A Note on a Singular Integral Equation

$\lambda\varphi(x)=\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(x-\xi)\varphi(\xi)d\xi$

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In this paper we consider such a singular integral equation as

$$(1)\qquad \lambda\varphi(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(x-\xi)\varphi(\xi)d\xi$$

under a wider hypothesis concerning the kernel than that which Wyle has given.

We shall say that $f(x)$ belongs to class $\mathfrak{M}$, in case f is a measurable function over $(-\infty, \infty)$ for which

$$\frac{1}{2B}\int_{-B}^{B}|f(x)|^2dx$$

is uniformly bounded in B.

Mr. N. Wiener proved :[1]

Theorem A. *Let $f(x)$ belong to $\mathfrak{M}$. Let $s(u)$ be defined as*

$$s(u)=s_1(u)+s_2(u),$$

$$s_1(u)=\underset{A\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]\frac{f(x)e^{-iux}}{-ix}dx,$$

$$s_2(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(x)\frac{e^{-iux}-1}{-ix}dx.$$

Let $K(x)$ be a bounded measurable function for which

$$\int_{-\infty}^{\infty}|K(x)|^2(1+x^2)dx<\infty.$$

Then if we put

$$t_1(u)=\underset{A\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]\frac{e^{-iux}}{-ix}dx\int_{-\infty}^{\infty}K(x-\xi)f(\xi)d\xi,$$

$$t_2(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{e^{-iux}-1}{-ix}dx\int_{-\infty}^{\infty}K(x-\xi)f(\xi)d\xi,$$

$$t(u)=t_1(u)+t_2(u),$$

(these functions really exist), *it will be seen that*

$$t(u+\varepsilon)-t(u-\varepsilon)-\left\{s(u+\varepsilon)-s(u-\varepsilon)\right\}\int_{-\infty}^{\infty}K(\xi)e^{-iu\xi}d\xi$$

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1. The Fourier integral and certain of its applications, Camb. (1933), pp. 164–185; especially Lemma 29_2 and 29_3.

$$=\underset{A\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K(\xi)d\xi\int_{-A}^{A}2f(x-\xi)\left[\frac{\sin\varepsilon x}{x}-\frac{\sin\varepsilon(x-\xi)}{x-\xi}\right]e^{-iux}dx.$$

Theorem B. *On the hypothesis of theorem A, if* $\xi K(\xi)$ *belongs to* L_1,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\left|t(u+\varepsilon)-t(u-\varepsilon)-\left\{s(u+\varepsilon)-s(u-\varepsilon)\right\}\int_{-\infty}^{\infty}K(\xi)e^{-iu\xi}d\xi\right|^2du=0.$$

Using these theorems, we shall prove the two following theorems which are analogous to A, B:

Theorem A′. *Let* $f(x)$ *belong to* $\mathfrak{M}$. *Let* $K(x)$ *be a measurable function for which*

$$K(x)=O(|x|),$$

and for any element $f\in\mathfrak{M}$,

$$\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}K(x-\xi)f(\xi)d\xi$$

$$\lim_{A\to\infty}\frac{1}{2A}\int_{x-A}^{x+A}K(x-\xi)f(\xi)d\xi$$

exist uniformly in x, *furthermore*

$$(2)\qquad \overline{\lim_{A\to\infty}}\frac{1}{4A^2}\int_{-A}^{A}|K(x)|^2(1+x^2)dx<\infty.$$

Then

$$t(u+\varepsilon)-t(u-\varepsilon)-\left\{s(u+\varepsilon)-s(u-\varepsilon)\right\}\cdot a(u)$$

$$=\lim_{A\to\infty}\underset{B\to\infty}{\text{l.i.m}}\frac{1}{2A}\int_{-A}^{A}K(\xi)d\xi\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}2f(x-\xi)$$

$$\times\left[\frac{\sin\varepsilon x}{x}-\frac{\sin\varepsilon(x-\xi)}{x-\xi}\right]e^{-iux}dx,$$

where $a(u)$, $t(u)$ *are defined as follows:*

$$a(u)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}K(\xi)e^{-iu\xi}d\xi,$$

$$t(u)=t_1(u)+t_2(u),$$

$$t_2(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}g(x)\frac{e^{-iux}-1}{-ix}dx,$$

$$t_1(u)=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_{1}^{B}+\int_{-B}^{-1}\right]\frac{g(x)e^{-iux}}{-ix}dx,$$

$$g(x)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}K(x-\xi)f(\xi)d\xi.$$

Theorem B′. *On the hypothesis of* theorem *A′, if*

$$\overline{\lim_{A\to\infty}}\frac{1}{2A}\int_{-A}^{A}|xK(x)|dx<\infty,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\left|t(u+\varepsilon)-t(u-\varepsilon)-\left\{s(u+\varepsilon)-s(u-\varepsilon)\right\}\cdot a(u)\right|^2du=0.$$

We shall begin with the following

Lemma. *Let* $\psi(x)$ *be a measurable function for which*

$$\frac{1}{2T}\int_{-T}^{T}|\psi(x)|\,dx$$

is bounded uniformly in T. *If* a *is any real number, and if the two limits*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\psi(x)dx,$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}\psi(x)dx$$

exist together, then they are equal.

