $|z|<R$. Second, let $M(r)=c$ (>0). For the present case we take c_0 (>1) such as $e^c=c_0$. Then we have

$$\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|f(re^{i\theta})|\,d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log c_0\,d\theta\geqq\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|c_0f(re^{i\theta})|\,d\theta=0\,;$$

because, as $|c_0f|\geqq|f|$, the set $E(|c_0f|\geqq 1)$ may be contained in the set $E(|f|\geqq 1)$. Thus by the same argument as in the first case, it is very easy to show that $f(z)=const.$

Now from the above stated properties of the function $M(r)$ we shall prove Carleman's Theorem I.

By the formula of Schlömilch[2] we have

$$\int_0^{2\pi}e^{\omega(\varphi)}d\varphi\leqq\exp.\left[\int_0^{2\pi}\omega(\varphi)d\varphi\Big/\int_0^{2\pi}e^{\omega(\varphi)}d\varphi\right],$$

$$i.\ e. \qquad \left[\int_0^{2\pi}e^{\omega(\varphi)}d\varphi\right]^{\int_0^{2\pi}e^{\omega(\varphi)}d\varphi}\leqq\exp.\int_0^{2\pi}\omega(\varphi)d\varphi\,;$$

but by the hypothesis, $\int_0^{2\pi}e^{\omega(\varphi)}d\varphi<const.$ While $\overset{+}{\log}|f(z)|<e^{\omega(\varphi)}$; hence, by (1), where $R\to\infty$, the function $M(r)$ is bounded for all values of r. On the other hand from the convexity of $M(r)$, unless $M(r)=const.$, it should be that $M(r)\to\infty$ as $r\to\infty$. Therefore necessarily $M(r)=const.$ for every r. This completes Theorem I.

In the sequel, for the purpose of the proof of Theorem II we define $F(z)$ as follows:

$$F(z)=f(z)\ [-\alpha\leqq\arg z\leqq\alpha]\,;\ =0\quad[\text{elsewhere}].$$

Then the function $\overset{+}{\log}|F(z)|$ is subharmonic. Hence from the theory of subharmonic functions by Montel,[3] an analogous integral with the one of Nevanlinna

$$(2)\qquad M_1(r)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|F(re^{i\theta})|d\theta=\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\overset{+}{\log}|f(re^{i\theta})|\,d\theta$$

must hold the same properties as $M(r)$. Thus by the hypotheses of theorem II, we have $M_1(r)=const.$ for every r.

Now let $M_1(r)=0$. Then if for a certain value of r it occurs

1. For this case our theorem becomes trivial by the classical Liouville Theorem.
2. See, *e. g.*, J. L. W. V. Jensen; Acta Math., t. **30** (1906), p. 178.
3. *Loc. cit.*

that $|f|>1$, we shall have $f(z)=const.$ throughout the angular region $-\alpha\leqq\varphi\leqq\alpha$; but otherwise we must have $|f(z)|\leqq 1$ throughout $-\alpha\leqq\varphi\leqq\alpha$.

Now since we can similarily discuss also the case where $M_1(r)=c(>0)$, it follows that Theorem II is correct.

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