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The PPP Approach to Robbins' Problem of Minimizing the Expected Rank

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1 Introduction

According to Bruss(2003), Robbins' problem can be described as follows: Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U[0,1]$ random variables, and let $\mathcal{F}_k=\sigma(X_1, X_2, \ldots, X_k)$, that is, the σ-algebra generated by $X_1, X_2, \ldots, X_k$. Further let R_k be the absolute rank of X_k among all $X_1, X_2, \ldots, X_n$, where ranks and observations are both defined in increasing order, i.e.,

$$R_k = R_k(n) = \sum_{i=1}^{n} \mathbf{I}(X_i \leq X_k).$$

Further let T_n be the set of all stopping rules adapted to $X_1, X_2, \ldots$, which means

$$T_n = \{\tau : \{t = k\} \in \mathcal{F}_k,\ \ 1 \leq k \leq n\}.$$

The problem is to find the value

$$V(n) = \inf_{\tau \in T_n} E[R_\tau],$$

including its asymptotic behavior as n tends to infinity, and also the stopping rule τ^* which achieves the value, that is,

$$\tau^* := \tau_n^* = \arg_{\tau \in T_n} \{\inf E[R_\tau]\}.$$

It is noted that we define the smaller observations to be the better ones(i.e., having smaller ranks), which is here more convenient. Why the above problem is called Robbins' problem is that, during the AMS Joint Summer Research Conference on Strategies for Sequential Search and Selection in Real Time (Amherst, June 1990), professor Herbert Robbins tried to get people interested in this problem.

In the classical secretary problem, solved by Lindley(1961), one is allowed to use rules that depend on the relative ranks of the observations (called the no-information problem) and the objective is to maximize the probability of selecting the observation of absolute rank 1(the best-choice problem). The full-information best-choice problem was solved by Gilbert and Mosteller(1966) and the no-information expected-rank problem was solved by Chow et al.(1964). Thus the Robbins' problem, i.e., the full-information expected-rank problem, would complete a two by two factorial design of secretary problems. To the best of my knowledge, there are three papers specifically on

Robbins' problem. These are Bruss and Ferguson(1993), (1996) and Assaf and Samuel-Cahn(1996) (abbreviated to AS).

This note, motivated by AS, has two objectives: One is to give Eq.(4.14) of AS again (referred to as Problem 1) and the other is to derive an explicit limiting expression (as $n \to \infty$) for the expected rank under the stopping rule considered in Remark 6.2 of AS (referred to as Problem 2) via the planar Poisson process approach.

2 PPP approach

Since X_i's i.i.d. uniform on [0,1]$\Rightarrow n(1-\max\{X_1,\dots,X_n\}) \to^D exp(1)$, it is clear that the desired setting is a so-called planar Poisson process, which will be abbreviated as PPP. This model has been widely used to study the limiting behavior of the full-information problems; see, e.g., Gnedin (1996), (2003) or Samuels (2004). Our preference is to have a Poisson process with unit rate on the space

$$\mathcal{T} \times \mathcal{Y} = [0,\ 1] \times [0, \infty).$$

This turns the problem upside down making Best=Smallest. We scan the process from left to right, and the best, 2nd best, etc., arrivals have values which are sums of i.i.d. exponential(1) random variables, and arrive at i.i.d. uniform[0, 1]times which are independent of the values.

It is easy to see that Problem 1 corresponds to stopping with the first atom (point) which lies below the curve $y = c/(1-t)$ on $\mathcal{T} \times \mathcal{Y}$. Thus we call this $c/(1-t)$ threshold rule. Note that, in AS, c and t is replaced by b and z respectively. The stopping rule to be considered in Problem 2 is simply described as the (a, c)-stopping rule, $0 < a < 1$, because this rule is the same as the $c/(1-t)$ threshold rule except that this rule only chooses a relatively-best atom if it appears before time a.

