On the Balmer Emission of the Planetary Nebulae

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Introduction

§ 1. Since 1927 when H. Zanstra[1] put forward the hypothesis of recombination mechanism to explain the phenomena of the Balmer Emission in the planetary nebulae, this theory has been discussed from different aspects and applied on various problems of astrophysics. One of the most important investigations is that of G. G. Cillié[2], who has attempted to account for the Balmer decrement, assuming the complete transparency of the nebula with respect to the subordinate lines. His theory seemed at first not to be well supported by the observations, so far as the observed decrement was often steeper than the theoretical prediction ; but this discrepancy has mostly been removed by the introduction of a correction for the space reddening, as recently proposed by L. Berman[3].

So far, the observed hydrogen emissions have been approximately explained in terms of the recombination theory. Notwithstanding there seem to exist in such current treatments of the problem certain very essential defects from the theoretical point of view, which may be enumerated as follows :—

a) The theoretical decrement is rather insensitive to the variation of the electron temperature T_e, and fails to explain the observed dispersion in the decrement ; for example, the intensity ratio of the first two Balmer emissions, H_α/H_β increases only from 2.7 to 2.8 as T_e increases from 5,000° to 10,000°, and even for T_e=50,000° it barely attains to 3.0, while for the nebula N. G. C. 7027, observed decrement amounts to 3.9, and this seems not to correspond to any reasonable electron temperature.

b) As has been pointed out by Cillié[2], the continuous Balmer emission is in some cases much weaker than would be expected from

1. H. Zanstra, ZS. f. Astrophysik. **2** (1931)1, 329.
2. G. G. Cillié, M. N. **92** (1932) 820, **96** (1936) 771.
3. L. Berman, Ibid. **96** (1936) 890.

the theory; i. e. discrete Balmer emission appears too strong compared with it. Such a discrepancy seems to be untenable, as will easily be seen, if we remember the fundamental mechanism of the theory that the continuous part of the Balmer emission is emitted by the direct capture of free electrons to the second level, while its discrete lines are caused by the transitions of electrons from the higher levels, to which they were directly captured, or to which they had cascaded down from still higher ones, having had captured therein.

c) If the free electrons in the nebulae are assumed to have a velocity distribution obeying the Maxwellian law, we are able to predict the intensity distribution of the continuous Balmer emission, provided the probability for capture is given as a function of the velocity of the impinging electron. The theory predicts a monotonous diminution of the emission intensity beyond the Balmer limit λ 3647 Å for any value of T_e. Observation available to verify this fact is that of T. L. Page[1], which is on the whole in line with the theory except that it often shows a hump at about λ 3530 Å. This may mean that the distribution of the electron velocities in the nebula does not exactly obey the Maxwellian law; or that the analysis of the observed data does not result in this law. On the other hand, as will be seen later (§ 7), mutual collision between electrons occurs in the nebula, much more frequently than recombination and ionization; therefore we are forced against Page's observation to conclude that the collision effect must keep the Maxwellian distribution, smoothing out the tendencies of both the increasing high speed electrons and the decreasing slow ones, caused by the photoelectric ionization and by the recombination with protons respectively. In any case, we can, as Page first has done, roughly derive the electron temperature from the slope of the observed continuous emission; but the temperature obtained in such a way seems to be the lower, the steeper the Balmer decrement, in the sense contrary to that resulted from the theory.

d) The method of temperature estimation above mentioned gives[1] 700°–1,600° and 1,200°–4,000° for N. G. C. 7027 and 7662 respectively; these objects belong to a group of nebulae, characterized by intensive nebulium emission, taking the intensity of H_β as standard; 2,700°–4,500° and 2,200°–11,000° for N. G. C. 6572 and I. C. 4593 belonging to another group with less intensive nebulium emission. These estimates, although

1. T. L. Page, M. N. **96** (1936) 604.

they were unexpectedly low, have been accepted generally without any detailed discussion. But the result seems not to be so trivial. Indeed, it is well known that the nebulium emission is only explainable by the excitation due to the electron impact[1]; but even this effect may be inadequate to account for the emission mechanism because in such a low temperature gas, electrons, which have sufficient kinetic energy to excite the nebulium elements are too scanty. For example, a temperature of 4,000° or higher seems to be necessary for electrons to excite the chief nebulium lines $\lambda\lambda$ 4959 and 5007 of O_{III}. Incidentally it may be interesting to remark that the observed temperature seems lower, as the nebulium lines appear brighter. The circumstances above stated are rather to be interpreted as such that the appearance of higher energy radiations such as $\lambda\lambda$ 3726 and 3729 of O_{II}, and $\lambda\lambda$ 3869 and 3968 of Ne_{III} etc., suggests a considerable order of magnitude for T_e.

§ 2. The effect of self-reversal has often been suggested to explain the anomalies enumerated in the preceding paragraph[2]. The reason consists in the tendency of considerable accumulation of atoms in levels 2S and 2P, in the former case on account of its metastability and in the latter on account of the denseness of the radiation of Lyman α,[3] to which the nebular matter is very opaque; this means that Cillié's assumption on the transparency of the nebula to the subordinate lines must be abandoned. But, as will be briefly discussed in §§ 24–26, from such a line of attack no straight solution of the problem seems to be obtained with any reasonable assumptions as to the nebular conditions. The problem, however, seems to be more naturally solved if we take into account the effect of collision.

In this paper a general theory will be formulated of the collisional excitation of the normal hydrogen atom by electron. In Chap. I, the emission mechanism of Balmer lines is treated generally, taking into account both the recombination and collision. For the practical application of the theory, it is necessary to know quantitatively the cross-sections for the inelastic collision inducing the excitation of the atom. For high speed collisions computations are rather simple, and fully discussed by many authors[4]; but the case immediately interesting for the nebular problem is that of the so-called slow collision, especially

1. I. S. Bowen. Ap. J. **67** (1928) 1.
2. V. Ambaruzumian. Pulkovo Circ. No. 6 (1935), M. N. **93** (1933) 50.
3. Cillié, *loc. cit.*
4. Cf. for instance, H. Bethe, Hb. d. Physik. Bd. XXIV/1. (1933) III.

so slow as to be just above the limit of excitation potential. In such a case, the mathematical expressions become so complicated that it is practically impossible to carry out numerical computations of them. In the last chapter of this paper, is a summarized method of approximate calculation together with results obtained. These results are used in the application of the theory on the nebulae (Chap. II and III), and it is shown that the very anomalies stated in § 1 can be explained. It is also found that a definite idea on the electron temperature of the nebulae is revealed.

I. General Theory

Ionization formula

§ 3. Degree of ionization in a hydrogen gas of temperature T_e, exposed to the diluted high temperature radiation is given by the well-known formula, in which two balancing processes have been considered: viz. photoelectric ionization and capture of free electrons. To include the collision effect, only a slight modification is required to the current formula.

Prior to our modified treatment, we may reproduce briefly, after S. Rosseland[1], the derivation of the current formula. Let N_1, N_c and N_e be the concentrations of hydrogen atoms in the normal, in the ionized state, and of free electrons respectively, neglecting all the intermediate states. For all transitions out of and into the given state, the conservation of energy is expressed by

$$N_1 a_{1c} = N_e N_c a_{c1},$$

where a_{1c} and a_{c1} are the transition probabilities due to the photoelectric ionization and recombination, multiplied by the energies associated therewith. Or, assuming $N_e = N_c$, viz. that the nebula is composed mainly of the hydrogen gas, we have

$$N_1 a_{1c} = N_e^2 a_{c1}. \tag{3.1}$$

In order to calculate a_{1c} and a_{c1}, we have to know the atomic absorption coefficient σ_ν in the ultraviolet region, and the probability of recombination. But it is sufficient to know one of them, since they are connected, as is well known, by the thermodynamical relations. Now in the nebula we have a field of the diluted temperature radiation, so that its density is

$$\frac{1}{\delta}\,\frac{8\pi h\nu^3}{c^3}\left(e^{\frac{h\nu}{kT_*}}-1\right)^{-1}$$

1. S. Rosseland, Theoretical Astrophysics. (1936) Chap. XII, § 116.

$$\delta = \frac{4r^2}{r_*^2}, \tag{3.2}$$

where r_*, r are the radius of the central star and of the nebulous shell, T_* the effective temperature of the nuclear star, and the other notations are as usual. On the other hand, we may safely assume that in the nebula, the velocity distribution of electrons is Maxwellian corresponding to a temperature T_e generally different from that of the central star. Thus Rosseland has derived the following expressions, approximating σ_ν by the Kramers' formula: $\sigma_\nu \backsim \nu^{-3}$,

$$\left.\begin{aligned} a_{1c} &= \frac{1}{\delta}\,\frac{8\pi\sigma_1}{c^2}kT_* e^{-h\nu_1/kT_*} \\ a_{c1} &= \frac{8\pi\sigma_1 kT_e}{\varpi c^2 f_e} \\ \sigma_\nu &= \sigma_1\nu^{-3} \qquad \nu > \nu_1 = \chi_1/h \\ \sigma_1 &= \frac{16\varepsilon^2\nu_1^2}{3\sqrt{3}mc} \\ f_e &= 2(2\pi mkT_e)^{3/2}h^{-3} \end{aligned}\right\} \tag{3.3}$$

where χ_1 means the ionization potential of the hydrogen atom, ν_1 the frequency of Lyman limit, and ϖ the ratio of the weight of the ionized to that of the normal state. Hence it follows from (3.1), the required formula of ionization;

$$\frac{N_e^2}{N_1} = \frac{1}{\delta}\,\frac{2\varpi(2\pi mkT_*)^{3/2}}{h^3}e^{-h\nu_1/kT_*}\sqrt{\frac{T_e}{T_*}}. \tag{3.4}$$

§ 4. To include the effect of collision in the formula, (3.1) must be replaced by

$$N_1 a_{1c} + N_1 N_e \gamma_{1c} = N_e^2 a_{c1}, \tag{4.1}$$

where γ_{1c} is the term of collisional ionization[1], which is expressed by

$$N_e \gamma_{1c} = \int_{v_1}^{\infty} h\nu \cdot Q(v) v \cdot N_e(v)\,dv, \tag{4.2}$$

with v = velocity of colliding electron,

v_1 = equivalent electron velocity of Lyman limit,

$N_e(v)dv$ = concentration of electrons in interval $(v, v+dv)$ per cm^3,

$Q(v)$ = effective area for ionization,

$h\nu$ = energy of the expelled electron.

