On the Hydrogen Emission near the Limit of the Balmer Series in the Solar Chromosphere

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Introduction

§ 1. Recently the Balmer emission of the solar chromosphere has been discussed in detail by D. H. Menzel and G. G. Cillié from the standpoint of the capture theory[1]. The theory is in beautiful accordance with the observation so far as it is concerned with the lower members; but this is not the case near the series limit. In fact the theory demands a clear-cut discontinuity at the series limit, while observation shows that the discrete lines merge into the continuous one quite smoothly as the quantum number n increases. Speaking precisely, the theoretical intensity of the continuous emission must be about twice as strong as that of the discrete lines near the series limit, so that the observed populations of large quantum states (except those of extraordinarily large n) are—taking that of free electrons as the standard[2]—twice as large as would be expected from the capture theory, or it may rather be said that they behave as if they were in thermodynamic equilibrium. Thus it seems that the simple application of the capture theory fails to explain the facts observed.

§ 2. The deficiency of the population above mentioned would be explained in terms of line excitations due to the direct solar radiation as has been suggested by Menzel and Cillié[1]. But study of the nature of the problem seems to suggest that there is something about the behaviour of the hydrogen atom in general, too important to be dealt with merely by making some working hypothesis in order to remove the discordance between the theory and observation. The completely isolated atom is an ideal one, so that the atomic electron, which revolves around the nucleus under the control of its Coulomb force, becomes more easily distorted by the existence of the surrounding electrons and

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1. D. H. Menzel and G. G. Cillié, Ap. J. **85** (1937) 88.
2. As for the exact expression, cf. § 3.

atoms as its quantum orbit becomes higher[1]. Thus as will be seen in the present paper, the atomic electrons with a large quantum number show a sort of *affinity*[2] much more strongly to the continuous state than to the lower discrete ones, so that their populations show a tendency to be adjusted to the Maxwell-Boltzmann's law, which always holds for the free electrons in the stellar atmosphere even when the population in the lower levels deviates widely from this law.

I. Observed Balmer Emission near the Series Limit in the Solar Chromosphere

§ 3. D. H. Menzel has measured the flash spectra taken by the expedition of the Lick observatory in August 1932[3]. They correspond to the observations at the height of 670, 900 and 1,500 km. of the chromosphere. The Balmer emissions beyond H_{31} merge into a continuous spectrum on the plates, grading into the true continuous Balmer emission without any noticeable discontinuity. From the relative intensities, both of discrete and of continuous emissions, we can derive the relative values of population, provided the transition probabilities A_{n2} are given: First, the population per c. c., N_n, in the discrete level n and the Balmer emission per c. c. per sec., E_n, associated with it, are related by the formula

$$E_n = N_n A_{n2} h\nu_{n2} . \qquad (3.1)$$

Second, as for the continuous states, the number of free electrons per c. c., N_e, is connected with the continuous Balmer emission per c. c. per sec. E_c per wave number σ beyond the series limit as follows[3]:

$$E_c d\sigma = N_e N_i \frac{C}{T_e^{3/2}} e^{-(\chi_2 - h\nu)/kT_e} d\sigma, \qquad (3.2)$$

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1. As to the disturbance of the electron in high levels by the surrounding atoms in the stellar atmosphere, W. H. Wright made some valuable suggestions about twenty years ago; cf. for example, *Nature*, **109** (1922) 810. In connection with this problem, the work of Prof. B. Arakatsu of the Institute for Physics of our University is very suggestive; he has taken by the ring discharge method in the laboratory a very beautiful photograph of the continuous hydrogen emission beyond about $n=7$; see, *B. Arakatsu, Memoirs Taihoku Imp. Univ.* **5** (1932) 1, 15. The aspect of it seems to be very similar to the continuous emission in the solar chromosphere. Prof. Arakatsu has recently put forward in a conversation with Prof. T. Araki the interesting opinion that the chromospheric emission should have some resemblance to the phenomenon in the ring discharge.

