On the Oscillation of Lake Water Generated by Wind Action

By **Toshichiro Takegami**

(Received October 10, 1938)

Abstract

In this paper the writer reports mathematical investigations for the oscillation of lake water produced by wind action. For form of basin he has considered the circle, the concentric circles, the sector and the fan-shape.

The method adopted may be described as follows:

1. The cylindrical co-ordinate is used.
2. It is assumed that the bottom of the lake is smooth.
3. It is assumed that the inertia terms and the horizontal convective terms are negligible.
4. Introducing the differential equation which determines the surface elevation, and is analogous to the differential equation for the oscillation of circular membrane in acoustics, he has solved it under the appropriate conditions. The equation obtained is

$$\frac{\partial^2\zeta}{\partial t^2}=\frac{\partial}{\partial r}\left(c^2\frac{\partial\zeta}{\partial r}\right)+\frac{c^2}{r}\frac{\partial\zeta}{\partial r}$$
$$+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(c^2\frac{\partial\zeta}{\partial\theta}\right)-\frac{1}{r}\frac{\partial}{\partial r}(rT_r)-\frac{1}{r}\frac{\partial T_\theta}{\partial\theta}$$

where ζ is the surface elevation from the undisturbed surface, r and θ are the radial and the angular co-ordinate, $c^2\equiv gh$ (h being depth of lake), and T_r and T_θ are respectively the radial and the tangential components of wind action.

Introduction

The oscillation of water in a circular or other geometrically regular basin generated by *the gradient of atmospheric pressure* has been investigated by many authors. Practically the oscillation in a basin is mainly generated by *wind action*, but except in a rectangular basin, no investigation of the osillation of water *generated by wind action* in a basin has been reported. In this paper, assuming that the bottom of a lake is smooth, and that the inertia terms and the horizontal convective terms in the equation of motion may be neglected, we shall introduce the differential equation which determines the oscillations generated by wind action when the form of basin is the circle, the concentric circles, the sector, or the fan-shape, and solve it under the suitable conditions.

The Fundamental Equation

Using the cylindrical co-ordinate the equation of motion, the equation of continuity and the boundary conditions at the surface and the bottom are expressed by

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}$$
$$=\nu\left(\frac{\partial^2 v_r}{\partial z^2}+\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$
$$-g\frac{\partial \zeta}{\partial r} \qquad \text{.........(1)},$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}$$
$$=\nu\left(\frac{\partial^2 v_\theta}{\partial z^2}+\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$
$$-g\frac{\partial \zeta}{r\partial \theta} \qquad \text{.........(2)},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \qquad \text{......(3)},$$

and

$$\left.\begin{aligned}&\mu\frac{\partial v_r}{\partial z}=T_r \quad \mu\frac{\partial v_\theta}{\partial z}=T_\theta \quad \text{at } z=0\\ &\mu\frac{\partial v_r}{\partial z}=0 \quad \mu\frac{\partial v_\theta}{\partial z}=0 \quad \text{at } z=-h\end{aligned}\right\} \qquad \text{......(4)}.$$

Here T_r and T_θ are respectively the radial and the tangential component of wind action, and the other notations have the usual meanings, and the vertical motion is only statically and not explicitly concerned.

If, in (1) and (2), the inertia terms $v_r\frac{\partial v_r}{\partial r}$, $v_r\frac{\partial v_\theta}{\partial r}$ etc. and the horizontal convective terms $\frac{\partial^2 v_r}{\partial r^2}$, $\frac{1}{r}\frac{\partial v_r}{\partial r}$ etc. may be neglected (especially, in the circular or the sectorial basin, the neglect of the terms multiplied by $\frac{1}{r}$ and $\frac{1}{r^2}$ seems quite arbitrary), then (1) and (2) become

$$\frac{\partial v_r\rho}{\partial t}=\nu\frac{\partial^2 v_r\rho}{\partial z^2}-g\frac{\partial \zeta\rho}{\partial r} \qquad \text{..............(1')},$$

$$\frac{\partial v_\theta\rho}{\partial t}=\nu\frac{\partial^2 v_\theta\rho}{\partial z^2}-g\frac{\partial \zeta\rho}{r\partial \theta} \qquad \text{..............(2')}.$$

Now integrate (1'), (2') and (3) with respect to z from $-h$ to 0, and take into account (4), then we have

$$\frac{\partial S_r}{\partial t}=rT_r-rgh\frac{\partial \zeta\rho}{\partial r} \qquad \dots\dots\dots\dots(1''),$$

$$\frac{\partial S_\theta}{\partial t}=T_\theta-gh\frac{\partial \zeta\rho}{r\partial \theta} \qquad \dots\dots\dots\dots(2''),$$

$$\frac{\partial S_r}{\partial r}+\frac{\partial S_\theta}{\partial \theta}=-r\rho\frac{\partial \zeta}{\partial t} \qquad \dots\dots\dots\dots(3'),$$

where $S_r=\int_{-h}^{0}\rho r v_r dz$ and $S_\theta=\int_{-h}^{0}\rho v_\theta dz.$