As to $h\nu$, it is admissible, to replace it by $h\nu_1$ in practice. If we put the mean value of $Q(v)$,

1. The converse process is neglected compared with $N_e^2 a_{c1}$; it is, indeed, a three body collision of two electrons and a proton, so that its frequency may be very small compared with that of simple capture $N_e^2 a_{c1}$, unless its probability is extremely large.

$$\bar{Q} \equiv \frac{\int_{v_1}^{\infty} h\nu Q(v) v N_e(v) dv}{h\bar{\nu} \int_{v_1}^{\infty} v N_e(v) dv}, \tag{4.3}$$

we get
$$N_e \gamma_{1c} = h\nu_1 \bar{Q} \int_{v_1}^{\infty} v N_e(v) dv ;$$

or
$$\gamma_{1c} = h\nu_1 \bar{Q} \phi(T_e), \tag{4.4}$$

with
$$\phi(T_e) \equiv \int_{v_1}^{\infty} v \frac{N_e(v)}{N_e} dv,$$

or, inserting Maxwell's expression in $N_e(v)$ and integrating,

$$\phi(T_e) = 4\left(\frac{k}{2\pi m}\right)^{1/2} T_e^{1/2}\left(1 + \frac{\chi_1}{kT_e}\right) e^{-\chi_1/kT_e}. \tag{4.5}$$

Now we are able to solve the equation (4.1):

$$\frac{N_e}{N_1} = \frac{1}{2}\left[\frac{\gamma_{1c}}{a_{c1}} + \sqrt{\left(\frac{\gamma_{1c}}{a_{c1}}\right)^2 + \frac{4}{N_1}\left(\frac{a_{1c}}{a_{c1}}\right)}\right], \tag{4.6}$$

which becomes by (3.3) and (4.4),

$$\frac{N_e}{N_1} = \frac{1}{2}\left[\frac{h\nu_1 \bar{Q} \phi(T_e)}{\frac{8\pi\sigma_1}{\varpi c^2 f_e} \cdot kT_e} + \sqrt{\left(\frac{h\nu_1 \bar{Q} \phi(T_e)}{\frac{8\pi\sigma_1}{\varpi c^2 f_e} \cdot kT_e}\right)^2 + \frac{4}{N_1} \cdot \frac{1}{\delta} \varpi f_e \left(\frac{T_*}{T_e}\right) e^{-\frac{h\nu_1}{kT_*}}}\right]. \tag{4.7}$$

When the collisional effect is negligible, it of course degenerates into the ordinary form (3.4). On the contrary, when it dominates,

$$\frac{N_e}{N_1} = \frac{h\nu_1 \bar{Q} \phi(T_e)}{\frac{8\pi\sigma_1}{\varpi c^2 f_e} \cdot kT_e}. \tag{4.8}$$

§ 5. To carry out numerically the computation of (4.7) or (4.8), we must know the values of $\bar{Q}$ and N_1.

First, as to the probabilities of collisional ionization, laborious calculations have been performed by H. S. W. Massey and C. B. O. Mohr[1]. At the series limit, it has a vanishing probability and rapidly increases with the velocity of colliding electron and attains a maximum of about πa_0^2, where a_0 means Bohr's radius of hydrogen. Generally speaking, $\bar{Q}$ in (4.3) depends on T_e, but for our purpose a good approximation is obtained, replacing $\bar{Q}$ by its mean value $\bar{Q} \doteqdot 0.2\pi a_0^2$.

Secondly, we assume, with Zanstra[2], that the ultraviolet radiation from the central star is almost completely absorbed by its enveloping hydrogen nebula. The atomic absorption coefficient beyond the Lyman

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1. Mott and Massey, The Theory of Atomic Collisions. (1933) Chap. XI § 3.
2. Zanstra, *loc. cit.*

limit amounts to 0.54×10^{-17}, so that for the almost complete absorption, (optical depth unity say), there must be $1/(0.54 \times 10^{-17})$ normal atoms per square cm., which corresponds to $N_1=6$ for a nebular shell of thickness 2,000 A. U. at the distance 10,000 A. U. from the nucleus, as is the case in the typical nebula.

§ 6. Thus adopting the values $\bar{Q}=0.2\,\pi a_0^2$ and $N_1=6$, we have carried out numerical computations, of which the results are reproduced in Table I.

Table I. Values of $\frac{N_e}{N_1}\times 10^{-2}$ $(N_1=6,\ \bar{Q}=0.2\pi a_0^2)$

T_*	log δ	T_e 8,000	10,000	12,500	15,000	17,500	20,000	22,500	25,000	30,000	40,000
20,000	13	1.68	1.82	1.82	1.95	2.16	2.28				
	14	0.518	0.575	0.58	0.620	0.695	0.85				
	15	0.164	0.182	0.184	0.205	0.268	0.456				
	16	0.052	0.058	0.060	0.075	0.146	0.375				
30,000	13	8.2	8.6	9.1	9.6	10.0	10.4	11.0	11.8	14.2	
	14	2.58	2.74	2.89	3.04	3.20	3.40	3.82	4.49	7.00	
	15	0.815	0.865	0.915	0.965	1.05	1.21	1.59	2.36	5.40	
	16	0.258	0.274	0.290	0.313	0.378	0.550	1.01	1.92	5.20	
40,000	13	16.4	18.2	18.2	19.2	20.2	20.6	21.8	23.0	25.4	37.0
	14	5.18	5.75	5.75	6.1	6.40	6.65	7.20	7.85	10.2	23.4
	15	1.64	1.82	1.82	1.93	2.06	2.23	2.62	3.32	6.00	21.2
	16	0.515	0.575	0.580	0.625	0.695	0.850	1.25	2.09	5.29	20.9
50,000	13	25.8	27.4	28.8	30.0	31.4	32.4	33.9	35.4	38.8	50.6
	14	8.15	8.65	9.1	9.52	9.95	10.1	11.0	11.8	14.2	26.6
	15	2.58	2.74	2.88	3.02	3.19	3.40	3.82	4.50	7.00	21.6
	16	0.815	0.865	0.915	0.99	1.05	1.21	1.58	2.36	5.40	20.9
60,000	13	40.8	43.4	44.8	48.0	49.5	51.5	53.5	56.5	59.5	73.0
	14	12.9	13.7	14.1	15.0	15.7	16.3	17.1	18.2	20.8	32.6
	15	4.08	4.34	4.48	4.76	5.00	5.30	5.75	6.45	8.80	22.6
	16	1.29	1.37	1.42	1.54	1.62	1.80	2.18	2.89	5.60	21.0

The heavy lines in the table indicate the very limit, of which the right hand side corresponds to the domain where the ionization is effected by collision in a perceptible amount—say, half as much as photoelectrically, or more, and as the electron temperature T_e increases, after passing this line to the right, the degree of ionization increases rapidly and becomes much higher than that given by the current formula (3.4).

§ 7. A short note will be added here on a basal assumption of our theory. We have assumed that a local electron temperature T_e exists at each point within the nebula; or, in detail, that the electron velocities are distributed in a Maxwellian form, so that their states are represented by one parameter T_e, at least in a first approximation and that capturing processes take place under this condition. The Maxwellian distribution is in this case disturbed by two functions, viz. the capture and the photoelectric ionization, and conserved by the mutual collision of electrons (cf. § 1 c).

In order to be able to make use of the parameter T_e, the latter process must be proved to be much powerful than the former. This is, indeed, the case; for example, in the nebula of ionization degree 100, a rough estimate shows that the mutual collision occurs as frequently as once a day, while the ionization or capture once a year, or less. Thus the assumption on the existence of the local electron temperature seems to work out to self-consistent results[1, 2].

Cyclic Equations

§ 8. With Zanstra[3] and Cillié[4], we assume the following conditions in the nebula:

(i) Nebula is mainly composed of hydrogen gas.

(ii) The photoelectric ionization is succeeded by a capture of free electron; if the electron is captured in a higher state, it cascades down till to the lowest state. (But the reverse process can be omitted[5]).

(iii) Nebula is transparent to the subordinate lines.

(iv) Nebula is so opaque to Lyman radiation that these radiations, if created in it, are completely reabsorbed and split into a quantum of Balmer lines and that of L_α, before leaving the nebula.

§ 9. The equation of quanta is then[4],

$$N_n \sum_{r=2}^{n-1} A_{n,r} = G_n(T_e) + \sum_{r=1}^{\infty} N_{n+r} A_{n+r,n}. \qquad (n=3, 4, \ldots\ldots) \qquad (9.1)$$

N_n means the number of atoms in the n-th state per c.c., $A_{n,r}$ the

1. J. H. Jeans, Dynamical Theory of Gases, (1916) § 353.

2. It would be worth while to remark here, that any start from a non-Maxwellian distribution will lead one to self-contradiction, even if one takes into account the dissipation of the kinetic energy induced by the nebulium excitation.