2. It will be defined quantitatively in § 10.

3. Menzel and Cillié, *loc. cit.*

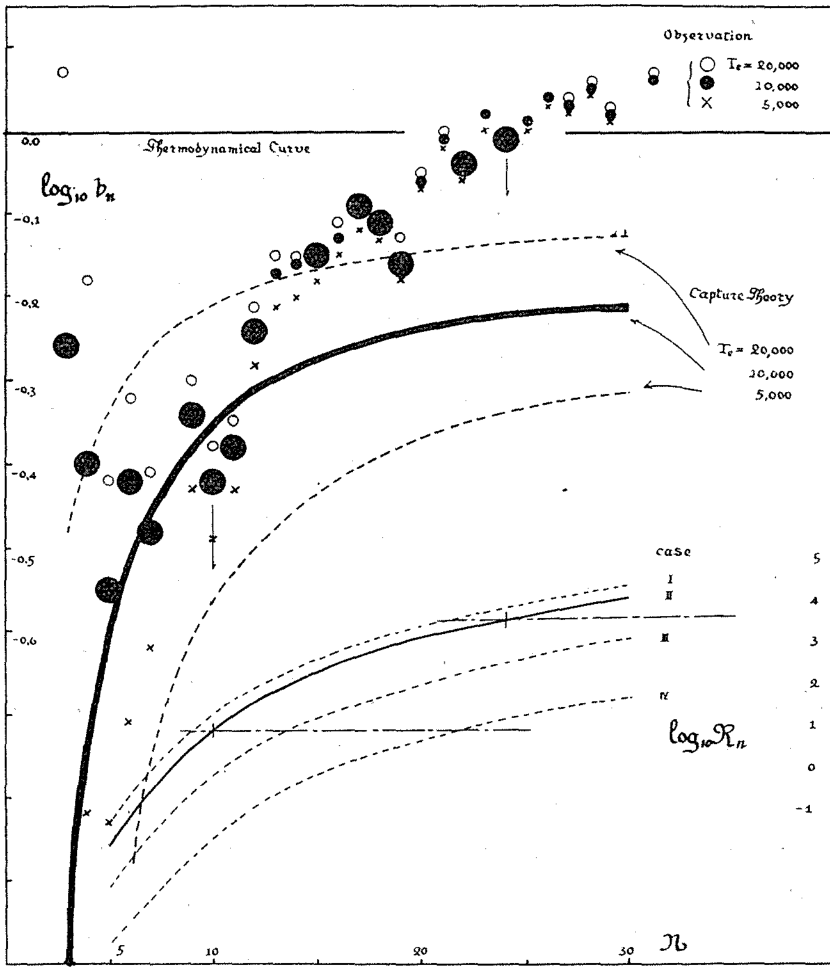
with
$$h\nu = hc\sigma = \frac{1}{2}mv^2 + \chi_2,$$
$$C \equiv \frac{2^6\pi^5\varepsilon^{10}}{(6\pi)^{3/2}m^{1/2}h^2c^2k^{3/2}},$$
where N_i means the number of protons per c.c. and χ_2 is the energy of ionization from the second state, v the velocity of free electron, T_e the electron temperature, and other notations are used as usual. Then we can derive the relative population N_n/N_eN_i by (3.1) and (3.2) from the value of E_n/E_c given by observation. Especially, if we take for E_c the intensity at the series limit, Ba_c, then (3.2) becomes
$$Ba_c d\sigma = N_eN_i\frac{C}{T_e^{3/2}}d\sigma, \tag{3.3}$$
owing to $\chi_2 = h\nu$. Thus we get finally
$$\frac{N_n}{N_eN_i} = \left(\frac{E_n}{Ba_c}\right)\frac{C}{T_e^{3/2}}\;\frac{1}{A_{n2}h\nu_{n2}}. \tag{3.4}$$

§ 4. To express the observed ratio N_n/N_eN_i, Menzel and Cillié have used another quantity[1]:
$$b_n \equiv \frac{N_n}{N_{n_{Te}}}, \tag{4.1}$$
replacing N_eN_i by an equivalent quantity $N_{n_{Te}}$, which means the population in state n in the thermal equilibrium of temperature T_e and which is connected to N_eN_i by Saha's formula:
$$\frac{N_eN_i}{N_{n_{Te}}} = \frac{(2\pi mkT_e)^{3/2}}{h^3}\cdot\frac{1}{n^2}e^{-\chi_n/kT_e}. \tag{4.2}$$
Inserting in (4.1) the expressions (3.4) and (4.2) for N_n and $N_{n_{Te}}$ respectively, we get
$$b_n = D\left(\frac{E_n}{Ba_c}\right)\frac{1}{n^2A_{n2}\sigma_{n2}}e^{-\chi_n/kT_e}, \tag{4.3}$$
$$D \equiv \frac{2^6\pi^5\varepsilon^{10}m}{3^{3/2}h^6c^3}.$$
These formulae are useful as expressing the degree to which the behaviour of hydrogen atoms deviates from the thermodynamic equilibrium[2] in the chromosphere.