By eliminating S_r and S_θ from $(1'')$, $(2'')$ and $(3')$ we get

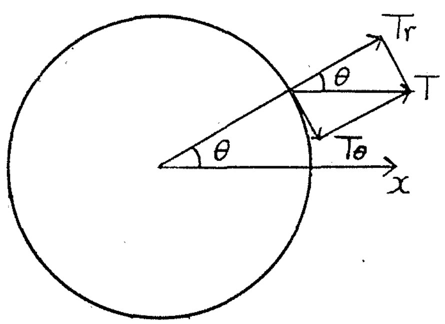
$$\frac{\partial^2\zeta}{\partial t^2}=\frac{\partial}{\partial r}\left(c^2\frac{\partial \zeta}{\partial r}\right)+\frac{c^2}{r}\frac{\partial \zeta}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(c^2\frac{\partial \zeta}{\partial \theta}\right)$$
$$-\frac{1}{r}\frac{\partial}{\partial r}(rT_r)-\frac{1}{r}\frac{\partial T_\theta}{\partial \theta} \qquad \dots\dots(5),$$

where $c^2\equiv gh$ and $\zeta\equiv\rho\zeta$. This is the required differential equation in the cylindrical co-ordinate which determines the surface elevation ζ. Thus the solution of this equation, under suitable conditions, will give the rising state of the oscillation of water when the forms of basin are respectively the circle, the concentric circles, the sector and the fan-shape. In the following article we shall give the solutions in a simple case when the wind action is uniform over the whole surface of basin and also the depth h is uniform.

1. Circular basin

As an example, we shall consider a simple case in which a uniform wind, its direction being x positive as seen in Fig. 1., suddenly blows over the surface of a circular basin (its radius being a). Now in order to solve (5) under the following initial and boundary conditions, i. e.,

Fig. 1.



$$\zeta=0,\ \frac{\partial \zeta}{\partial t}=0 \quad \text{when } t=0,$$

$$c^2\frac{\partial \zeta}{\partial r}=T_r=T\cos\theta$$

at $r=a$,

we shall separate ζ into a stationary part ζ_1 and a varying part $-\zeta_2$ which satisfy $\zeta=\zeta_1-\zeta_2$ and respectively the following two sets of equations and conditions (A) and (B)

$$
\text{(A)}\left\{\begin{array}{ll}
c^2\left\{r^2\dfrac{\partial^2\zeta_1}{\partial r^2}+r\dfrac{\partial\zeta_1}{\partial r}+\dfrac{\partial^2\zeta_1}{\partial\theta^2}\right\}=r\dfrac{\partial}{\partial r}(rT_r)-r\dfrac{\partial T_\theta}{\partial\theta} & \\
\qquad =r\dfrac{\partial}{\partial r}(rT\cos\theta)-r\dfrac{\partial}{\partial\theta}(T\sin\theta)=0, & \\
c^2\dfrac{\partial\zeta_1}{\partial r}=T\cos\theta & \text{at } r=a, \\
\zeta_1 \text{ is finite} & \text{at } r=0,
\end{array}\right.
$$

$$
\text{(B)}\left\{\begin{array}{ll}
\dfrac{\partial^2\zeta_2}{\partial t^2}=c^2\left\{\dfrac{\partial^2\zeta_2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\zeta_2}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\zeta_2}{\partial\theta^2}\right\}, & \\
\zeta_2=\zeta_1 \text{ and } \dfrac{\partial\zeta_2}{\partial t}=0 & \text{when } t=0, \\
c^2\dfrac{\partial\zeta_2}{\partial r}=0 & \text{at } r=a, \\
\zeta_2 \text{ is finite} & \text{at } r=0.
\end{array}\right.
$$

The general solution of (A) is

$$\zeta_1=\left(Ar+\frac{B}{r}\right)\cos\theta+C,$$

and the integration constant A, B and C in the right can be determined from the boundary conditions and the invariableness of water quantity in the basin $\left(\int_0^{2\pi}\int_0^a\zeta_1 r\,dr\,d\theta=0\right)$. Thus we have

$$\zeta_1=\frac{T}{c^2}r\cos\theta.$$

And the solution of (B), which satisfies the conditions, is

$$\zeta_2=\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}\left\{A_{ns}\cos n\theta+B_{ns}\sin n\theta\right\}J_n\left(\lambda_{ns}\frac{r}{a}\right)\cos\left(\lambda_{ns}\frac{ct}{a}\right)$$

where J_n (n being a positive integer) is the Bessel function of the first kind, λ_{ns} are the roots of $J'_n(\lambda)=0$ which has the infinite number of real roots, and