Problem 1. (Expected rank under $c/(1-t)$ threshold rule)

We shifts an infinite vertical detector in the positive direction of t and choose the first atom encountered that is located under $c/(1-t)$ thresholds. Note that the chosen atom is not necessarily relatively-best. Let R denote the (absolute) rank of the atom chosen in this way and (T, Y) denote the coordinates of this atom (see Fig.1), which we sometimes call the state of the atom. It is easy to see that the density function of T is given by

$$f_T(t) = c(1-t)^{c-1}, \quad 0 < t < 1, \tag{1}$$

and conditional on $T = t$, Y is uniformly distributed on $(0, c/(1-t))$. Let $R(t, y)$ denote the rank of the atom chosen at state $(T, Y) = (t, y)$. Then we have

$$E[R] = \int_0^1 \left\{ \int_0^{\frac{c}{1-t}} E[R(t,y)] \frac{1-t}{c} dy \right\} f_T(t) dt. \tag{2}$$

The following lemma yields $E[R(t, y)]$.

Lemma 1.1

$$E[R(t,y)] = \begin{cases} 1 + (1-t)y, & \text{if } 0 \le y < c \\ 1 + (1-t)y + (y-c) + c\log\left(\dfrac{c}{y}\right), & \text{if } c \le y \le \frac{c}{1-t} \end{cases}$$

Proof. Let the state of the chosen atom be (t, y) . Then it can be distinguished into two cases i.e., Case (i) and Case (ii) depending on $0 \leq y < c$ or $c \leq y \leq \frac{c}{1-t}$ for each t (see. Figures 2 and 3). Let $N(t, y)$ be the number of atoms included in the shaded region in each figure. Then evidently $R(t, y) = 1 + N(t, y)$. Since the area of the shaded region is given by

$$\lambda(t,y) = \begin{cases} (1-t)y & 0 \leq y < c \\ (1-t)y + \int_0^{1-\frac{c}{y}} \left(y - \frac{c}{1-t}\right) dt = (1-t)y + (y-c) + c\log\left(\frac{c}{y}\right), & c \leq y \leq \frac{c}{1-t} \end{cases} \tag{3}$$

and $N(t, y)$ is a Poisson random variable with parameter $\lambda(t, y)$ from PPP assumption. We have

$$\begin{aligned} E[R(t,y)] &= 1 + E[N(t,y)] \\ &= 1 + \lambda(t,y), \end{aligned}$$

which, combined with (3), completes the proof.

We have from Lemma 1.1,

$$\begin{aligned} \int_0^{\frac{c}{1-t}} E[R(t,y)]\frac{1-t}{c}dy &= \left(\frac{1-t}{c}\right)\left\{\int_0^{\frac{c}{1-t}}[1+(1-t)y]dy + \int_c^{\frac{c}{1-t}}[(y-c)+c\log\left(\frac{c}{y}\right)]dy\right\} \\ &= \left(\frac{1-t}{c}\right)\left\{\frac{c(c+2)}{2(1-t)} + \frac{c^2}{1-t}\left[\frac{(2-t)t}{2(1-t)} + \log(1-t)\right]\right\} \\ &= \frac{c+2}{2} + c\left\{\frac{(2-t)t}{2(1-t)} + \log(1-t)\right\}. \end{aligned}$$

Hence, applying this and (1) to (2) yields

$$\begin{aligned} E[R] &= \int_0^1 \left[\frac{c+2}{2} + c\left\{\frac{(2-t)t}{2(1-t)} + \log(1-t)\right\}\right] c(1-t)^{c-1}dt \\ &= \left(\frac{c+2}{2}\right) + c^2\left\{\int_0^1 \frac{t(2-t)(1-t)^{c-2}}{2}dt + \int_0^1 (1-t)^{c-1}\log(1-t)dt\right\} \\ &= \left(\frac{c+2}{2}\right) + c^2\left\{\frac{1}{(c-1)(c+1)} + \left(-\frac{1}{c^2}\right)\right\} \\ &= 1 + \frac{c}{2} + \frac{1}{c^2-1}, \end{aligned}$$

which coincides with Eq.(4.14) of Assaf and Samuel-Cahn.

Remark: In a similar way, the expected rank under $c/(1-t)^2$ threshold rule can be calculated to yield

$$E(R) = \frac{7}{6} + c^{-1} + \frac{2}{3}c - \frac{c^2}{6} + \frac{1}{6}c^3 e^c I(c),$$

where

$$I(c) = \int_1^\infty \frac{e^{-cu}}{u} du.$$

Problem 2. (Expected rank under (a,c)-stopping rule)

The (a,c)-stopping rule, $0 \le a \le 1$, is the same as the $c/(1-t)$ threshold rule expect that this rule only chooses a relatively-best atom if it appears before time a (this rule is referred to as in Remark 6.2 of Assaf and Samuel-Cahn).