3. Zanstra, *loc. cit.*

4. Cillié, *loc. cit.*

5. It follows from the condition (iii).

transition probability from the n-th to the r-th state. By the processes of recombination and collision, the atom is brought to the n-th state; let the number of such atoms be $G_n(Te)$ per c.c., per sec. The number of atoms which come down to this state from the higher ones is represented by the second term in the right side of the equation. Thus the equation balances the number of quanta arriving at this state against that of leaving there by the downward transitions[1], the complete opaque of the nebula to the Lyman radiation being introduced in summing up $A_{n,r}$ from $r=2$ to $n-1$.

For our purpose, or to evaluate the ratio $H\alpha : H\beta$, exhaustive calculation is unnecessary, and hence we shall neglect states higher than the fifth. Then (9.1) becomes

$$N_n\sum_{r=2}^{n-1}A_{n,r}=G_n(Te)+\sum_{r=1}^{n+r=5}N_{n+r}\,A_{n+r,n}\,. \qquad (n=3,4,5)$$

This is a set of simultaneous equations with respect to N's, of which the solution is

$$\begin{aligned} 10^8N_3 &= 2.28\ G_3+1.189\ G_4+1.106\ G_5 \\ 10^8N_4 &= \qquad\qquad\ \ 5.84\ \ G_4+2.12\ \ G_5, \end{aligned} \qquad (9.2)$$

where numerical values of the transition probabilities have been taken from the work of D. H. Menzel and C. L. Pekeries[2].

§ 10. It is now convenient to give separate expressions in G-terms both for the capture and collision. To see clearly their dependency on the electron density, we put

$$G_n(Te)=Ne^2\mathfrak{C}_n(Te)+NeN_1\mathfrak{D}_n(Te)$$

or

$$=NeN_1\left[\frac{Ne}{N_1}\mathfrak{C}_n(Te)+\mathfrak{D}_n(Te)\right]. \qquad (10.1)$$

(i) The part of capture $\mathfrak{C}_n(Te)$ is nothing but that of Cillié[3], which he derived from the J. A. Gaunt's absorption coefficient[4]:

$$\mathfrak{C}_n(Te)=\frac{2^9\pi^5}{(6\pi)^{3/2}}\ \frac{\varepsilon^{10}}{m^2c^3h^3}\left(\frac{m}{k}\right)^{3/2}\cdot\frac{1}{Te^{3/2}}\ \frac{1}{n^3}e^{\chi_n/kTe}[-E_i(-\chi_n/kTe)]$$

$$=3.22\times10^{-6}\cdot M_n(Te). \qquad (10.2)$$

(ii) The collisional term is

$$\mathfrak{D}_n(Te)=\int_{v_n}^{\infty}Q_n(v)\frac{v\cdot Ne(v)}{Ne}dv,$$

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1. The photoelectric ionization from states other than the normal one are negligible, owing to the smallness of the population therein. Similarly, the same holds as for the excitation or de-excitation by collision.
2. D. H. Menzel and C. L. Pekeries, M. N. **96** (1936) 77.
3. Cillié, *loc. cit.*
4. J. A. Gaunt, Phil. Trans. A. **229** (1929) 200.

where v_n = electron velocity equivalent to the excitation potential of the n-th state, $\chi_1-\chi_n$.

$Ne(v)dv$ = concentration of electrons in the interval $(v, v+dv)$ per c.c.,

$Q_n(v)$ = effective cross-section for the excitation of normal atoms to the n-th state and for the collision of v-electron.

If we put, as before,

$$\bar{Q}_n \equiv \frac{\int_{v_n}^{\infty} Q_n(v) v Ne(v) dv}{\int_{v_n}^{\infty} v Ne(v) dv}, \tag{10.3}$$

it becomes

$$\mathfrak{D}_n(Te) = \bar{Q}_n \cdot \phi_n(Te), \tag{10.4}$$

with

$$\phi_n(Te) = \int_{v_n}^{\infty} v \cdot \frac{Ne(v)}{Ne} dv.$$

Inserting for $Ne(v)$ the Maxwell expression corresponding to Te, we have

$$\phi_n(Te) = 4\left(\frac{k}{2\pi m}\right)^{1/2} Te^{1/2}\left(1+\frac{\chi_1-\chi_n}{kTe}\right)e^{-(\chi_1-\chi_n)/kT_e}, \tag{10.5}$$

or, χ's being expressed in e.V.,

$$\phi_n(Te) = 6.22 \times 10^5 Te^{1/2}\left(1+\frac{11580(\chi_1-\chi_n)}{Te}\right)e^{-11580(\chi_1-\chi_n)/T_e}. \tag{10.6}$$

§ 11. $Q_n(v)$ increases with v, from zero at the excitation limit v_n.[1] $\bar{Q}_n$ is also a function of Te; but as in the case of $\bar{Q}$ in the ionization formula, (cf. § 5), we shall approximate it by the following constant values:

n	3	4	5
$\bar{Q}_n$	0.2	0.1	0.06

$\pi a_0^2 = 0.88 \times 10^{-16}$ cm^2. being taken as the unit. $\bar{Q}_3$ and $\bar{Q}_4$ have been taken from the data given in the Chap. IV, while $\bar{Q}_5$ has been extrapolated both from those for lower states in our calculations and those for higher velocity impact computed by many authors[2]; yet $\bar{Q}_5$ is fortunately of less importance for the calculation of $H\alpha/H\beta$.

Both $\mathfrak{C}_n$ and $\mathfrak{D}_n$ have been computed for some values of Te and are given tabulated as follows:

1. Cf. Chap. IV, Table X.
2. Cf. Bethe, *loc. cit.*

Table II. $\mathfrak{C}(_nT_e)$ and $\mathfrak{D}_n(T_e)$

$n \diagdown T_e$	8,000	10,000	12,500	15,000	17,500	20,000	22,500	25,000	30,000
$\mathfrak{C}_n(T_e)\cdot 10^{14}$									
3	5.64	4.80	4.05	3.51	3.09	2.74	2.49	2.28	1.94
4	3.65	3.04	2.52	2.15	1.88	1.66	1.49	1.35	1.13
5	2.52	2.08	1.70	1.43	1.24	1.09	0.97	0.87	0.725
$\mathfrak{D}_n(T_e)\cdot 10^{12}$									
3	0.000	0.015	0.218	1.285	4.64	11.98	24.60	43.3	104.0
4	0.000	0.003	0.059	0.388	1.44	4.03	9.06	16.6	40.7
5	0.000	0.001	0.029	0.196	0.781	2.09	4.68	8.82	22.2

Balmer Decrement

§ 12. The population N_n and the Balmer emission E_n associated with it are connected by

$$E_n = A_{n2} h\nu_{n2} \cdot N_n, \tag{12.1}$$

where A_{n2} represents the probability of spontaneous transition from the n-th to the 2-nd state and ν_{n2} the frequency of the associated emission. The intensity ratio of $H\alpha$ to $H\beta$ is given by

$$\frac{E_3}{E_4} = \frac{A_{32}\nu_{32}}{A_{42}\nu_{42}} \cdot \frac{N_3}{N_4}. \tag{12.2}$$

If we use, as one often does, $H\alpha/H\beta$ instead of E_3/E_4, we have,

$$\frac{H\alpha}{H\beta} = 3.88 \times \frac{N_3}{N_4}. \tag{12.3}$$

Here N's are given by (9.2) in terms of G's, which in turn are given by (10.1). In (10.1), both $\mathfrak{C}$ and $\mathfrak{D}$ are functions of T_e only; however, to construct G by adding them, $\mathfrak{C}$ must be multiplied by the degree of ionization N_e/N_1, so that the Balmer decrement depends, in general, not only on T_e but also on T_*, δ, implicitly through N_e/N_1. In a special case, where either $\mathfrak{C}$ or $\mathfrak{D}$ is negligible, relative values of G's, and consequently of N's, and therefore $H\alpha/H\beta$, are all dependent solely on T_e.

II. Applications to the Model Nebulae

§ 13. In this chapter we will apply the result of our theory to the observed nebulae. For this purpose the objects are in general too remote and faint to enable us to make sufficient observation thereof, and therefore we must satisfy ourselves with some preliminary data.

Observational data on the Balmer decrement have been recently compiled by Berman[1]. He has classified the observed nebulae in three groups, according to the relative brightness of the principal nebular line, N_2, in comparison with $H\beta$, and shown statistically that the brighter the nebulium emission, the steeper is the decrement and the higher the temperature T_* of the unclear star. The data required for our purpose are reproduced from his work in the first six columns of Table III.[2]

Table III.

Group	N_2	$H\alpha$	$H\beta$	$H\gamma$	T_e	T_*	r in A. U.	δ
I	4.37	3.9	1.00	0.45	1,000	60,000	40,000	4.0×10^{16}
II	2.97	—	1.00	0.48	——	45,000	20,000	$3.0_2 \times 10^{15}$
III	1.81	2.77	1.00	0.50	2,500	30,000	10,000	$1.4_5 \times 10^{14}$

The values of T_* in the table have been determined, referring mainly to Zanstra's observation.[3] The radius of nebulae, r, has been derived from distance estimation by Berman[1], combined with H. D. Curtis' measurement of the apparent diameter[4]; although its values show a considerable dispersion among the individual objects in each group, we have, for definiteness, adopted the figures as in the table. As to the dilution factor δ, assuming with Zanstra[3], $M_* = 0.0^m$, for the absolute bolometric magnitude of the nuclear star, we can determine the radius of the central star r_* by the well-known formula:

$$\log_{10} r_*/r_\odot = 8.5 - 2.\log_{10} T_*,$$

where $r_\odot$ is that of the Sun; thus combined with r above adopted, it determines $\delta = 4r^2/r_*^2$, of which the values are tabulated in the last column.