$$A_{ns}=\frac{1}{\pi a^2}\frac{\varepsilon_n\lambda^2_{ns}}{(\lambda^2_{ns}-n^2)J_n^2(\lambda_{ns})}\int_0^a\int_0^{2\pi}\zeta_1(\alpha,\varphi)J_n\left(\lambda_{ns}\frac{\alpha}{a}\right)\cos n\varphi\,\alpha\,d\alpha\,d\varphi,$$

$$B_{ns}=\frac{1}{\pi a^2}\frac{\varepsilon_n\lambda^2_{ns}}{(\lambda^2_{ns}-n^2)J_n^2(\lambda_{ns})}\int_0^a\int_0^{2\pi}\zeta_1(\alpha,\varphi)J_n\left(\lambda_{ns}\frac{\alpha}{a}\right)\sin n\varphi\,\alpha\,d\alpha\,d\varphi,$$

$$\varepsilon_0=1,\quad \varepsilon_1=\varepsilon_2=\cdots\cdots=2.$$

Since, in our case, T is independent of t and x, A_{ns} and B_{ns} reduce to very simple forms, i. e.,

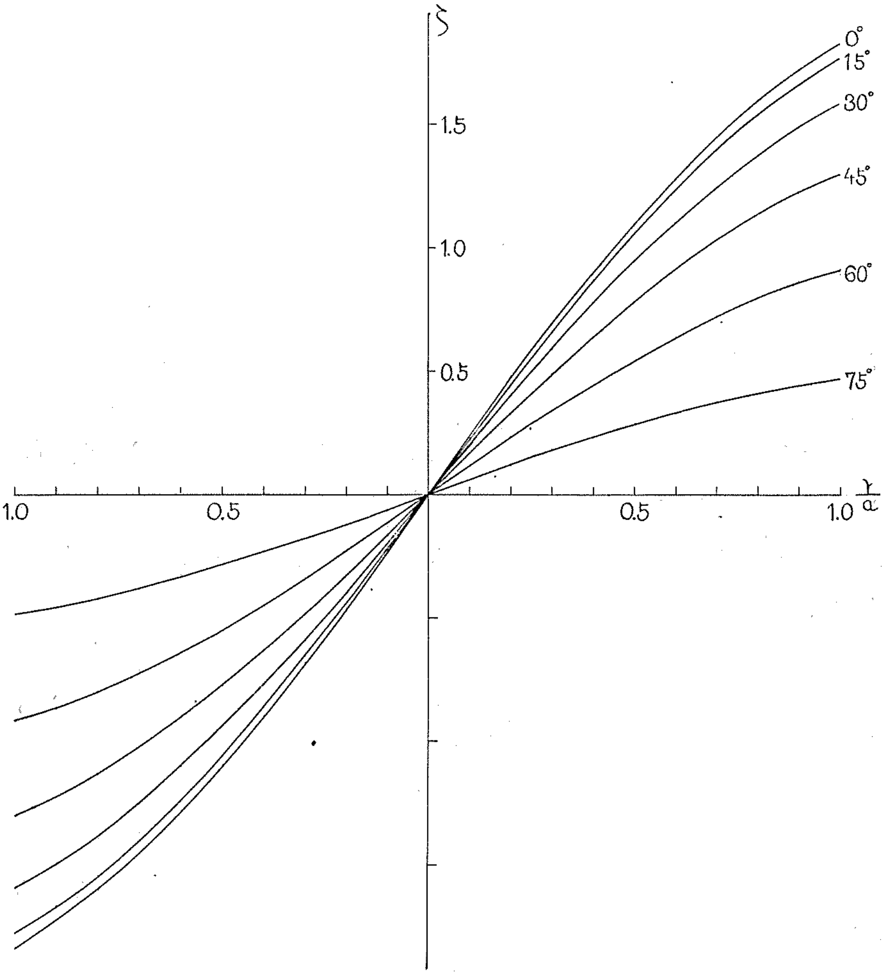
$$A_{ns}=0 \quad \text{for} \quad n\neq 1, \quad B_{ns}=0 \quad \text{for all values of } n,$$

and
$$A_{1s}=\frac{2T}{c^2}\frac{a}{\lambda_{1s}(\lambda_{1s}^2-1^2)J_1^2(\lambda_{1s})}\int_0^{\lambda_{1s}}\beta^2 J_1(\beta)d\beta$$
$$=\frac{2Ta}{c^2}\frac{1}{(\lambda_{1s}^2-1^2)J_1(\lambda_{1s})},\quad\left(\beta\text{ being }\frac{\lambda_{1s}a}{a}\right),$$

therefore
$$\zeta_2=\frac{2Ta}{c^2}\cos\theta\sum_{s=1}^{\infty}\frac{1}{(\lambda_{1s}^2-1^2)J_1(\lambda_{1s})}J_1\left(\lambda_{1s}\frac{r}{a}\right)\cos\left(\lambda_{1s}\frac{ct}{a}\right).$$

And our required solution is

Fig. 2. (a)