Menzel's observation has been plotted in the figure by the black circles, and the unblended or slightly blended lines are emphasized by

1. Menzel and Cillié, *loc. cit.*

2. In this note, the word "*thermodynamic equilibrium*" is used as the synonym of *Maxwell-Boltzmann's distribution law*, and does not imply the Kirchhoft-Planck's condition for the interaction between matter and radiation.



larger circles. We have chosen for T_e, 10,000°, which corresponds to the value estimated by Menzel[1] from the intensity gradient of the continuous Balmer emission.

For $n > 31$ the emissions are apparently continuous on the plate. But in this region, the estimate of N_n from the measured intensity $Id\sigma$ has also been given by Menzel and Cillié[1], i. e.

$$b_n = \frac{I}{Bac}, \qquad \text{for large } n. \tag{4.4}$$

The observed intensity I is constant for large n and merges into

1. Menzel and Cillié, *loc. cit.*

the true continuous emission beyond the series limit[1]. Hence we have

$$I=B_{ac} \quad \text{or} \quad b_n=1 \quad . \tag{4.5}$$

Consequently in the figure, the sequence of black circles must be extended to run along the thermodynamical line $\log_{10}b_n=0.0$ with increasing n beyond 31.

§ 5. We can also derive the theoretical values of b_n, if we insert in E_n/B_{ac} of (4.3) its theoretical values. In the case of thermodynamic equilibrium, we have evidently $b_n=1$ independent of n, that is, all b_n's are on the horizontal line at $\log_{10}b_n=0.0$ in the figure.

Prediction of b_n according to the capture theory can be obtained by solving the cyclic equations of Cillié[2]; its values computed by J. G. Baker and D. H. Menzel[3] for $T_e=10{,}000°$ are reproduced by the heavy line in the figure. Those for $T_e=5{,}000°$ and $20{,}000°$ are also represented for the purpose of comparison.

§ 6. As is clear in the figure, observation agrees well with the capture theory for lower members, except the lowest two: $H\alpha$ and $H\beta$. It then deviates gradually from the theoretical curve with increasing n and attains finally the thermodynamic limit $\log_{10}b_n=0.0$ over $n \doteqdot 20$.

Such deficiency of theoretical population would be, as Menzel and Cillié (*loc. cit.*) have considered, supplied by the augment due to direct stimulation from the photospheric radiation. But it must be remembered in these lines of attack that the amount of supplement depends closely on a number of parameters, so that the agreement of the observed chromospheric population with the thermodynamic one seems to be merely an accident. Therefore, isn't it more natural to seek some other solution of the problem, one that implies a physical meaning in this agreement itself?

It may be said in this connection that it is not necessarily meant by $b_n=1$ the Kirchhoff-Planck's condition holds in the chromosphere, but merely that the chromospheric population is Boltzmannian; it is indeed possible that the population behaves like a capture population for the lower members, and like a thermodynamic for the higher ones.

II.

§ 7. The atomic electron moves in a quantized orbit around its nucleus controlled by the Coulomb force, the mean radius of the elec-

1. Menzel and Cillié, *loc. cit.*
2. G. G. Cillié, M. N. **92** (1932) 823.
3. J. G. Baker and D. H. Menzel, Ap. J. **88** (1938) 52.

tronic orbit a_n increasing rapidly in proportion to the square of quantum number n; i. e. for example, $a_1 \doteqdot 10^{-8}$ cm. and $a_{30} \doteqdot 10^{-5}$ cm. Therefore it rapidly gets rid of the control of the nuclear force as n increases and becomes more easily distorted by the outer field[1].

As to the continuous state, we have various observational evidence which indicates that the velocity distribution of the free electrons is Maxwellian, namely, that it obeys thermodynamic law, even when the populations for the discrete levels deviate strongly from Boltzmannian. This means that the mutual collisions between free electrons take place much more frequently than interactions between discrete and continuous states. These circumstances may be also understood theoretically; indeed, the cross-section for the mutual elastic collision of electrons which results in an exchange of their momenta is of the order $\gtrsim \pi a_1^2 \doteqdot 10^{-16}$ cm^2., while that for capture is much smaller, viz. $\sim 10^{-22}$ cm^2. and, since both processes are proportional to N_e^2 per c. c. in the case $N_e = N_i$, the difference of the cross-sections immediately gives their frequency-difference. Other processes, such as the ionization of the normal atom by the electron impact, contribute nothing; for the free electrons which have sufficient kinetic energy to ionize the atom, are too scarce, and, as for the ionization from the higher states, the populations are too small to bring out any effect. In any case it may be said that the free electrons show a stronger affinity for one another than for the bounded electrons so that they form a Maxwellian distribution.