§ 14. From (12.3), (9.2) and (10.1), by means of Tables I and II, we can now derive the Balmer decrement for each group, taking Te as a parameter. Results of calculations are summarized in Tables IV and V, and illustrated in Fig. 1.

In Table IV, we have calculated, besides G, the quantity

$$R_n(Te) \equiv \mathfrak{D}_n(Te) \Big/ \frac{Ne}{N_1} \mathfrak{C}_n(Te), \qquad (14.1)$$

which indicates, in a certain sense, the relative importance of collision

1. Berman, *loc. cit.*
2. As to T_e, which has been estimated by Page from the intensity distribution of the continuous Balmer emission, some discussions will be deferred to § 23.
3. Zanstra, *loc. cit.*
4. H. D. Curtis, Publ. Lick Obs. **13** (1918) Part II.

Table IV.

	Te	8,000	10,000	12,500	15,000	17,500	20,000	22.500	25,000
Group I	N_e/N_1	0.63	0.68	0.71	0.77	0.80	0.90	1.09	1.44
	G_3	0.814	0.804	0.775	1.10	2.05	4.68	10.72	24.2
	R_3		0.003	0.08	0.48	1.88	4.8	9.0	13.2
	G_4	0.529	0.506	0.473	0.565	0.846	1.79	4.19	9.61
	R_4		0.001	0.03	0.23	0.96	2.7	5.6	8.6
	G_5	0.360	0.348	0.320	0.360	0.510	0.995	2.25	5.21
	R_5		0.001	0.02	0.18	0.79	2.1	4.4	7.0
Group II	Ne/N_1	1.20	1.31	1.36	1.49	1.80	2.13	2.58	3.96
	G_3	2.96	2.97	2.75	3.48	6.58	13.62	28.8	74.5
	R_3		0.002	0.04	0.25	0.84	2.1	3.8	4.8
	G_4	1.915	1.08	1.71	1.93	3.16	5.80	12.0	31.3
	R_4		0.001	0.02	0.12	0.42	1.14	2.4	3.1
	G_5	1.305	1.28	1.16	1.25	1.93	3.38	6.66	17.5
	R_5			0.01	0.09	0.35	0.9	1.87	2.6
Group III	Ne/N_1	2.14	2.28	2.40	2.54	2.76	3.18	4.18	6.20
	G_3	9.40	8.99	8.40	9.32	13.02	23.7	52.7	128.0
	R_3		0.002	0.02	0.14	0.55	1.37	2.4	3.1
	G_4	6.1	5.69	5.29	5.35	6.55	10.68	23.0	55.6
	R_4		0.001	0.01	0.07	0.28	0.76	1.45	2.0
	G_5	4.16	3.89	3.60	3.49	4.15	6.37	13.14	31.8
	R_5			0.01	0.05	0.23	0.60	1.15	1.63

(G's must be multiplied by 10^{-8} and N_e/N_1 by 10^2.)

Table V. $H\alpha/H\beta$

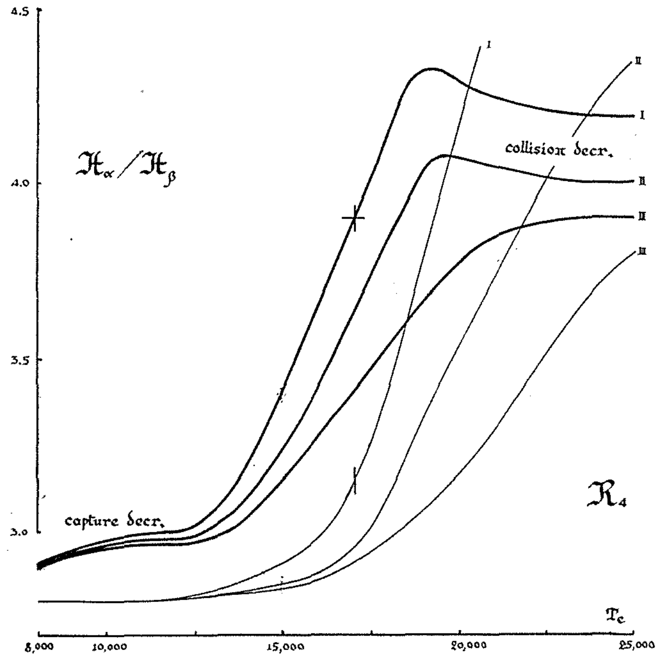
	Te	8,000	10,000	12,500	15,000	17.500	20,000	22,500	25,000
Group	I	2.91	2.98	3.02	3.42	4.02	4.30	4.21	4.19
	II	2.90	2.96	2.98	3.24	3.74	4.07	4.02	4.00
	III	2.90	2.96	2.965	3.16	3.47	3.77	3.88	3.89

and capture. In Fig. 1, $H\alpha/H\beta$ is represented by thick lines and the ratio R_4 by sharp ones.

The effect of collision is, as is clearly seen from the figure, practically negligible in the region $Te < 12{,}500°$, so that the current (pure capture) theory is applicable; the decrement in this region is so-called "capture-decrement".

Beyond this limit of Te, the collision becomes effective, and $H\alpha/H\beta$ begins to deviate rapidly from the capture-decrement, increasing till it reaches very high value, say ca. 4.3, 4.1 and 3.9 respectively for group

Fig. 1



I, II, and III, at about $T_e=20{,}000°$. Its highest value may or may not be a maximum, according to the different conditions of the model; but in either case, it approaches asymptotically to a limiting value 3.9 for high temperature over 30,000° (cf. § 15).

In the region $T_e=20{,}000°$, the decrement depends chiefly upon the collision; hence we will call it conventionally "collision-decrement" as distinguished from the "capture-decrement".

§ 15. We can determine the electron temperature in nebulae from the observed Balmer decrement. For group I, the observed decrement is steepest, i. e. $H\alpha/H\beta=3.9$ (cf. Table III). From Table V or Fig. 1, it corresponds $T_e \doteqdot 17{,}000°$ and $R_4=0.8$[1].

Such a determination of T_e is quite impossible from the pure capture theory; for the decrement in this case is so insensitive to the variation of T_e that it is not accountable at all for the observed high decrement of group I (cf. § 1 a). These circumstances may also be understood from the very formula of Cillié[2], which expresses the dependence of

1. Some remarks as to such a high temperature: cf. § 24.
2. Cillié, M. N. **92** (1932) 820, eq. (16).

the capture probability on the velocity of the incoming electron. Parenthetically it may be interesting to refer here to the fact that E. C. G. Stückelberg and P. M. Morse[1], in their probability calculation of slow electrons, have come to the conclusion that relative cross-section of the atom for each state is, in the first approximation, indifferent to the electron velocity.

Now the collision effect not only makes the decrement itself steep but also its dependency on T_e very sensitive, so that the high decrement observed in the nebulae of group I can be accounted for and also the electron temperature thereof can be determined. This comes from the fact that, with increasing quantum number n, the cross-section of the atom decreases more rapidly in the case of collisional excitation than in the case of capture, and moreover this tendency is accelerated by the multiplication of ϕ_n-factor, which is nearly parallel to the number of active electrons.

As for the properties of the "collision-decrement", a clear idea would be obtained if we consider the limiting case where the capture effect is completely excluded. In this case, the decrement has at $T_e \sim \infty$ a minimum value of 3.9, to which it tends asymptotically, as T_e increases, starting from ∞ at $T_e \sim 0$. This fact is due to the circumstances, that the diminution of ϕ_n with increasing n is gradually relieved as T_e increases (cf. (10.4)). The curves in Fig. 1 have been obtained by combining both decrements in this case and in the current theory; the relative weights of this combination being the degree of ionization N_e/N_1, on which the behaviour of the curves depends.

Similarly we can treat the groups II and III. In group II, as $H\alpha/H\beta = 3.4$ (Table III), we have $T_e = 16{,}000°$ and $R_4 = 0.2$. The observation for group III fits the capture decrement, so that T_e must be about $12{,}500°$ or less, the collision playing no essential rôle.

§ 16. We have tried in the present chapter, to give the theoretical interpretation of the high decrement observed in groups I and II. Thereby, a method has been found for estimating the electron temperature of the nebula from the observed decrement. But another method of T_e-estimation is suggested from our theory; namely, this can be done by comparing the intensity of any discrete line with that of continuous Balmer emission, as the intensity ratio of these two sorts of emission is intimately connected with R_n (cf. § 22).

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1. E. C. G. Stückelberg and P. M. Morse, Phys. Rev. **36** (1930) 16.

Observed value of this ratio is in some nebulae of group I much larger than theoretically predicted by Cillié (cf. § 21); this fact may, however, be reasonably explained, if we reflect upon the circumstances that the collision contributes nothing to the continuous emission, which always remains in a pure capture spectrum (cf. § 1 b).

III. N. G. C. 7662

§ 17. In the preceding chapter we have considered the model nebulae. To be able to apply our theory to the actual nebulae, we must have thorough knowledge on various properties thereof, of which accurate observations are desirable. But only of the bright nebula N. G. C. 7662, have we any considerable amount of data; these have been compiled by T. L. Page[1] and L. Berman[2]. It may be interesting to verify the present theory as applied to this nebula, or to infer some physical conditions in this object from the theory.

The nebula is an object of vis. mag. $8.^m4$, known as "Ring nebula in Andromeda"; its position for 1900 is: $AR = 23^h\ 21.^m1$, $\delta = +41^\circ\ 59'$.

It is a somewhat elongated disc of apparent diameter $14'' \times 17''$, surrounded by the fragmentally detached faint outer ring of $28'' \times 32''$. According to Berman (*loc. cit.*) its distance is 1,180 parsecs, so that we get $r = 10{,}000$ A. U. for the mean diameter of the inner ring.