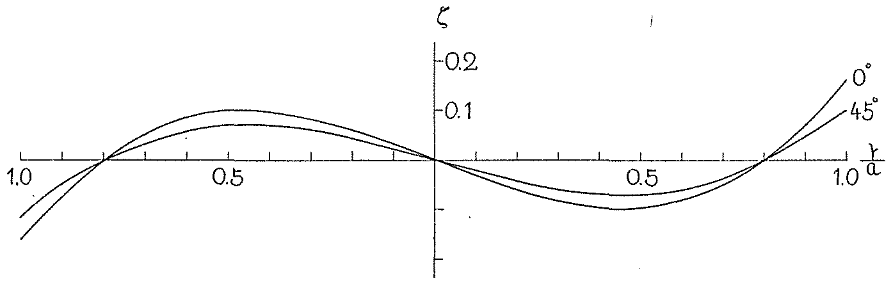
$$\zeta=\frac{T}{\rho gh}a\cos\theta\left\{\frac{r}{a}-2\sum_{s=1}^{\infty}\frac{1}{(\lambda_{1s}^2-1^2)J_1(\lambda_{1s})}J_1\left(\lambda_{1s}\frac{r}{a}\right)\cos\left(\lambda_{1s}\frac{ct}{a}\right)\right\},$$

or if T varies with the time, from Nomitsu's extension[1] of the Duhamel theorem, we get

$$\zeta=\frac{2c}{\rho gh}\sum_{s=1}^{\infty}\frac{\lambda_{1s}}{(\lambda_{1s}^2-1^2)J_1(\lambda_{1s})}\cos\theta J_1\left(\lambda_{1s}\frac{r}{a}\right)\int_0^t T(\tau)\sin\frac{\lambda_{1s}c}{a}(t-\tau)d\tau.$$

After carrying out the numerical calculations, we get Figs. 2 (a) and (b) which give respectively the sum of ζ_1 and the first harmonic

Fig. 2. (b)



of the elevation $-\zeta_2$, at two epochs $\frac{\pi}{1.841}\frac{a}{c}$ and $\frac{2\pi}{1.841}\frac{a}{c}$, in the seven vertical sections which pass the centre of basin and have an angular interval of 15° to each other, where 1.841 is a root λ_{11} of $J_1'(\lambda)=0$ and $\frac{Ta}{\rho gh}$ is taken as the unit of ζ. But since the elevations in the other half sections are symmetrical, in respect to the line $\theta=0$, to these figures, they need not be given.

2. Concentric Circular Basin

In this case the sets of equation and conditions corresponding to (A) and (B) in the previous article are

$$(\mathrm{A}')\begin{cases} r^2\frac{\partial^2\zeta_1}{\partial r^2}+r\frac{\partial\zeta_1}{\partial r}+\frac{\partial^2\zeta_1}{\partial\theta^2}=0 \\ c^2\frac{\partial\zeta_1}{\partial r}=T\cos\theta \quad \text{at} \quad r=a \quad \text{and} \quad r=b, \end{cases}$$

1. Proc. Imp. Acad. Tokyo, **11**, 359 (1935).

$$(\mathrm{B}')\begin{cases} \dfrac{\partial^2\zeta_2}{\partial t^2}=c^2\left\{\dfrac{\partial^2\zeta_2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\zeta_2}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\zeta_2}{\partial\theta^2}\right\} & \\ \zeta_2=\zeta_1 \text{ and } \dfrac{\partial\zeta_2}{\partial t}=0 & \text{when } t=0 \\ c^2\dfrac{\partial\zeta_2}{\partial r}=0 & \text{at } r=a \text{ and } r=b, \end{cases}$$

where a and b are respectively the inner and the outer radius of the concentric circle.

The solution of (A′) is obviously

$$\zeta_1=\frac{T}{c^2}r\cos\theta.$$

And the solution of (B′), after some calculation, is represented by

$$\zeta_2=\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}\left\{A_{ns}\cos n\theta+B_{ns}\sin n\theta\right\}K_n\left(\lambda_{ns}\frac{r}{a}\right)\cos\left(\lambda_{ns}\frac{ct}{a}\right),$$

where $$K_n\left(\lambda_{ns}\frac{r}{a}\right)=\frac{J_n\left(\lambda_{ns}\frac{r}{a}\right)}{J_n'(\lambda_{ns})}-\frac{Y_n\left(\lambda_{ns}\frac{r}{a}\right)}{Y_n'(\lambda_{ns})},$$