Let A be the best atom above the threshold and S be the time as shown in Figures 4 and 5 (Figures 4 and 5 correspond to the Case (1): $s \ge a$ and the Case (2): $S < a$ respectively).

The density function of S is given by

$$f_S(s) = \frac{cs}{(1-s)^{c+2}} e^{-\frac{cs}{1-s}}, \quad 0 < s < 1 \tag{4}$$

As in problem 1, we denote by R the rank of the atom chosen under the (a,c)-stopping rule and by (T,Y) the state of this atom. Let also $R(t,y), N(t,y)$ and $\lambda(t,y)$ be defined similarly. First we calculate $E[R(t,y)]$.

Lemma 2.1 (Case (1): $s \ge a$)

This case is further distinguished into three cases, i.e., Cases (1a),(1b), and (1c) as shown in Figures 6, 7 and 8.

$$R(t,y) = \begin{cases} 1 + N(t,y), & \text{for Cases (1a) and (1b)} \\ 2 + N(t,y), & \text{for Case (1c).} \end{cases}$$

$$E[R(t,y)] = \begin{cases} 1 + (1-t)y, & \text{for Cases (1a) and (1b)} \\ 2 + (1-t)y + g(y,s), & \text{for Case (1c)} \end{cases}$$

where $g(y,s)$, the area of D in Figure 8, is given by

$$g(y,s) = \left[(y-c) + c\log\left(\frac{c}{y}\right)\right] - \left[\frac{cs}{1-s} + c\log(1-s)\right]$$

Lemma 2.2 (Case (2): $s < a$)
This case is further distinguished into four cases i.e., Case (2a),(2b),(2c), and (2d) as shown in Figures 9, 10, 11 and 12.

$$R(t,y) = \begin{cases} 1 + N(t,y), & \text{for Cases (2a) and (2b)} \\ 2 + N(t,y), & \text{for Cases (2c) and (2d)} \end{cases}$$

$$E[R(t,y)] = \begin{cases} 1 + (1-t)y, & \text{for Cases (2a) and (2b)} \\ 2 + (1-t)y + a\left(y - \frac{c}{1-s}\right), & \text{for Case (2c)} \\ 2 + (1-t)y + a\left(\frac{c}{1-a} - \frac{c}{1-s}\right) + g(y,a), & \text{for Case (2d)} \end{cases}$$

The distribution of T conditional on S is given in the following lemma

Lemma 2.3
[I] Case (1): $S = s \geq a$.

$$f_{T|S}(t) = f_T(t) = c(1-t)^{c-1}, \quad 0 < t < 1$$

[II] Case (2): $S = s < a$.

$$f_{T|S=s}(t) = \begin{cases} c(1-t)^{c-1}, & 0 \leq t < s \\ c(1-s)^{c-1} e^{-\frac{c(t-s)}{1-s}}, & s \leq t < a \\ c(1-t)^{c-1} \left(\frac{1-a}{1-s}\right)^{-c} e^{-\frac{c(a-s)}{1-s}}, & a \leq t \leq 1 \end{cases}$$

We can compute $E[R]$ by conditioning on S, that is,

$$E[R] = \int_0^1 E[R \mid S = s] f_S(s) ds. \tag{5}$$

By the way, we can now compute $E[R \mid S = s]$ from Lemmas 2.1-2.3 as follows.

Lemma 2.4
[I] Case (1): $S = s \geq a$.

$$\begin{aligned} E[R \mid S = s] &= \int_0^1 \left\{ \int_0^{\frac{c}{1-t}} E[R(t,y)] \frac{1-t}{c} dy \right\} f_{T|S=s}(t) dt \\ &= \frac{c+2}{2} + \frac{c}{(c-1)(c+1)} (1-s)^{c-1}. \end{aligned}$$

[II] Case (2) : $S = s < a$.