The data of spectroscopic observations will be referred to later in due course.

§ 18. The effective temperature T_* of the nucleus star is estimated by various methods; but they are all the same in principle. Among them, the original ones of Zanstra are based on the condition for the continuity of quanta and the equation of energy, viz. the method of hydrogen recombination and that of nebulium excitation. His formulations are as follows[4]:

Method I (hydrogen recombination)

$$\int_{x_1}^{\infty} \frac{x^2}{e^x - 1} dx = \sum_{Balmer} \frac{x^3}{e^x - 1} A_\nu \qquad (18.1)$$

Method II (nebulium excitation)

$$\int_{x_1}^{\infty} \frac{x^3}{e^x - 1} dx - x_1 \int_{x_1}^{\infty} \frac{x^2}{e^x - 1} dx = \sum_{nebulium} \frac{x^4}{e^x - 1} A_\nu \qquad (18.2)$$

1. Page, *loc. cit.*
2. Berman, *loc. cit.*
3. Curtis, *loc. cit.*
4. Zanstra, *loc. cit.*

$$A_\nu \equiv \frac{L_p}{\nu \dfrac{\partial L_s}{\partial \nu}}, \qquad x \equiv \frac{h\nu}{kT_*},$$

where L_p and L_s are the intensities of the nebular emission and continuous nuclear spectrum. To avoid the intensity L_s, which is very difficult to measure in the technics of observation, R. H. Stoy[1] has proposed another method, combining (18.1) and (18.2):

Method III (line ratio)
$$\frac{\displaystyle\int_{x_1}^{\infty} \frac{x^3}{e^x - 1} dx - x_1 \int_{x_1}^{\infty} \frac{x^2}{e^x - 1} dx}{\displaystyle x' \int_{x_1}^{\infty} \frac{x^2}{e^x - 1} dx} = \frac{\displaystyle\sum_{nebulium} L_p}{\displaystyle\sum_{Balmer} L_p\left(\frac{\nu'}{\nu_p}\right)}, \qquad (18.3)$$

where ν' denotes a certain frequency suitably chosen, say, that of H_β, and x' corresponds to ν'.

Page has applied these formulae to his observations[2]; the results are inserted in the second column of Table VI.

Table VI. Nuclear Temperature of N. G. C. 7662.

Method	Direct (Page)	Corrected for collision
I. Hydrogen recombination	73,000°	70,000°
II. Nebulium excitation	67,000	85,000
III. Line ratio	32,000	75,000

§ 19. As is seen from the table, the value derived by the third method seems to be too small compared with the other two, even if the difficulty of the observational technic is taken into account. This discrepancy seems to be removable by introducing a correction for collisional effect.

i) Each term under $\sum$ in (18.1) represents the number of Balmer quanta. This, however, should not be equated to the number of quanta in the ultraviolet region, if the collision effect is not negligible, viz. we have not to take L_p itself, but the very fraction of it, which corresponds to the contribution from capture[3]. For our nebula, it amounts ca. 1/3, because this contribution appears about half that of collision, as is easily inferred from the observation of E_n/Ba_c (cf. § 22). Usually, the continuous Balmer emission is neglected in the summation of quanta. But

1. R. H. Stoy, M. N. **93** (1933) 588.
2. Page, *loc. cit.*
3. When the collisional ionization is effective, this fraction must further be reduced to that part of capture which balances with the photoelectric ionization. But it is not the case in our nebula.

this is not the case in the present consideration ; in fact, it is as large as about 1/2 of the whole discrete Balmer quanta due to capture (cf. C_{nl} in Table X), and consequently the right hand side of (18.1) must be multiplied by $\frac{1}{3}\times 2$. Thus Page's result is reduced to about 70,000°.

ii) As for the second method, each term on the left hand side of (18.2) represents the energy liberated by the spontaneous transition succeeding to the collisional excitation. Thus in our case, where the hydrogen emission does not consist purely of capture spectrum, the summation must be extended to those parts of Balmer and Lyman emission which come from the collisional excitation[1]. To see the order of magnitude of this excitation, we take the contribution from the Balmer emission as $5\times\frac{2}{3}\doteqdot 3$, normalizing the intensity of its whole energy to 5 ; that of Lyman α as 30, taking as ten times that of Balmer[2], and the total nebulium emission as 30, twice that of N_1+N_2 (cf. Table III). We should then adopt for the total sum of the nebulium emission $33+30$ instead of 30 ; that is, the second side of the equation (18.2) must be doubled. Consequently, Page's result must be raised to about 85,000°. A discrepancy of 20% may not be so grave, if we remark that T_* is very sensitive for its high value to the variation of A_ν.

iii) The value obtained by the third method has been corrected in the same way, multiplying the numerator by $\frac{1}{3}\times 2$ and the denominator by 2 in the right side of (18.3). The corrected value, 75,000°, seems to lessen the discrepancy.

§ 20. Now we tentatively employ 70,000° for T_*, according to the

Table VII. Theoretical Balmer Decrement for N. G. C. 7662.

T_e	15,000	17,500	20,000	22,500	25,000	30,000
$N_e/N_1\ \times 10^{-2}$	3.9	3.78	4.00	4.42	5.12	7.52
$H\alpha/H\beta$	3.12	3.36	3.68	3.85	3.95	4.04
R_3	0.09	0.40	1.09	2.24	3.71	7.11
R_4	0.05	0.20	0.605	1.31	2.40	4.79
R_5	0.04	0.17	0.48	1.09	1.98	4.07

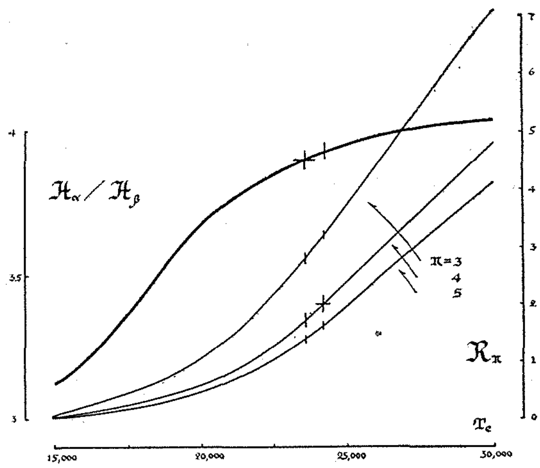
1. This means that the dissipation of the kinetic energy of the electron is caused by the inelastic collision not only with nebulium elements but also with hydrogen atom. (also cf. foot-note 3 of the preceding page.)

2. Contributions from higher series are all neglected, owing to the smallness of their energy.

discussion in the preceding paragraph. Then, we obtain as the dilution factor $\frac{1}{\delta}=2.25\times10^{-16}$, assuming $M_{*}=0.^{m}0$ as before (cf. § 13).

Then, we are able to calculate the Balmer decrement from (12.3), (9.2), and (10.1), taking T_e as a parameter. The results are given in Table VII and illustrated in Fig. 2.

Fig. 2. N. G. C. 7662



§ 21. The electron temperature T_e can be determined either from the observed Balmer decrement or from the intensity ratio of the discrete to the continuous Balmer emission. The observed data are as in Table VIII.

Table VIII. Observed Balmer Decrement of N. G. C. 7662.

	Balmer decrement	$\log_{10} E_n/Ba_c$						
		Observational (Page)		Theoretical (Cillié)				
		W 4.9″	E 4.9″	1,000°	5,000°	10,000°	20,000°	50,000°
N_2	3.66							
$H\alpha$	(3.9)	—	—	—	3.38	3.60	3.77	3.98
$H\beta$	1.0	4.02	3.76	2.40	2.95	3.14	3.31	3.50
$H\gamma$	0.45	3.67	3.41	2.13	2.65	2.84	3.00	3.14
$H\delta$	0.23	3.38	3.12	1.92	2.42	2.60	2.74	2.91

In the table, the values of Balmer decrement are those of Berman[1], which have been corrected by him for space reddening. $H\alpha$ is absent, but if we judge from the intensities of $H\gamma$ and $H\delta$, it seems to bear the character of collisional decrement. Thus we will assume for it the mean value of the first group (cf. Table I).

Table VIII also contains the intensity ratio of the discrete Balmer E_n to the continuous emission Ba_c per unit frequency at the series limit.

1. Berman, *loc. cit.*

Here the observed values are not the original ones of Page[1], who observed the continuous emission at 4.9″ West and East of the nucleus, but those obtained after correcting for space reddening, according to the Berman's formula.

Theoretical ratio is due to Cillié[2], who derived it as follows :—The energy emitted per sec. per c.c. in a frequency range (ν, $\nu+d\nu$) beyond the series limit is given by

$$N_e^2 \frac{2^6\pi^5}{(6\pi)^{3/2}} \cdot \frac{\varepsilon^{10}}{m^2c^3h^2}\left(\frac{m}{kT_e}\right)^{3/2} e^{(\chi_2-h\nu)/kT_e} d\nu.$$

At the head of the series limit, it becomes per unit wave number

$$Ba_e = [\overline{2}3{\cdot}902] N_e^2/T_e^{3/2}.$$

Combining it with (12.1), he found

$$\frac{E_n}{Ba_e} = [22{\cdot}098] \frac{N_n}{N_e^2} A_{n2} h\nu_{n2} T_e^{3/2}. \qquad (21.1)$$

§ 22. In the case of pure capture spectrum, N_n/N_e^2 is uniquely determined by (9.2) and (10.1) as a function of T_e. For our nebula, the collision, as will be found later, does not yet affect the degree of ionization, so that the denominator remains the same as Cillié's value for it. Therefore only N_n increases with T_e. But the observed ratio is about 3 times that given by (21.1) for $T_e=20{,}000^\circ$; and thus it is suggested N_n must be raised by the same factor. Hence we may conclude that the collisional excitation surpasses about twice that of capture[3]; thus we may roughly take for $n=4$, $R_4=2$, which corresponds to $T_e=24{,}500^\circ$ and $H\alpha/H\beta=3.93$. On the contrary, if we start from the decrement, its value 3.9 gives $T_e=23{,}500^\circ$ and $R_4=1.7$. The accordance of the above results may be permitted as satisfactory, if we reflect upon the approximate nature of our method and various difficulties on the observational side.