The above considerations suggest that the atomic electron obeys atomic law, only in the domain where the nuclear force predominates, and it is easily adjusted to the thermodynamic law in the outer region, where the mutual interaction cancels the nuclear force. The border of these two regions depends on the external disturbances, above all on the electron density and does not always lie at $n \sim \infty$, although it is not in general sharply defined, but may consist of some breadth, wherein the electrons would show a mixed character.

In order to see the affinity of the bound electrons to the continuous domain, we will, as a criterion, compare the frequency of the collisional

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1. According to H. Bethe (Hb. d. Physik. XXIV/1. Kap. 3 Ziff. 3), the mean distance of the nearest electron or proton from the atom is given by $r = \frac{1}{2}\left(\frac{3}{4\pi}\frac{1}{2N_e}\right)^{1/3}$, so that, e. g. the orbit a_n where the nuclear Coulomb force surpasses 10 times that of the nearest electron or proton corresponds to $a_n \doteqdot 10^{-8} n^2 = \frac{1}{\sqrt{10}} \cdot \frac{1}{2}\left(\frac{3}{4\pi}\frac{1}{2N_e}\right)^{1/3}$. At the base of the chromosphere, $N_e \doteqdot 3 \times 10^{11}$ and hence we have $n \doteqdot 30$.

ionization from n-th state with the probability of the downward transition from the same level. Of course, this is inadequate for such criterion, unless the collisional ionization implies at the same time its inverse process, viz. the three body collision of two electrons and a proton, resulting in the capture of one of the former by the latter, because these two processes, acting simultaneously, are able to adjust the populations to the Maxwell-Boltzmann's distribution. In the present note, we shall not give any theoretical explanation of this point, but the adequateness of our choice for the criterion may be accepted in view of the fact that the collisional ionization and its inverse process are both considered as interactions between the loosely bound and free electrons in a weak field of nuclear force which exactly correspond to the mutual collisions in the case of free electrons.

§ 8. By the method of Born's approximation, H. Bethe[1] has carried out the computation of the cross-section for collisional ionization by the quantum mechanics. Born's approximation essentially restricts its application to the case where the kinetic energy of the orbital electron is much smaller than that of the colliding electron. This is, in fact, the case for the states of large n, since the square of the velocity of the orbital electron is given by

$$v_n^2 = 4.78 \times 10^{16} n^{-2}, \qquad (8.1)$$

while the mean thermal velocity corresponding to $T_e = 10{,}000^\circ$ is

$$\bar{v}^2 = 4.56 \times 10^{15}, \qquad (8.2)$$

in c. g. s. unit[2]. To simplify our order-estimation we further approximate the Bethe's formula to the classical one of J. J. Thomson[3]:

$$Q_n^{(i)}(v) = \frac{2\pi\varepsilon^4}{mv^2} \cdot \chi_n^{-1}, \qquad (8.3)$$

where χ_n means the ionization potential from state n and v the velocity of colliding electron. Inserting for v the mean thermal velocity for $T_e = 10{,}000^\circ$, (8.2), we get

$$Q_n^{(i)}(\bar{v}) = 3.68 \times 10^{-15} n^2 \text{ cm}^2., \text{ for large } n, \qquad (8.4)$$

or, in terms of πa_1^2, it becomes

$$Q_n^{(i)}(\bar{v}) = 42\, \pi a_1^2 \cdot n^2. \qquad (8.5)$$

§ 9. i) *Frequency of ionization rom n-th state*, S_n, is given approximately by

$$S_n = Q_n^{(i)}(\bar{v}) \cdot \bar{v} \cdot N_e, \qquad (9.1)$$

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1. H. Bethe, Ann. d. Physik. **5** (1930) 325.
2. We use the c. g. s. unit throughout this paper, unless otherwise stated.
3. J. J. Thomson, Phil. Mag. **23** (1912) 449.