$$
\begin{aligned}
E[R \mid S=s] &= \int_0^s \left\{ \int_0^{\frac{c}{1-t}} E[R(t,y)] \frac{1-t}{c} dy \right\} f_{T|S=s}(t)dt \\
&+ \int_s^a \left\{ \int_0^{\frac{c}{1-s}} E[R(t,y)] \frac{1-s}{c} dy \right\} f_{T|S=s}(t)dt \\
&+ \int_a^1 \left\{ \int_0^{\frac{c}{1-t}} E[R(t,y)] \frac{1-t}{c} dy \right\} f_{T|S=s}(t)dt \\
&= \frac{c+2}{2} - \frac{(1-s)^c}{2} \\
&+ \frac{(1-s)^c}{2(c+1)} e^{-c\left(\frac{a-s}{1-s}\right)} \left[(2c+1) + \frac{(c+1)c^2}{(c-1)(1-a)} - c\frac{c+3+(c-1)a}{1-s} + \frac{c^2 a(1-a)}{(1-s)^2} \right]
\end{aligned}
$$

Applying Lemma 2.4 to (5) yields the explicit form of $E[R]$.

Theorem 2.5

$$
\begin{aligned}
E[R] &= \frac{c+1}{2} + \left[\frac{1}{2} + \frac{1+(c-1)a}{(c^2-1)(1-a)} \right] e^{-\frac{ca}{1-a}} \\
&+ \frac{1}{2(c^2-1)a^2(1-a)} \left[\left\{ c^2(c+1) - (c^3+c^2+c-1)(1-a) - c(c-1)(1-a)^3 \right\} e^{-ca} \right. \\
&\left. - \left\{ 2c + (c^3+2c^2-6c+1)(1-a) - c(c^2+3c-4)(1-a)^2 \right\} e^{-\frac{ca}{1-a}} \right] \\
&+ \frac{c}{2} \left[e^c I\left(\frac{c}{1-a}, c \right) - \frac{c^2(c+1)+(2c^2-c-1)(1-a)}{(c^2-1)(1-a)} I\left(\frac{ca}{1-a}, ca \right) \right],
\end{aligned}
$$

where

$$
I(\beta, \alpha) = \int_\alpha^\beta \frac{e^{-u}}{u} du
$$

Unfortunately, numerical experiences show that (a,c)-stopping rule gives no significant improvement over $(0,c)$-stopping rule, for example, $E[R] = 2.33044$ for $a = 0.42, c = 1.95$, while $E[R] = 2.33182$ for $a = 0, c = 1.95$.

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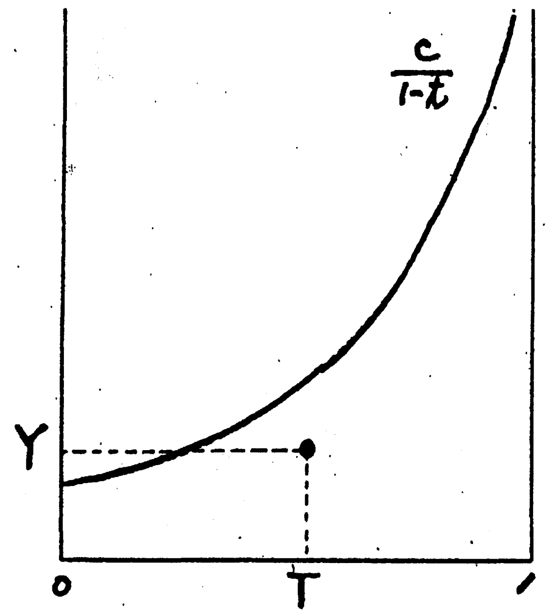
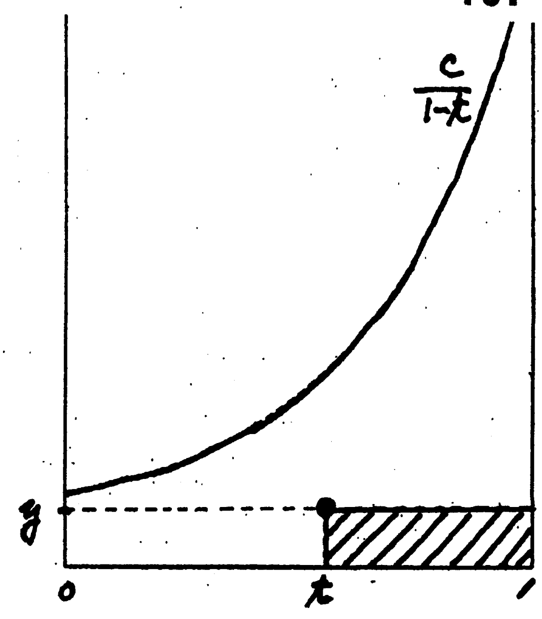
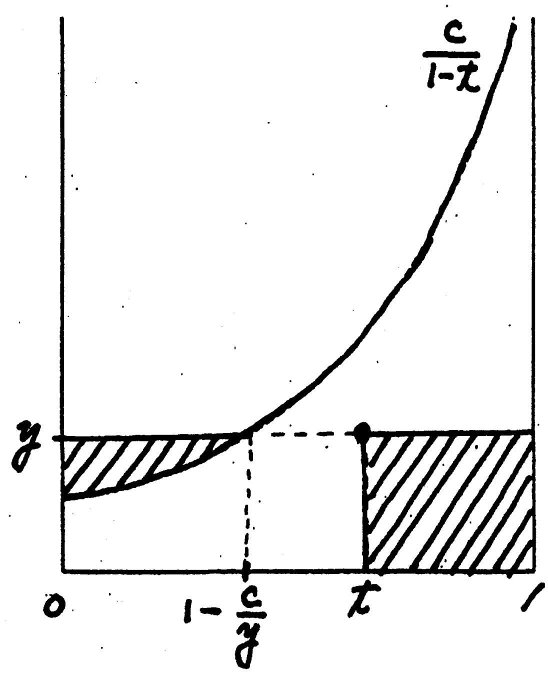