Y_n (n being a positive integer) is the Bessel function of the second kind, λ_{ns}'s are the roots of $J_n'\left(\lambda\frac{b}{a}\right)Y_n'(\lambda)-J_n'(\lambda)Y_n'\left(\lambda\frac{b}{a}\right)=0$,

$$A_{ns}=\frac{1}{\pi a^2}\cdot\frac{\varepsilon_n\lambda^2_{ns}}{(\lambda^2_{ns}-n^2)K_n^2(\lambda_{ns})-\left(\lambda^2_{ns}\frac{b^2}{a^2}-n^2\right)K_n^2\left(\lambda_{ns}\frac{b}{a}\right)}$$
$$\times\int_b^a\int_0^{2\pi}\zeta_1(\alpha,\varphi)K_n\left(\lambda_{ns}\frac{\alpha}{a}\right)\cos n\varphi\,\alpha\,d\alpha\,d\varphi,$$

$$B_{ns}=\frac{1}{\pi a^2}\cdot\frac{\varepsilon_n\lambda^2_{ns}}{(\lambda^2_{ns}-n^2)K_n^2(\lambda_{ns})-\left(\lambda^2_{ns}\frac{b^2}{a^2}-n^2\right)K_n^2\left(\lambda_{ns}\frac{b}{a}\right)}$$
$$\times\int_b^a\int_0^{2\pi}\zeta_1(\alpha,\varphi)K_n\left(\lambda_{ns}\frac{\alpha}{a}\right)\sin n\varphi\,\alpha\,d\alpha\,d\varphi,$$

$$\varepsilon_0=1,\ \varepsilon_1=\varepsilon_2=\ldots\ldots\ldots\ldots\ldots=2.$$

But since in our case $\zeta_1=\dfrac{T}{c^2}\alpha\cos\varphi$,

$$A_{ns}=0 \text{ for } n\neq 1 \text{ and } B_{ns}=0 \text{ for all values of } n,$$

and by using the recurring formulae of Bessel function, A_{1s} may be reduced to the following simple form:

$$A_{1s}=\frac{1}{a^2}\frac{2\lambda_{1s}}{(\lambda_{1s}^2-1^2)K_1^2(\lambda_{1s})-\left(\lambda_{1s}^2\frac{b^2}{a^2}-1^2\right)K_1^2\left(\lambda_{1s}\frac{b}{a}\right)}\int_b^a\frac{T}{c^2}\alpha^2K_1\left(\lambda_{1s}\frac{\alpha}{a}\right)d\alpha$$

$$=\frac{Ta}{c^2}\frac{2\left\{K_1(\lambda_{1s})-\frac{b}{a}K_1\left(\lambda_{1s}\frac{b}{a}\right)\right\}}{(\lambda_{1s}^2-1^2)K_1^2(\lambda_{1s})-\left(\lambda_{1s}^2\frac{b^2}{a^2}-1^2\right)K_1^2\left(\lambda_{1s}\frac{b}{a}\right)},$$

therefore we have

$$\zeta_2=\sum_{s=1}^{\infty}A_{1s}\cos\theta K_1\left(\lambda_{1s}\frac{r}{a}\right)\cos\left(\lambda_{1s}\frac{ct}{a}\right).$$

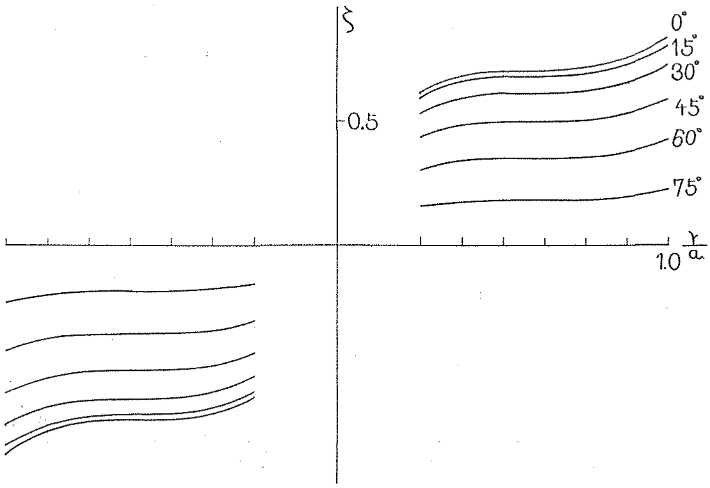
Thus our required solution is

$$\zeta=\frac{Ta}{\rho gh}\cos\theta\left\{\frac{r}{a}-2\sum_{s=1}^{\infty}\frac{K_1(\lambda_{1s})-\frac{b}{a}K_1\left(\lambda_{1s}\frac{b}{a}\right)}{(\lambda_{1s}^2-1^2)K_1^2(\lambda_{1s})-\left(\lambda_{1s}^2\frac{b^2}{a^2}-1^2\right)K_1^2\left(\lambda_{1s}\frac{b}{a}\right)}\right.$$
$$\left.\times K_1\left(\lambda_{1s}\frac{r}{a}\right)\cos\left(\lambda_{1s}\frac{ct}{a}\right)\right\},$$

or if T varies with the time, from Nomitsu's extension of the Duhamel theorem, we have