where $Q_n^{(c)}(\bar{v})$ the cross-section given by (8.4). As for N_e we employ the current formula[1]:

$$N_e(z)=N_e(0)e^{-\frac{1}{2}\alpha z} \tag{9.2}$$

with

$$N_e(0)=3\times 10^{11}\ \text{cm.}^{-3},$$
$$\alpha=1.54\times 10^{-8}\ \text{cm.}^{-1}, \tag{9.3}$$

where z is the height above the base of the chromosphere. Strictly speaking, $N_e(z)$ in (9.1) should be reduced to that fraction of it which has kinetic energy sufficient to ionize the atom in state n: viz. $\frac{1}{2}mv^2>\chi_n$, but this correction can be neglected for large n in practice, since χ_n is sufficiently small; moreover, as we receive, in the case of eclipse observation, the light integrated throughout the line of sight, we should also replace $N_e(z)$ by some mean value of it. However, this correction is small, so we shall neglect it. Thus we find, inserting (9.2) and (8.4) in (9.1)

$$S_n=7.45\times 10^4 e^{-\frac{1}{2}\alpha z} n^2, \tag{9.4}$$

S_n is a function of z; in order to see this dependence our further discussion will be referred to four places at different height above the base of the chromosphere such as in Table I.

Table I

case	I	II	III	IV
z (cm.)	0	10^8	4×10^8	8×10^8

Case II nearly corresponds to the height of Menzel's measurement (cf. § 3) and case IV to the top of the hydrogen chromosphere.

ii) *Mean lifetime*, T_n, (reciprocal of transition probability A_n), in state n is fairly proportional to $n^{9/2}$:[2]

$$T_n\propto n^{9/2}; \tag{9.5}$$

it amounts to 8.8×10^{-8} sec. for $n=5$, hence we get for any n

$$T_n=8.8\times 10^{-8}\left(\frac{n}{5}\right)^{9/2}\text{sec.} \tag{9.6}$$

The numerical values of T_n, $Q_n^{(c)}(\bar{v})$ of (8.4) and S_n are given in Table II for each case of Table I. As is seen from the table, the atomic electron has an extraordinarily long lifetime for large n; as n increases,

1. Cf. A. Unsöld, Physik der Sternatmosphären, Kap. XVII § 98.
2. H. Bethe, Hb. d. Physik. XXIV/1. Kap. 3 Ziff. 42.

T_n gradually approaches in its order to that of the metastable state of nebulium element.

Table II Logarithmic Values of T_n, $Q_n^{(i)}(\bar{v})$ and S_n

n	5	10	15	20	25	30	40
$\log_{10} T_n$	−7.06	−5.70	−4.91	−4.35	−3.91	−3.66	−2.99
$\log_{10} Q_n^{(i)}$	−13.04	−12.43	−12.08	−11.83	−11.64	−11.48	−11.23
$\log_{10} S_n$ I	6.28	6.87	7.23	7.48	7.60	7.83	8.04
II	5.94	6.55	6.89	7.14	7.34	7.49	7.75
III	4.94	5.55	5.89	6.14	6.34	6.49	6.75
IV	3.62	4.22	4.58	4.83	5.02	5.18	5.43

§ 10. In order to see the *affinity* of state n to the continuous region (cf. § 7) we define as a criterion for it the quantity R_n by

$$R_n \equiv \frac{S_n}{A_n} = S_n T_n. \qquad (10.1)$$

Inserting for S_n and T_n the expressions (9.4) and (9.6) we get for $\log_{10} R_n$ the equation:

$$\log_{10} R_n = -5.33 - Z + 6.5 \log_{10} n,$$
$$Z \equiv 3.34 \times 10^{-9} z. \qquad (10.2)$$

This relation is represented in the text-figure for each case of § 9. If $R_n = 1$, the chance of spontaneous downward transition is equal to that of the ionization; the quantum number n corresponding to such circumstances is obtained from (10.2): n=ca. 7, 8, 11 and 18 respectively for cases I, II, III, IV and the higher the state above these values of n, the more the population comes to be influenced by outer disturbance, approaching the range of statistics of free electrons, namely Maxwell-Boltzmann's law (cf. § 7)[1].

