On the Condition of the Set which might be a Derived Set

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"*Let E be a metric space and M an isolated set contained in* E_β[1], *then there exists a set A such that*

$$M_\alpha{}^{2}=A_\beta."$$

For any point $x \varepsilon M$, there are neighbourhoods of x which contain no points of M except x itself. We shall denote the upper bound of the radii of these by ρ_x (>0), then we have

$$U_x\left(\frac{\rho_x}{2}\right)U_{x'}\left(\frac{\rho_{x'}}{2}\right)=0 \qquad \text{for } x \neq x' \ (x \varepsilon M, \ x' \varepsilon M).$$

Because, if $U_x\left(\frac{\rho_x}{2}\right)$ and $U_{x'}\left(\frac{\rho_{x'}}{2}\right)$ have a common point y, it follows that $yx<\frac{\rho_x}{2}$, $yx'<\frac{\rho_{x'}}{2}$ and so

$$xx' \leqq xy+yx'<\frac{\rho_x}{2}+\frac{\rho_{x'}}{2} \leqq \max. \ (\rho_x, \ \rho_{x'}).$$

This inequality contradicts the definition of ρ_x.

Now let A^x ($x \varepsilon M$) be a set as follows:

$$\begin{cases} x=\lim\limits_{n\to\infty} x_n, \ x_n \varepsilon E, \\ A^x=\{x_1, \ x_2, \dots, \ x_n, \dots\}^{3} \ \subseteqq \ U_x\left(\frac{\rho_x}{5}\right), \\ A^x_\beta=\{x\}. \end{cases}$$

(The existence of A^x follows from the fact that $x \varepsilon E_\beta$.)

Then, it is clear that

$$A^x A^{x'}=0 \quad \text{for } x \neq x' \ (x, \ x' \varepsilon M)$$

and we may prove that

$$A_\beta = M_\alpha$$

where

$$A \equiv \sum_{x \varepsilon M} A^x.$$

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1. E_β means the derived set of E.
2. M_α means the closure of the set M.
3. $\{x_1, \ x_2, \dots, \ x_n \dots\}$ means the set which consists of the points $x_1, \ x_2, \dots, \ x_n \dots$.

Since $x \varepsilon A_\beta^x \leqq A_\beta$ for any point of M, we have

(1) $$M_\alpha \leqq A_{\beta\alpha} = A_\beta.$$

On the other hand, for any point y of A_β and for any positive number $\rho > 0$, there is, since $U_y(\rho)A \supset 0$, at least one point $x \varepsilon M$ such that $U_y(\rho)A^x \supset 0$.

If there exist a point $x \varepsilon M$ such as

$$U_y(\rho)A^x \supset 0, \quad \frac{\rho_x}{5} \geqq \rho$$

for a suitable $\rho > 0$, we have

$$xy \leqq xz + zy < \frac{\rho_x}{5} + \rho \leqq \frac{2}{5}\rho_x$$

where $z \varepsilon U_y(\rho)A^x$. While, for a point $\xi \varepsilon (A - A^x)$, there exists a point $x' \varepsilon M$ such that $\xi \varepsilon U_{x'}\left(\frac{\rho_{x'}}{5}\right)$, $x \neq x'$. Hence it follows that

$$y\xi \geqq x\xi - xy \geqq \frac{\rho_x}{2} - \frac{2}{5}\rho_x = \frac{\rho_x}{10} \quad (>0),$$

for $\xi \bar{\varepsilon} U_x\left(\frac{\rho_x}{2}\right)$. In this inequality, ρ_x being independent of ξ, we have

$$y \bar{\varepsilon} (A - A^x)_\beta.$$

And so $y \varepsilon A_\beta^x$ for $y \varepsilon A_\beta$, $A_\beta = A_\beta^x \dot{+} (A - A^x)_\beta$. Therefore, $y \varepsilon M$ for $A_\beta^x = \{x\}$.

If there be no point $x \varepsilon M$ such that

$$A^x U_y(\rho) \supset 0, \quad \frac{\rho_x}{5} \geqq \rho$$

for any positive $\rho > 0$, there exists at least one point $x \varepsilon M$ such that $U_y\left(\frac{1}{n}\right) A^x \supset 0$, $(n = 1, 2, 3, \ldots)$ for $y \varepsilon A_\beta$. And so, for a point $x_{n,y}$ among them and for a point $z_n \varepsilon U_y\left(\frac{1}{n}\right) A^{x_{n,y}}$, we have

$$yx_{n,y} \leqq yz_n + z_n x_{n,y} < \frac{1}{n} + \frac{\rho_{x_{n,y}}}{5} < \frac{2}{n}$$

$$\therefore \ y = \lim_{n \to \infty} x_{n,y}$$

$$\therefore \ y \varepsilon M_\alpha \quad [\because \ x_{n,y} \varepsilon M].$$

Therefore we have $y \varepsilon M_\alpha$ for all points y of A_β and so

(2) $$A_\beta \leqq M_\alpha.$$

The inequalities (1) and (2) prove the theorem.

"*Let E be a metric space. The necessary and sufficient conditions that a set H shall be equal to a derived set of some suitable set, are* *that it is closed in E and does not contain any isolated points of space E.*"

It is clear that these conditions are necessary. We will show that they are also sufficient.

Let F be a set which satisfies the conditions of this theorem, we have $F \equiv E_\beta$ [$\because$ $FE_j = 0$, $E = E_h + E_j$,[1] $E_h = E_\beta E = E_\beta$].

Now

$$F = F_k + F_s,^{2} \quad F_j \equiv F_s \equiv F_{j\alpha},$$

and from the result which we have had, it is proved that there exists a set A such that

$$F_{j\alpha} = A_\beta,$$

for F_j is an isolated set. Hence it follows that

$$F = (A \dot{+} F_k)_\beta.$$

For, since $(A \dot{+} F_k)_\beta = A_\beta \dot{+} F_{k\beta}$, $F_k \equiv F_{k\beta}$, we have

$$(A \dot{+} F_k)_\beta = A_\beta \dot{+} F_{k\beta} \supseteqq F_{j\alpha} \dot{+} F_k \supseteqq F_s + F_k = F.$$

On the other hand,

$$(A \dot{+} F_k)_\beta = A_\beta \dot{+} F_{k\beta} \equiv F,$$

for $A_\beta = F_{j\alpha} \equiv F_\alpha = F$, $F_{k\beta} \equiv F_\beta \equiv F$.

Therefore we have

$$(A \dot{+} F_k)_\beta = F.$$

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1. $E_h = EE_\beta$, and E_j means the set of isolated points of E.
2. F_k means the nucleus of F, and F_s means the complement of F_k with respect to F.