§ 23. The intensity distribution of the continuous Balmer emission led Page to a conclusion of much lower temperature (1,200°–4,000° for the present nebula) than ours; it moreover suggests a non-Maxwellian distribution of the electron velocities in the nebula (cf. § 1 c and d). As an approach to the explanation of this anomaly, it is suggested that slow electrons created by the inelastic collision are captured[4], before they are adjusted to the thermal velocity, disturbing the Maxwellian distri-

1. Page, *loc. cit.*
2. Cillié, *loc. cit.*
3. In fact, G_n is the main term for N_n (cf. (9.2)).
4. The capture probability is inversely proportional to the kinetic energy of the colliding electron.

bution. They may be effective in the direct vicinity of the series limit and be able to change the slope of the continuous emission[1]; though they can, by no means, affect the whole amount of capture. In other words, it may happen in the nebula that the velocity distribution *effective for capture* is non-Maxwellian.

§ 24. *Metastability of 2S state.* In the preceding paragraphs, we have been concerned only with the principal quantum number n, neglecting all the finer structure of hydrogen atom. Especially, we have pretended ignorance on the metastability of $2S$ state, which may have caused an accumulation of atoms in this level and may therefore have violated our starting assumption of the transparency of the nebula to the subordinate lines[2].

In N. G. C. 7662, for example, there are 500×6 electrons per c.c. (cf. Table VIII), but among them only 1% have energy enough to excite the atom to $2S$ state: thus there are 30 active electrons and 6 normal atoms per c.c. If we assume 2×10^8 cm. sec.$^{-1}$ as the mean electron velocity and $\pi a_0^2 \doteqdot 10^{-16}$ cm^2, as the effective cross-section (cf. Table X), we get for the frequency of such excitations, $30\times 2\times 10^8\times 10^{-16}=6\times 10^{-7}$ sec.$^{-1}$ On the other hand, the spontaneous transition occurs once every several months[3] or with the frequency of the order 10^{-7} sec., a figure which is rather smaller than the former. Thus if we put aside for a moment the removal of atoms from $2S$ state by quantum absorption, it may be concluded that the population N_{2S} is mainly adjusted by balancing the excitation $1S\rightarrow 2S$ against the de-excitation $2S\rightarrow 1S$ through the collisional processes; that is N_{2S} obeys Boltzmann's law:

$$\frac{N_{2S}}{N_{1S}}=e^{-h(\chi_1-\chi_2)/kT_e},$$

the values of which are given in the second row of the following table.

T_e	10,000	15,000	20,000	25,000	30,000
N_{2S}/N_{1S}	7.8×10^{-6}	3.9×10^{-4}	2.8×10^{-3}	0.91×10^{-2}	2.0×10^{-2}
$\omega_{2S,3P}\cdot N_{2S}/N_{1S}$	1.6	7.9×10	5.6×10^2	1.8×10^3	4.0×10^3

According to the usual notation, let $\omega_{2S,3P}$ be the ratio of the absorption coefficient for $2S-3P$ of $H\alpha$ to that of ultraviolet radiation

1. If so, in the comparison of the discrete to the continuous emission (§ 21), it would not be preferable to make use of the intensity of continuous emission at the series limit. It is best to use the integrated emission beyond the series limit, but may be difficult observationally as an actual problem.

2. As for the self-reversal, M. Kurihara has exhaustively discussed the nature of this effect in the case of B_e stars; (these Memoirs **21** (1938) 89.)

3. Bethe, *loc. cit.* § 43.

beyond the Lyman limit. As it depends slightly on T_e and T_*, we will put $\omega_{2S,3P}=2\times10^5$, with M. Kurihara[1]. In our investigation we have assumed that the optical depth is unity for the ultraviolet radiation, so that $\omega_{2S,3P}\frac{N_{2S}}{N_{1S}}$ represents directly the optical depth for $2S-3P$ of $H\alpha$. For $T_e<10{,}000°$, $\omega_{2S,3P}\frac{N_{2S}}{N_{1S}}<1$ as shown in the above table. Therefore the nebula may be considered as transparent to $H\alpha$ in this case. Thus so far as the nebulae of group III are concerned, our assumption of transparency is justified. But this is not the case for the other groups; there it seems, indeed, that the absorptions take place from the $2S$ state.

§ 25. So far we have treated the atom as an isolated existence. But at the same time we have to take account of the influences of the environment; its disturbances cause, more or less, the mutual coherences of pure states, and thus the metastable state has a tendency to lose its proper character. As for such external influences, there are in our case powerful mechanisms, which remove the atom from $2S$ state.

i) The transition $2^2S_{1/2}\rightarrow2^2P_{1/2}$ is induced by the electron impact. The effective area for the mechanism can be estimated in the same way as in Chap. IV; a rough calculation shows that it amounts to at least 100, perhaps 10^4 times that of de-excitation (of the order πa_0^2). Thus the values of N_{2S}/N_{1S} given above may be reduced by the same ratio.

ii) There is a sort of Stark effect caused by the electric field due to the surrounding free electrons and protons. There the disturbed wave functions are given by the linear combinations of $2^2S_{1/2}$- and $2^2P_{1/2}$-state. Thus $2^2S_{1/2}$-state loses its metastability[3]; the transition probability to $1^2S_{1/2}$-state becomes in our case of the order 1 sec.$^{-1}$ Therefore, if only this effect is considered, the population N_{2S} is given roughly by

$$\frac{N_{2S}}{N_{1S}}\doteqdot\frac{6\times10^{-7}}{1}\doteqdot10^{-6}.$$

Therefore

$$\omega_{2S,3P}\frac{N_{2S}}{N_{1S}}<1.$$

These circumstances would prevent the accumulation of atoms in that very state.

§ 26. Some authors[4] have suggested that considerable amount of atoms may be accumulated to $2P$ state, owing to the large density of

1. M. Kurihara, *loc. cit.*
2. Bethe, *loc. cit.* § 43.
3. Cf. Bethe, *loc. cit.* especially § 43.
4. Ambarzumian, Cillié, Page, Berman, *loc. cit.*

L_α-radiation. But simple estimation shows that this radiation is not so dense as to cause the self-reversal of Balmer lines. Indeed, Ambarzumian[1], in his order-estimation of this effect, has employed as the dilution factor $\delta=10^{13}$ for $T_*=40{,}000^\circ$–$50{,}000^\circ$, which seems, however, to be too small (cf. Table III). If we adopt our correct value $\delta=10^{15}$, the population of atoms in $2P$ state is reduced to $1/100$ of his value. Thus it becomes $N_{2P}/N_{1S} \doteqdot 5\times 10^{-7}$, which makes the optical depth $\omega_{2P,3D}\dfrac{N_{2P}}{N_{1S}}$ smaller than unity.

IV. Excitation of Hydrogen Atom by Slow Electron Impact

The collisional excitation of hydrogen atom immediately interesting to the nebular condition is the case of slow electron impact; indeed, electrons have in the nebula very slow thermal velocity, so that few of them exceed the limit of Lyman series, 13.53 e. V. The effective cross-section for excitation can be easily estimated, if the electron velocity corresponds to 100 e. V. or more. For slow collision, however, we must treat very complicated expressions, since both the distortion of electron wave and the exchange effect between colliding and atomic electrons become large.

In this chapter we will try to make an order-estimation in such a case, neglecting all but the effect of exchange; the method is in principle the same as that of Massey and Mohr.[2]

§ 27. *Effective cross-section.* We treat the problem statically and start with Schrödinger's equation for the system of two electrons:

$$(\nabla_1^2+\nabla_2^2)\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)+\frac{8\pi^2 m}{h^2}\left(E_1+E_2-\varepsilon^2\left[\left(\frac{1}{r_{12}}-\frac{1}{r_1}\right)-\frac{1}{r_2}\right]\right)\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=0, \tag{27.1}$$

where suffixes 1 and 2 are referred to the colliding and atomic electrons respectively, and ε means the elementary charge, and r_{12}, the distance between two electrons.

As a first approximation, the atomic electron must satisfy the wave equation of hydrogen:

$$\nabla_2^2\psi(\boldsymbol{r}_2)+\frac{8\pi^2 m}{h^2}\left(E_2+\frac{\varepsilon^2}{r_2}\right)\psi(\boldsymbol{r}_2)=0. \tag{27.2}$$

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1. Ambarzumian, *loc. cit.*
2. Mott, Proc. Roy. Soc. **125** (1929) 222.
 Massey and Mohr, ibid. **132** (1931) 605, **136** (1932) 289, **139** (1933) 187, **140** (1933) 613, **146** (1934) 880.

Let $\psi_N(\boldsymbol{r}_2)$ be a solution of this equation, viz. an eigen-function corresponding to the energy level χ_N, each quantum state (n, l, m) being represented by a single letter N.