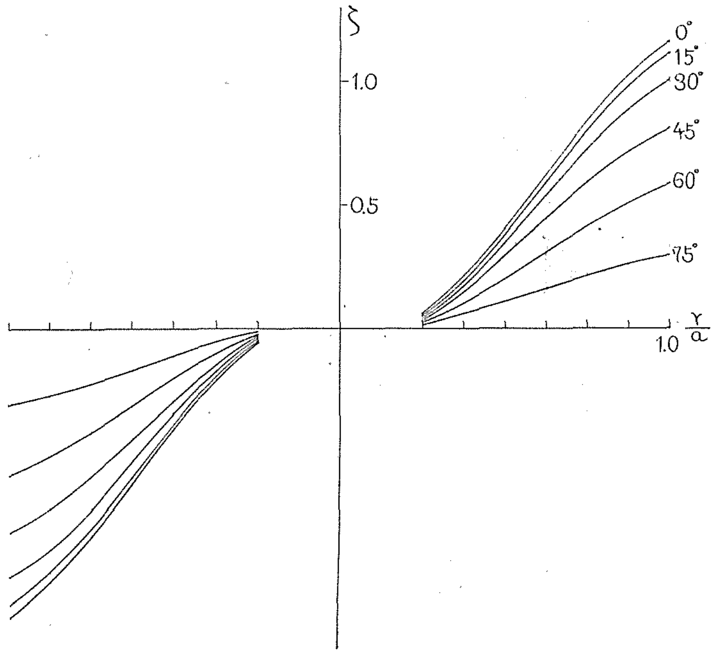
$$\zeta=\frac{2c}{\rho gh}\cos\theta\sum_{s=1}^{\infty}\frac{\left\{K_1(\lambda_{1s})-\frac{b}{a}K_1\left(\lambda_{1s}\frac{b}{a}\right)\right\}\lambda_{1s}}{(\lambda_{1s}^2-1^2)K_1^2(\lambda_{1s})-\left(\lambda_{1s}^2\frac{b^2}{a^2}-1^2\right)K_1^2\left(\lambda_{1s}\frac{b}{a}\right)}$$
$$\times K_1\left(\lambda_{1s}\frac{r}{a}\right)\int_0^t T(\tau)\sin\frac{\lambda_{1s}c}{a}(t-\tau)d\tau.$$

Fig. 3. (a)



As an example, we consider a case of $\frac{b}{a}=3$, and after carrying out the numerical calculations, we get Figs. 3 (a) and (b) which give respectively the sum of ζ_1 and the first harmonic of the elevation $-\zeta_2$, at two epochs $\frac{\pi}{5.205}\frac{a}{c}$ and $\frac{2\pi}{5.205}\frac{a}{c}$, in the seven vertical sections which pass the centre of the basin and have an angular interval of 15° to each other, where 5.205 is a root λ_{11} of $J_1'\left(\lambda\frac{b}{a}\right)Y_1'(\lambda)-J_1'(\lambda)\times Y_1'\left(\lambda\frac{b}{a}\right)=0$, and $\frac{Ta}{\rho gh}$ is taken as the unit of ζ. But since the

Fig. 3. (b)



elevations in the other half sections are in symmetry, in respect to the line $\theta=0$, to these figures, they need not be given.

3. Sectorial Basin

Consider the sectorial basin surrounded by two lines $\theta=0$, $\theta=\alpha$ and an arc (its radius being a), and the direction of wind making an angle θ_0 with the line $\theta=0$. Then the sets of equations and conditions are

$$
(\mathrm{A}'')\begin{cases} r^2\dfrac{\partial^2\zeta_1}{\partial r^2}+r\dfrac{\partial\zeta_1}{\partial r}+\dfrac{\partial^2\zeta_1}{\partial\theta^2}=0 & \\ c^2\dfrac{\partial\zeta_1}{\partial r}=T\cos(\theta-\theta_0) & \text{at } r=a \\ \zeta_1 \text{ is finite} & \text{at } r=0 \\ c^2\dfrac{\partial\zeta_1}{r\partial\theta}=T\sin\theta_0 & \text{at } \theta=0 \\ c^2\dfrac{\partial\zeta_1}{r\partial\theta}=-T\sin(\alpha-\theta_0) & \text{at } \theta=\alpha, \end{cases}
$$

$$
(\mathrm{B}'')\begin{cases} \dfrac{\partial^2\zeta_2}{\partial t^2}=c^2\left\{\dfrac{\partial^2\zeta_2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\zeta_2}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\zeta_2}{\partial\theta^2}\right\} & \\ \zeta_2=\zeta_1 \text{ and } \dfrac{\partial\zeta_2}{\partial t}=0 & \text{when } t=0 \\ c^2\dfrac{\partial\zeta_2}{\partial r}=0 & \text{at } r=a \\ \zeta_2 \text{ is finite} & \text{at } r=0 \\ c^2\dfrac{\partial\zeta_2}{r\partial\theta}=0 & \text{at } \theta=0 \text{ and } \theta=\alpha. \end{cases}
$$