Fig. 1



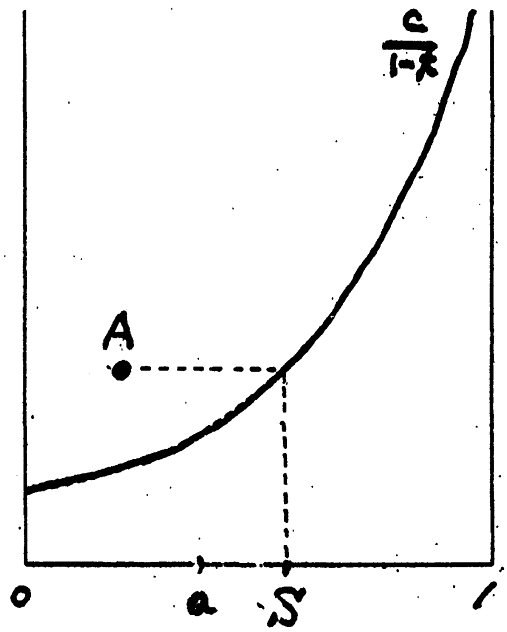
Case(i) : $0 \leq y < c$

Fig. 2



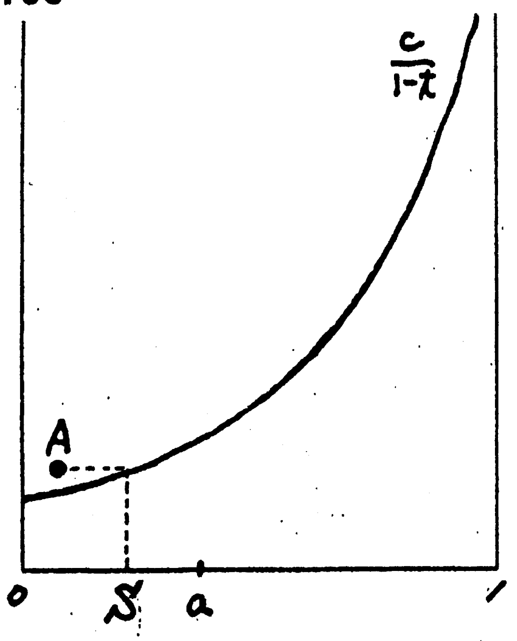
Case (ii) : $c \leq y \leq \frac{c}{1-t}$