Suppose now for a moment, two electrons are distinguishable and seek solution of such a form as

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2)=(\textstyle\sum+\int)_N\psi_N(\boldsymbol{r}_2)F_N(\boldsymbol{r}_1), \tag{27.3}$$

in which for large r holds

$$F_N(\boldsymbol{r})\sim e^{ik_1z}+\frac{e^{ik_1r}}{r}f_I(\theta) \quad \text{for normal state, } N=I,$$

$$\sim \qquad \frac{e^{ik_Nr}}{r}f_N(\theta) \quad \text{otherwise.} \tag{27.4}$$

By summation and integration in (27.3) are included both all the discrete and the continuous state N; and

$$k_1=\frac{2\pi mv_1}{h}, \qquad k_N^2=\frac{8\pi^2m}{h^2}[E_1-(\chi_N-\chi_I)]=\left(\frac{2\pi mv_N}{h}\right)^2; \tag{27.5}$$

thus e^{ik_1z} means the incident electron wave of velocity v_1 along z-axis and of unit density per c.c. and the scattered electron after the excitation of the atom in its N-th state is represented by the spherical wave $r^{-1}e^{ik_Nr}f_N(\theta)$, where θ is the angle of scattering (Fig. 3).

Fig. 3 Direct Excitation

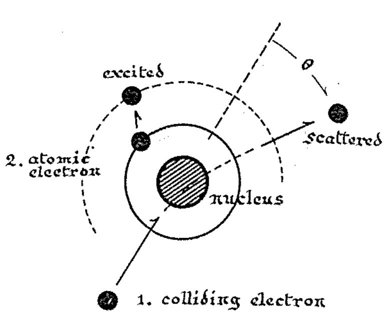
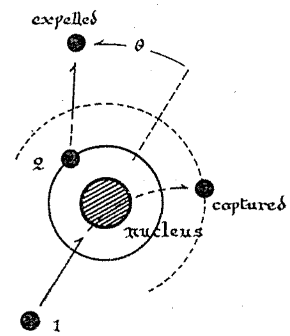


Fig. 4 Exchange Effect



Then, if the incident beam is such that one electron falls on cm², per sec., the number of electrons $I_N(\theta)d\omega$, which are scattered with velocity v_N into an angle element $d\omega$ per sec. is given by

$$I_N(\theta)d\omega=\frac{v_N}{v_1}|f_N(\theta)|^2=\frac{k_N}{k_1}|f_N(\theta)|^2. \tag{27.6}$$

Now, we define the total cross-section as

$$Q_N\equiv\int I_N(\theta)d\omega=2\pi\int_0^\pi I_N(\theta)\sin\theta d\theta. \tag{27.7}$$

Now, it follows from (27.1) and (27.3), taking account of (27.2),

$$(\sum+\int)_N\psi_N(\boldsymbol{r}_2)\left[\nabla_1^2+\frac{8\pi^2m}{h^2}(E_1+E_2-(\chi_N-\chi_l))\right]F_N(\boldsymbol{r}_1)$$
$$=\frac{8\pi^2m}{h^2}\left[\varepsilon^2\left(\frac{1}{r_{12}}-\frac{1}{r_1}\right)\right]\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2);$$

multiplying it by $\psi_N^*(\boldsymbol{r}_2)$ and integrating over the whole $\boldsymbol{r}_2$ space,

$$(\nabla_1^2+k_N^2)F_N(\boldsymbol{r}_1)=\frac{8\pi^2m}{h^2}\int\varepsilon^2\left(\frac{1}{r_{12}}-\frac{1}{r_1}\right)\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)\psi_N^*(\boldsymbol{r}_2)dv_2. \qquad (27.8)$$

§ 28. *Exchange effect.* When the eigen-function corresponds to the continuous state, it also gives some information about the probability that the incident electron is captured and the atomic electron ejected (Fig. 4). But in order to see it more clearly, we expand (27.3) in the alternative form:

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=(\sum+\int)_N\psi_N(\boldsymbol{r}_1)G_N(\boldsymbol{r}_2), \qquad (28.1)$$

assuming that G_N has the asymptotic form:

$$G_N\sim\frac{e^{ik_Nr}}{r}g_N(\theta) \qquad \text{for large } r. \qquad (28.2)$$

Then, in the same way as before, we obtain the equation for G_N:

$$(\nabla_2^2+k_N^2)G_N(\boldsymbol{r}_2)=\frac{8\pi^2m}{h^2}\int\varepsilon^2\left(\frac{1}{r_{12}}-\frac{1}{r_2}\right)\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)\psi_N^*(\boldsymbol{r}_1)dv_1. \qquad (28.3)$$

Equations (27.8) and (28.3) determine the function F_N and G_N.

§ 29. As $\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$ contains F_N and G_N, the equations (27.8) and (28.3) are of integro-differential type. They are easily reduced to the integral equation. But it is still difficult to obtain their strict solutions; hence some adequate approximation to $\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$ is required.

The following calculations are based on the *Born's approximation*; that is, neglecting all the disturbances of electron waves and of the atomic field, we take for $\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$ the following form:

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)=e^{ik_1\boldsymbol{n}_0\boldsymbol{r}_1}\psi_l(\boldsymbol{r}_2), \qquad (29.1)$$

where $\boldsymbol{n}_0$ is the unit vector in the direction of positive z-axis. Then (27.8) and (28.3) become

$$(\nabla_1^2+k_N^2)F_N(\boldsymbol{r}_1)=\frac{8\pi^2m\varepsilon^2}{h^2}\int\left(\frac{1}{r_{12}}-\frac{1}{r_1}\right)e^{ik_1\boldsymbol{n}_0\boldsymbol{r}_1}\psi_l(\boldsymbol{r}_2)\psi_N^*(\boldsymbol{r}_2)dv_2, \qquad (29.2)$$

$$(\nabla_2^2+k_N^2)G_N(\boldsymbol{r}_2)=\frac{8\pi^2m\varepsilon^2}{h^2}\int\left(\frac{1}{r_{12}}-\frac{1}{r_2}\right)e^{ik_1\boldsymbol{n}_0\boldsymbol{r}_1}\psi_l(\boldsymbol{r}_2)\psi_N^*(\boldsymbol{r}_1)dv_1. \qquad (29.3)$$

In the integration of (29.2), the term coming from $1/r_1$ vanishes owing to the orthogonality of the wave functions.

The solutions for F_N and G_N are obtained by applying Green's theorem, viz. we get, for large r,

$$F_N(\boldsymbol{r}) \sim \frac{e^{ik_N r}}{r} f_N(\theta),$$

$$G_N(\boldsymbol{r}) \sim \frac{e^{ik_N r}}{r} g_N(\theta),$$

with

$$f_N(\theta) = \frac{2\pi m\varepsilon^2}{h^2} \iint e^{i(k_1 \boldsymbol{n}_0 - k_N \boldsymbol{n})\boldsymbol{r}_1} \left(\frac{-1}{r_{12}}\right) \psi_I(\boldsymbol{r}_2) \psi_N^*(\boldsymbol{r}_2) dv_1 dv_2, \tag{29.4}$$

$$g_N(\theta) = \frac{2\pi m\varepsilon^2}{h^2} \iint e^{i(k_1 \boldsymbol{n}_0 \boldsymbol{r}_1 - k_N \boldsymbol{n} \boldsymbol{r}_2)} \left(\frac{1}{r_2} - \frac{1}{r_{12}}\right) \psi_I(\boldsymbol{r}_2) \psi_N^*(\boldsymbol{r}_1) dv_1 dv_2, \tag{29.5}$$

where $\boldsymbol{n}$ is the unit vector in the direction of $\boldsymbol{r}$.

§ 30. So far, we have assumed that the two electrons are distinguishable. Now we introduce the condition that they are indistinguishable and obey Fermi-Dirac's statistics ; then, for the unpolarized beam of the impinging electrons, both the functions f_N and g_N are combined to give the scattering intensity as

$$I_N(\theta) d\omega = \frac{k_N}{k_1} \left[\frac{3}{4} |f_N(\theta) - g_N(\theta)|^2 + \frac{1}{4} |f_N(\theta) + g_N(\theta)|^2 \right] d\omega, \tag{30.1}$$

and Q_N is given by (27.7).

Final formulae and Numerical results

§ 31. If the reduction of our formulae is treated in detail, it will be easily seen that the states with negative magnetic quantum number ($m < 0$) have the same excitation probabilities as the positive ones. Consequently we restrict ourselves to the case $m \geqslant 0$.

The normalized wave function for hydrogen is[1]

$$\psi_N = \psi_{nlm} = N_{nlm} \left(\frac{2r}{na_0}\right)^l e^{-r/na_0} L_{n+l}^{2l+1} \left(\frac{2r}{na_0}\right) \bar{P}_l^m(\cos\theta)\ e^{im\phi}$$

$$N_{nlm} = \frac{1}{\sqrt{4\pi}} \frac{\left(\frac{2}{na_0}\right)^{3/2}}{(n+l)!} \sqrt{\frac{(n-l-1)!}{(n+l)!\ n}}, \tag{31.1}$$

where L's are Laguerr polynomials and $\bar{P}$'s normalized spherical harmonics.

Now, we have to insert these expressions in (29.4) and (29.5).

§ 32. For $f_N(\theta)$, after some complicated mathematical reductions we obtain finally

$$f_N(\theta) = \sqrt{\pi}\, a_0 \cdot \Theta_n(\theta) \Omega_{nlm} \bar{P}_l^m(\cos\psi) I_{nl}(\zeta), \tag{32.1}$$

1. Cf. Condon and Shortley, The Theory of Atomic Spectra. (1935) Chap. V.

with

$$\cos\psi = (k_1 - k_n \cos\theta)/K$$
$$K^2 = k_1^2 + k_n^2 - 2k_1 k_n \cos\theta$$
$$\Theta_n(\theta) = (a_0 K)^{-5/2}$$
$$\Omega_{nlm} = i^{l+m+2} \frac{1}{(n+l)!}\sqrt{2n\frac{(n-l-1)!}{(n+l)!}}$$
$$\zeta = \frac{na_0}{2}K \qquad (32.2)$$
$$I_{nl}(\zeta) = \int_0^\infty e^{-\xi(1+n)/2}\xi^{l+3/2} L_{n+l}^{2l+1}(\xi) J_{l+1/2}(\zeta\xi)\, d\xi \qquad (32.3)$$

where J's are Bessel functions. Integrals (32.3) can be evaluated elementarily and their values are given in Table IX.