It would be interesting to make a definite formulation about this point from the statistical point of view. But we shall here prefer the reverse course, trying to obtain empirically by the aid of Menzel's data the relation between R_n and the "effective" sphere of nuclear force, viz. the domain in which the populations are effectively controlled by

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1. It must be remembered that the "influence" here considered means that on the population and not the distortion of the quantum state itself; these two kinds of distortion are not the same thing, though many investigators do not make such distinction: The order-estimation shows that the outer influence appears appreciable in the population much earlier than in the distortion of the state itself.

As for the latter effect, Stark-splitting of the state becomes to be of equal width with the energy difference with the neighbouring state for n=ca. 40 at the base of the chromosphere and for n=ca. 90 at the top of it; thus, beyond this limit states form a continuum.

the nuclear force. The observation shows that the chromospheric population in case II begins to deviate from the capture one at about $n=10$ and reaches the thermodynamic one at ca. $n=24$ (cf. text-figure), the former corresponding to

$$\log_{10} R_n = 0.83 \quad \text{or} \quad R_n = 6.8, \tag{10.3}$$

and the latter to

$$\log_{10} R_n = 3.3 \quad \text{or} \quad R_n = 2000. \tag{10.4}$$

They are both plausible figures. The uncertainty of R_n comes mainly from that of $N_e(z)$ which appears as a factor in S_n, and when only the order of $N_e(z)$ is considered, the same is true for R_n, too. On the other hand, trivial approximations made in § 9 have a tendency to give a larger value to R_n. Therefore we can say nothing about it but that $R_n \sim 1$ and $R_n \sim 1000$ respectively for the point of deviation n_c from the capture population and for the adjoining point n_t to the thermodynamic one.

§ 11. In short, we can say that the simple application of the capture theory begins to fail beyond n corresponding to $R_n \sim 1$, and for the state in which $R_n \sim 100$ its population roughly obeys Maxwell-Boltzmann's law and for $R_n \sim 1000$ the population is completely adjusted to this law. In this way the quantity R_n may be considered a criterion to determine the range of adaptability of the capture theory, which has been hitherto believed to be always applicable when the states are discrete.

The theoretical calculation made in this note may be verified observationally as follows: Assuming (10.3) and (10.4), from (10.2) we can express n_c and n_t as functions of the height Z:

$$\log_{10} n_c = \frac{1}{6.5}(6.16 + Z), \tag{11.1}$$

$$\log_{10} n_t = \frac{1}{6.5}(8.62 + Z). \tag{11.2}$$

Thus if our reasoning is right, the population must coincide with the prediction from the capture theory till to $n=14$ & 23 respectively for cases III and IV and then gradually deviate from that and reach the thermodynamic one at about $n=34$ & 55 respectively (compare with case II of text figure). It is desirable to have observation made with respect to these points.

It must be noted that the relations (11.1) and (11.2) are quite independent of the uncertainty of N_e, since they have been derived relative to case II, so that the above considerations may be verified without any ambiguity.

Summary

It has been shown that in the solar chromosphere the hydrogen atoms in high energy levels are strongly affected by the outer influence due to the existence of other atoms and electrons and that their populations follow Boltzmann's law as the free electrons do, so that the capture theory is adequate to determine the population only for the states of small quantum number, which are under the complete control of the nuclear force.

In order to characterize the outer influence, a criterion R_n has been defined as the ratio of the frequencies of collisional ionization S_n and of the spontaneous transition A_n in state n, and has been given as a function of electron density. With this criterion the chromospheric observation has been discussed. The results are as follows:

i) R_n varies considerably with height in the chromosphere.

ii) At the height of 1,000 km., where detailed observations have been made, our criterion seems to give the right order of figures.

It gives me pleasure to thank Prof. Dr. Toschima Araki for his earnest guidance and his interest in the progress of the present work, and to Dr. Michinori Kurihara for his helpful advice at all times. Finally I also express my deep indebtness to "Ishida Ikuëi-kwaï" for financial support.

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A Note added on 21 Dec. 1938. — In the course of printing this paper, I received a paper on the similar subject by Prof. A. Pannekock, which was published in M. N. **98** (1938) No. 9. p. 694. He treats mainly the distortion of energy levels of large quantum numbers and the effect of the overlapping of neighbouring discrete emissions. There he expresses an opinion that *since the stimulated transitions up and down between the highest levels, the ionized state included, are more important than the spontaneous transitions, the partition between these levels will probably be nearer to the thermodynamical equilibrium than to the case of a pure capture spectrum.* His idea is in accord with ours, except that the alteration of partition is considered there to be due to the stimulation by radiation.