Fig. 3



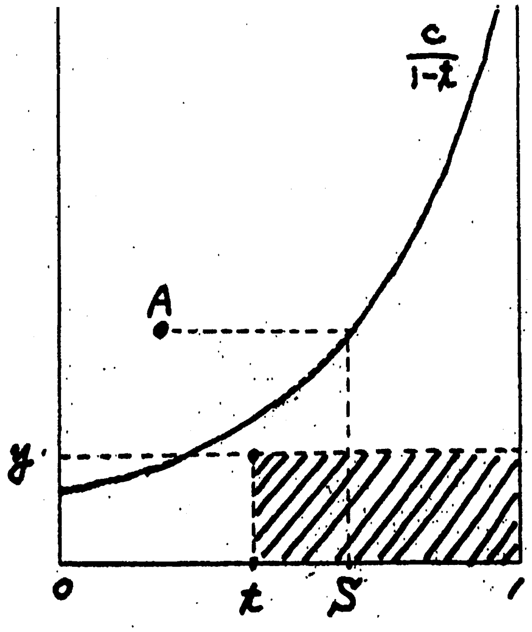
Case (1) : $S \geq a$

Fig. 4



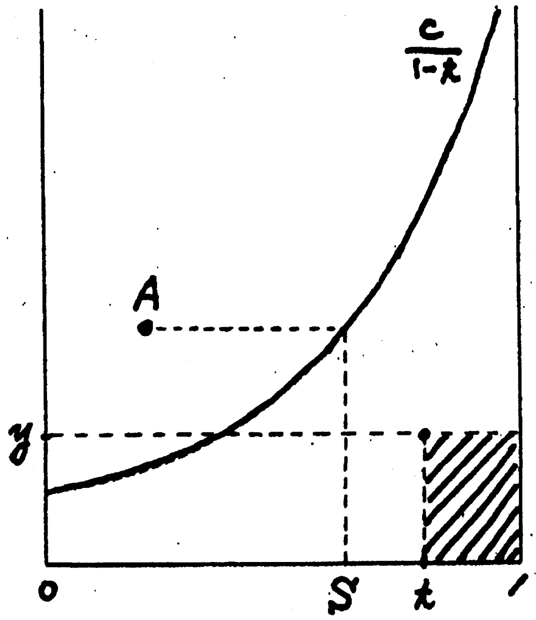
Case (2) : $S < a$

Fig. 5



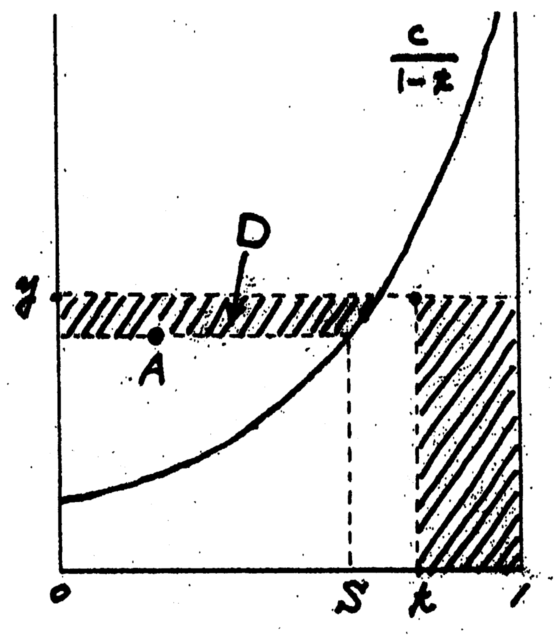
Case (1a): $t < s$

Fig. 6



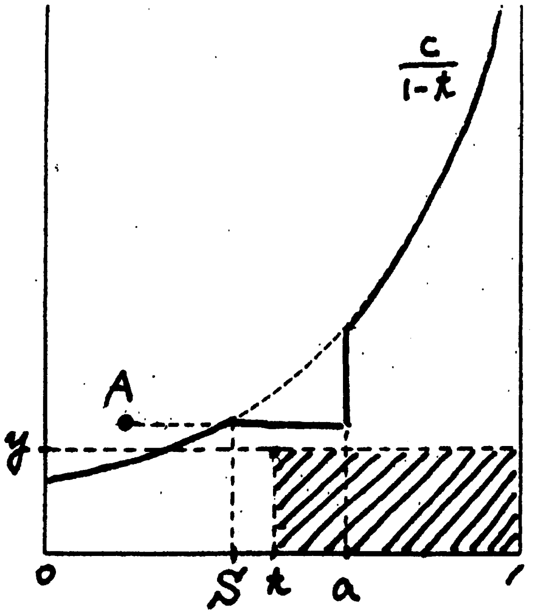
Case (1b): $t \geq s$, $0 \leq y \leq \frac{c}{1-s}$