The solution of (A″) is

$$
\zeta_1=\frac{Ta}{c^2}\left\{\frac{r}{a}\cos(\theta-\theta_0)-\frac{2}{3}\,\frac{\sin(\alpha-\theta_0)+\sin\theta_0}{\alpha}\right\},
$$

and the solution of (B″) is

$$
\zeta_2=\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}A_{ns}\cos n\theta J_n\left(\lambda_{ns}\frac{r}{a}\right)\cos\left(\lambda_{ns}\frac{ct}{a}\right),
$$

where $n\equiv\dfrac{m\pi}{\alpha}$, λ_{ns}'s are the roots of $J_n'(\lambda)=0$ and

$$
A_{ns}=\frac{2}{\alpha a^2}\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda_{ns}^2}{(\lambda_{ns}^2-n^2)J_n^2(\lambda_{ns})}\int_0^a\!\!\int_0^\alpha \zeta_1(\beta,\varphi)J_n\left(\lambda_{ns}\frac{\beta}{a}\right)\cos n\varphi\,\beta\,d\beta\,d\varphi.
$$

Thus the required solution is given by subtracting the above ζ_2 from ζ_1.

As an example, we took up a case where $\theta_0=0$ and $\alpha=\dfrac{2\pi}{3}$, and carried out the numerical calculation for a term of $m=1$ and $s=1$. Then this term becomes

$$
[\zeta_2]_{m=1,\,s=1}=0.660\frac{Ta}{\rho gh}\,\frac{J_{\frac{3}{2}}\left(\lambda_1\frac{r}{a}\right)\cos\frac{3}{2}\theta\,\cos\left(\lambda_1\frac{ct}{a}\right)}{\left(\lambda_1^2-\left(\frac{3}{2}\right)^2\right)J_{\frac{3}{2}}^2(\lambda_1)}\left\{\frac{5}{4\lambda_1}\int_0^{\lambda_1}\sqrt{\frac{2}{\pi\gamma}}\cos\gamma\,d\gamma\right.
$$

$$-\frac{3\sqrt{3}}{4\pi}\int_0^{\lambda_1}\sqrt{\frac{2}{\pi\gamma}}\sin\gamma d\gamma-\frac{5}{2}\frac{\cos\lambda_1}{\sqrt{\lambda_1}}+\frac{\sqrt{3}-2\pi}{2\pi}\sqrt{\lambda_1}\sin\lambda_1\Bigg\},$$

$$\gamma\equiv\lambda_1\frac{\beta}{a}\quad\text{and}\quad\lambda_1=2.463\ \text{is a root of}\ J_{\frac{3}{2}}'(\lambda)=0.$$

And the definite integrals $\int_0^{\lambda_1}\sqrt{\frac{2}{\pi\gamma}}\cos\gamma d\gamma$ and $\int_0^{\lambda_1}\sqrt{\frac{2}{\pi\gamma}}\sin\gamma d\gamma$ in the above expression are respectively reduced to a well known rapidly convergent series

Fig. 4. (a)

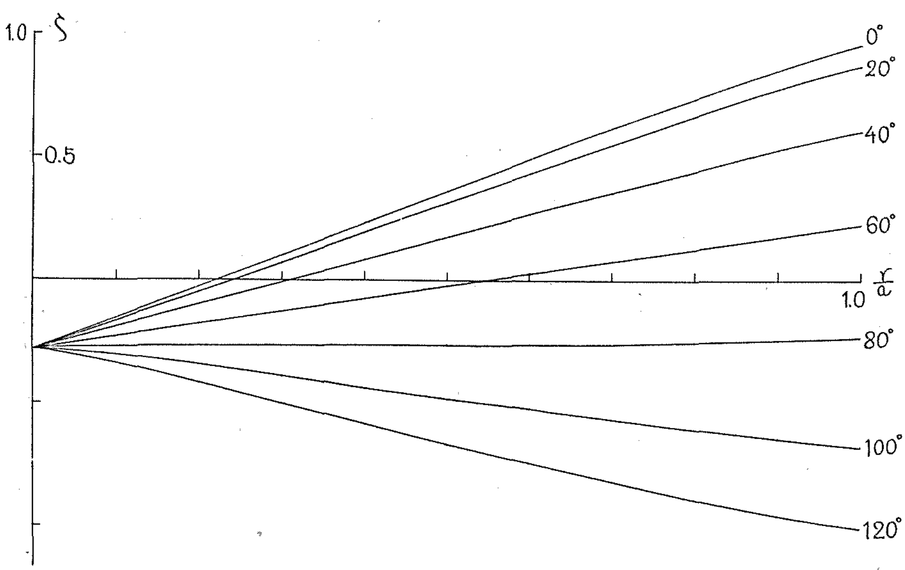
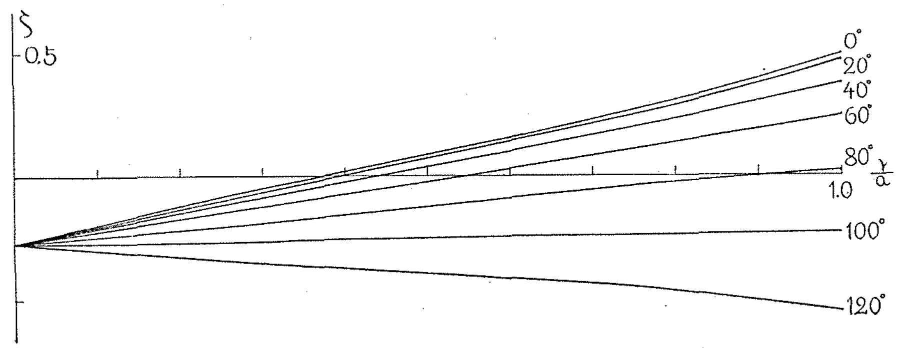