Table IX.

(n,l)	$-\sqrt{\frac{\pi}{2}}\cdot I_{nl}(\zeta)$	$-\sqrt{\frac{\pi}{2}}\cdot I^{\circ}_{nl}(\zeta)$
(2,0)	$16\zeta^{5/2}\left(\frac{9}{4}+\zeta^2\right)^{-3}$	$2\zeta^{1/2}(4\zeta^2-1)\left(\frac{1}{4}+\zeta^2\right)^{-3}$
(2,1)	$72\zeta^{3/2}\left(\frac{9}{4}+\zeta^2\right)^{-3}$	$24\zeta^{3/2}\left(\frac{1}{4}+\zeta^2\right)^{-3}$
(3,0)	$36\zeta^{5/2}(3\zeta^2+4)(4+\zeta^2)^{-4}$	$9\zeta^{1/2}\left(6\zeta^4-4\zeta^2+\frac{3}{8}\right)\left(\frac{1}{4}+\zeta^2\right)^{-4}$
(3,1)	$576\zeta^{3/2}(3\zeta^2+4)(4+\zeta^2)^{-4}$	$144\zeta^{3/2}(4\zeta^2-1)\left(\frac{1}{4}+\zeta^2\right)^{-4}$
(3,2)	$11520\zeta^{5/2}(4+\zeta^2)^{-4}$	$2880\zeta^{5/2}\left(\frac{1}{4}+\zeta^2\right)^{-4}$
(4,0)	$48\zeta^{5/2}(16\zeta^4+13\zeta^2+575)\left(\frac{25}{4}+\zeta^2\right)^{-5}$	$\zeta^{1/2}(384\zeta^6-672\zeta^4+168\zeta^2+21)\left(\frac{1}{4}+\zeta^2\right)^{-5}$
(4,1)	$1800\zeta^{3/2}(16\zeta^4+88\zeta^2+125)\left(\frac{25}{4}+\zeta^2\right)^{-5}$	$120\zeta^{3/2}(80\zeta^4-56\zeta^2+5)\left(\frac{1}{4}+\zeta^2\right)^{-5}$
(4,2)	$69120\zeta^{5/2}(8\zeta^2+25)\left(\frac{25}{4}+\zeta^2\right)^{-5}$	$34560\zeta^{5/2}(4\zeta^2-1)\left(\frac{1}{4}+\zeta^2\right)^{-5}$
(4,3)	$4838400\zeta^{7/2}\left(\frac{25}{4}+\zeta^2\right)^{-5}$	$967680\zeta^{7/2}\left(\frac{1}{4}+\zeta^2\right)^{-5}$

§ 33. Reduction of $g_N(\theta)$ is much more complicated. To avoid laborious calculations, we introduce the following approximations:

i) *Slow collision*: Close above the excitation potential, k_N is small compared with k_1, namely, the colliding electron dissipates most of its kinetic energy in exciting the atom. We may, therefore, neglect k_N in the exponential factor of (29.5), thus the g_N-terms become independent of the scattering angle θ and the excitation probability vanishes owing to the factor $e^{-im\phi}$, except $m=0$.

$$g_N(\theta)=\sqrt{\pi}\,a_0\cdot\eta^{-1/2}\Xi_{nl}I_{nl}(\eta), \qquad m=0 \tag{33.1}$$
$$\eta=\frac{na_0}{2}k_1$$
$$\Xi_{nl}=i^{l+2}\frac{n}{(n+l)!}\sqrt{\frac{2l+1}{2}\cdot\frac{(n-l-1)!}{(n+l)!}}. \tag{33.2}$$

Here, the extra term coming from $1/r_2$ in the brackets of (29.5) has been omitted, though it surpasses the retained term (33.1) in the present case. It is, indeed, as in the case of $f_N(\theta)$, a term, which ought to vanish if the exact expression was employed for the wave function of incoming electron instead of plane wave approximation. The discrepancy may originate from the circumstances that the neglection of k_N means, in fact, for this term, complete omission of the phase factor for the second electron.

ii) *Moderate velocity collision*: Though of no direct use, we will here deal with this case for the sake of completeness. Now, the integral value of (29.5) comes mainly from the domain where r_1 and r_2 are small, so that we may put $e^{-r_2/a_0}=1$ in $\psi_l(\boldsymbol{r}_2)$. Then we get

$$g_N(\theta)=\sqrt{\pi}\,a_0\cdot\Theta_n^0(\theta)\Omega_{nlm}\bar{P}_l^m(\cos\psi)I_{nl}^0(\zeta), \tag{33.3}$$
$$\Theta_n^0=(a_0K)^{-1/2}(a_0k_n)^{-2}, \tag{33.4}$$

Table X. $Q_{nl}(v)$ and C_{nl}

	$Q_{nl}(v)$							C_{nl}
(n,l) v	1.95	2.0	2.1	2.2	2.3	2.4	2.5	
$1S$								0.227
1								0.227
$2S$	0.14	0.18	0.23	0.26	0.29	0.31	0.32	0.0335
$2P$	0.46	0.63	0.88	1.07	1.26	1.30	1.38	0.109
2	0.60	0.81	1.11	1.33	1.55	1.61	1.70	0.143
$3S$			0.024	0.041	0.051	0.058	0.064	0.0114
$3P$			0.077	0.13	0.19	0.22	0.26	0.0403
$3D$			0.0084	0.016	0.021	0.024	0.026	0.0520
3			0.109	0.187	0.261	0.302	0.350	0.104
$4S$				0.022	0.032	0.039	0.046	0.0053
$4P$				0.042	0.061	0.072	0.084	0.0190
$4D$				0.0072	0.0089	0.011	0.012	0.0318
$4F$				—	—	—	—	0.0254
4				0.071	0.102	0.122	0.142	0.0814

$$I_{nl}^{0}(\zeta)=\int_{0}^{\infty} e^{-\xi/2}\xi^{l+3/2}L_{n+l}^{2l+1}(\xi)J_{l+1/2}(\zeta\xi)d\xi. \tag{33.5}$$

Thus elementary integration is also possible as in (32.3). Final expressions are shown in Table IX.

§ 34. *Numerical results.* We can now calculate the cross-section Q_N, integrating $I_N(\theta)$ over all directions, which, in turn, is given by f_N and g_N in (29.4) and (29.5), and by k_N in (27.5).

Rough calculations were performed and reproduced in Table X. As for the absolute values, only their orders must be consulted, but their relative values for different states N may be rather accurate.

In the table, cross-section $Q_{nl}(v)$ is given in units of πa_0^2, and v in 1000 km. per sec. We have also inserted recombination probabilities, which have been taken from Stückelberg and Morse's paper[1]. There, a quantity C_{nl} is defined by the formula of cross-section : $q_{nl}=C_{nl}\cdot 10^{-20}/V$, V being the electron velocity in volts.

Summary

It has been pointed out that the effect of collisional excitation of normal hydrogen atoms cannot be disregarded in the emission mechanism of the Balmer lines in the planetary nebulae; that it accounts for the high decrement of Balmer emission and that it gives at the same time, a definite idea of the electron temperature.

(I) 1) In §§ 3–5, the current formula of ionization is modified in order to include the collision effect; the modified formula changes slightly the degree of ionization in the nebular conditions (§ 6).

2) In connection with this, we have discussed the velocity distribution of the electrons in details, and concluded that it must obey the Maxwellian law, at least in close approximation (§ 7).

3) Cillié's equation has been modified to include the effect of collisional excitation in §§ 8–11.

(II) 4) Using the formulae above obtained, Balmer decrements have been derived for model planetary nebulae, which are chosen so as to correspond to Berman's classification. Theoretical decrement changes from 2.9 to 4.3 with electron temperature in agreement with observations (§§ 13 & 14).

5) Balmer decrement is sensitively connected with the electron temperature, so that the observed decrement fixes the latter. It falls between 10,000°–25,000° for most nebulae (§ 14).

1. Stückelberg and Morse, *loc. cit.*

(III) In connection with Page's observations of the continuous Balmer emission, detailed discussions have been made about N. G. C. 7662, especially about various questions which arose about it.

6) Discrete Balmer emission is a mixture of capture and collision spectra, while the continuous Balmer emission always remains in a pure capture spectrum. From this point of view, a correction has been made of the Zanstra and Stoy methods of determination of nuclear temperature, and applied to the same nebula N. G. C. 7662 (§§ 18 & 19). Thus it seems that the systematic discrepancy between Zanstra's estimate and Stoy's is improved.

7) That the Balmer continuous emission appears fainter than has been expected from the recombination theory has been also explained from the same point of view (§§ 16, 21 & 22).

8) Finally our starting assumption that the nebula is transparent to the subordinate lines has been criticized: (i) the metastability of $2S$ state loses its proper character, owing to the disturbances from the outer fields (§§ 24 & 25); (ii) the accumulation of atoms in $2P$ state due to the denseness of Lyman α radiation, seems not to be so large as to cause the self-reversal.

(IV) In Chap. IV, the excitation of the normal hydrogen atom by slow electron impact has been treated wave-mechanically; the method of approximation has been explained and the final formulae with their numerical results have been given.

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