Fig. 7



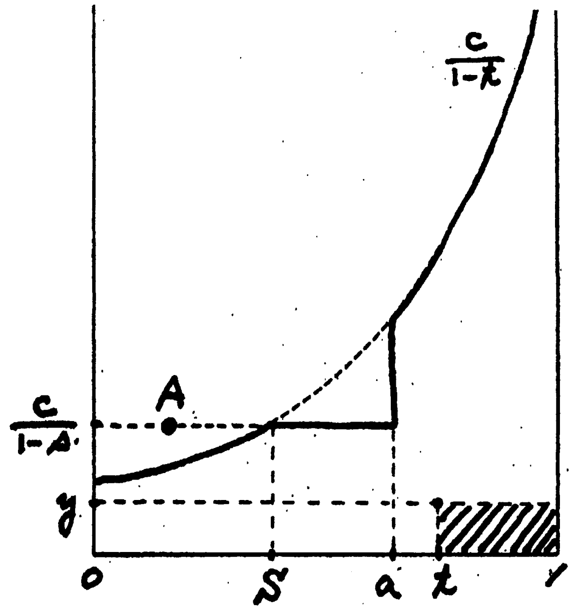
Case (1c): $t \geq s$, $\frac{c}{1-s} < y \leq \frac{c}{1-t}$

Fig. 8



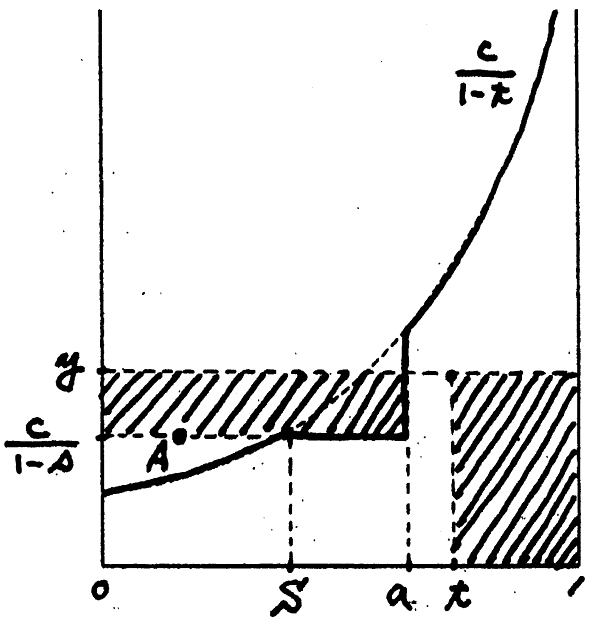
Case(2a): $t < a$

Fig. 9



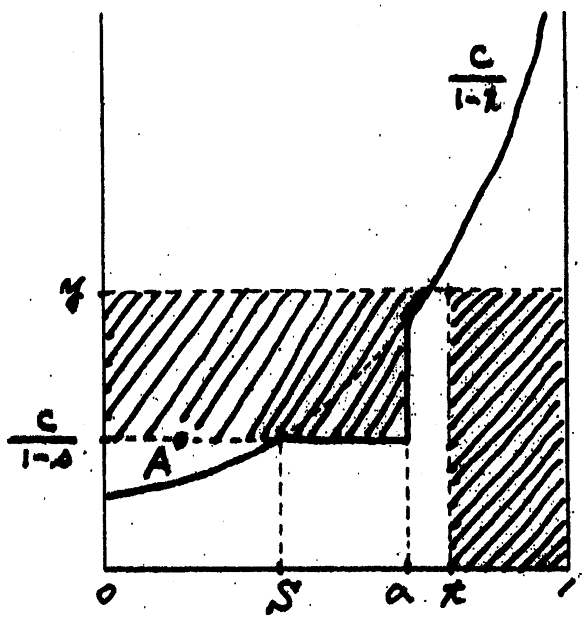
Case(2b): $t \geq a,\ 0 \leq y < \frac{c}{1-s}$

Fig. 10



Case(2c): $t \geq a,\ \frac{c}{1-s} \leq y < \frac{c}{1-a}$

Fig. 11



Case(2d): $t \geq a,\ \frac{c}{1-a} \leq y \leq \frac{c}{1-t}$

Fig. 12