Fig. 4. (b)



$$2\left\{J_{\frac{1}{2}}^2(\lambda_1)+J_{\frac{5}{2}}^2(\lambda_1)+J_{\frac{9}{2}}^2(\lambda_1)+\cdots\cdots\right\} \text{ and } 2\left\{J_{\frac{3}{2}}^2(\lambda_1)+J_{\frac{7}{2}}^2(\lambda_1)+J_{\frac{11}{2}}^2(\lambda_1)+\cdots\cdots\right\}.$$

Calculating $\zeta_1-[\zeta_2]_{\substack{s=1\\m=1}}$, at two epochs $\frac{\pi}{2.463}\frac{a}{c}$ and $\frac{2\pi}{2.463}\frac{a}{c}$, by these formulae, Figs. 4 (a) and (b) are obtained.

When the wind action T varies with the time, the solution may be obtained by using, also, Nomitsu's extension of the Duhamel theorem, i. e.,

$$\zeta=\frac{2c}{\alpha a^3}\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda_{ns}^3 J_n\left(\lambda_{ns}\frac{r}{a}\right)\cos n\theta}{(\lambda_{ns}^2-n^2)J_n^2(\lambda_{ns})}\int_0^a\int_0^\alpha\int_0^t\zeta_1(\tau,\beta,\varphi)J_n\left(\lambda_{ns}\frac{\beta}{a}\right)$$
$$\times\cos n\varphi\sin\frac{\lambda_{ns}c}{a}(t-\tau)\beta d\beta d\varphi d\tau.$$

4. Fan-shaped Basin

In this case it is sufficient that the conditions—ζ_1 and ζ_2 are finite at $r=0$ in (A″) and (B″)—are respectively replaced with the conditions $c^2\frac{\partial\zeta_1}{\partial r}=T\cos(\theta-\theta_0)$ at $r=b$ and $c^2\frac{\partial\zeta_2}{\partial r}=0$ at $r=b$.

Then
$$\zeta_1=\frac{Ta}{c^2}\left\{\frac{r}{a}\cos(\theta-\theta_0)-\frac{2}{3}\frac{a^3-b^3}{a(a^2-b^2)}\cdot\frac{\sin(\alpha-\theta_0)+\sin\theta_0}{\alpha}\right\}$$

and
$$\zeta_2=\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}A_{ns}\cos n\theta L_n\left(\lambda_{ns}\frac{r}{a}\right)\cos\left(\lambda_{ns}\frac{ct}{a}\right),$$

where λ_{ns}'s are the roots of $J_n'(\lambda)J_{-n}'\left(\lambda\frac{b}{a}\right)-J_n'\left(\lambda\frac{b}{a}\right)J_{-n}'(\lambda)=0$, n is $\frac{m\pi}{\alpha}$,

$$A_{ns}=\frac{2}{\alpha a^2}\frac{\lambda_{ns}^2}{(\lambda_{ns}^2-n^2)L_n^2(\lambda_{ns})-\left(\lambda_{ns}^2\frac{b^2}{a^2}-n^2\right)L_n^2\left(\lambda_{ns}\frac{b}{a}\right)}$$
$$\times\int_b^a\int_0^\alpha\zeta_1(\beta,\varphi)L_n\left(\lambda_{ns}\frac{r}{a}\right)\cos n\varphi\,\beta d\beta\,d\varphi,$$

and
$$L_n=\frac{J_n\left(\lambda\frac{r}{a}\right)}{J_n'(\lambda)}-\frac{J_{-n}\left(\lambda\frac{r}{a}\right)}{J_{-n}'(\lambda)}.$$

Since the numerical calculations of these formulae are very tedious, we have given only the formal solution without the figures corresponding to those given in the preceding cases.

Now it must be noted that, if the equations of motion and of continuity are expressed by another curvilinear co-ordinate such as the elliptic cylinder or the parabolic cylinder etc., we may give, in a form similar to that of the present paper, the differential equation for ζ in another geometrically regular basin. Especially since, when the basin is elliptic, the solution of the differential equation may be applied to the generating stage of oscillation in Osaka bay, it will be of great interest. But since it is very tedious and difficult to obtain the solution, we shall leave the matter to expert mathematicians.

In conclusion the writer wishes to express his sincere thanks to Prof. T. Nomitsu for his kind advice and encouragement during the study.