Let us define

$$E=E_1(\psi\geqq 0)+E_2(\psi<0),$$

$$\psi_1(x)=\psi(x) \quad \text{for } x\in E_1; \quad =0 \quad \text{for } x\bar{\in} E_1,$$

$$\psi_2(x)=-\psi(x) \quad \text{for } x\in E_2; \quad =0 \quad \text{for } x\bar{\in} E_2.$$

Then we are able to define the integrals of $\psi(x)$, $|\psi(x)|$ by the equations

$$\int_{-T}^{T}\psi(x)dx=\int_{-T}^{T}\psi_1(x)dx-\int_{-T}^{T}\psi_2(x)dx,$$

$$\int_{-T}^{T}|\psi(x)|\,dx=\int_{-T}^{T}\psi_1(x)dx+\int_{-T}^{T}\psi_2(x)dx.$$

Now, let us put

$$\delta(T)=\frac{1}{2T}\int_{-T}^{T}\psi(x)dx,$$

$$\delta_{1,2}(T)=\frac{1}{2T}\int_{-T}^{T}\psi_{1,2}(x)dx.$$

Clearly, $\delta(T)$, $\delta_1(T)$ and $\delta_2(T)$ are all positive and bounded uniformly in T. Hence there may exist such a suitable sequence $\{T_n\}$ that

$$\delta_1(T_n)\uparrow\alpha,$$

α being finite.

Since $\lim_{T\to\infty}\delta(T)=\lim_{n\to\infty}\delta(T_n)$ exists, for the sequence $\{T_n\}$

$$\lim_{n\to\infty}\delta_2(T_n)=\lim_{n\to\infty}\left\{\delta_1(T_n)-\delta(T_n)\right\}=\lim_{n\to\infty}\delta_1(T_n)-\lim_{n\to\infty}\delta(T_n)$$

may exist. However, we see that if $\varphi(x)\geqq 0$, a is any real number, and if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x)dx=A,$$

then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}\varphi(x)dx=A.^{1}$$

By using this fact, we obtain

$$\lim_{n\to\infty}\delta_{1,2}(T_n)=\lim_{n\to\infty}\frac{1}{2T_n}\int_{-T_n+a}^{T_n+a}\psi_{1,2}(x)dx\,;$$

accordingly

$$\lim_{n\to\infty}\frac{1}{2T_n}\int_{-T_n}^{T_n}\psi(x)dx=\lim_{n\to\infty}\frac{1}{2T_n}\int_{-T_n+a}^{T_n+a}\psi(x)dx.$$

From the hypothesis, it will clearly be seen that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\psi(x)dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}\psi(x)dx.$$

Thus we have completed the proof of the lemma.

Now in the sequel, to prove theorem A′, we shall introduce a function ${}_Af(x)$, defined as follows:

$${}_Af(x)=f(x)\quad[|x|<A]\,;\quad =0\quad[|x|\geqq A].$$

Let $K_A^*(x)$ be defined as

$$K_A^*(x)=\frac{1}{2A}{}_AK(x),\quad[-\infty,\ \infty].$$

Then it is clear that, for all A, $K_A^*(x)$ is bounded, and

$$\int_{-\infty}^{\infty}K_A^*(x-\xi)f(\xi)d\xi=\frac{1}{2A}\int_{x-A}^{x+A}K(x-\xi)f(\xi)d\xi.$$

By means of Schwarz's inequality,

$$\left[\frac{1}{2A}\int_{-A}^{A}|K(x-\xi)f(\xi)|d\xi\right]^2$$

$$\leqq\int_{-A}^{A}\frac{|f(\xi)|^2d\xi}{1+(x-\xi)^2}\frac{1}{4A^2}\int_{-A}^{A}|K(x-\xi)|^2\left\{1+(x-\xi)^2\right\}d\xi$$

$$\leqq\int_{-\infty}^{\infty}\frac{1+\xi^2}{1+(x-\xi)^2}\frac{|f(\xi)|^2}{1+\xi^2}d\xi\frac{1}{4A^2}\int_{x-A}^{x+A}|K(\xi)|^2(1+\xi^2)d\xi\,;$$

however, on account of the hypothesis (2),

$$\overline{\lim_{A\to\infty}}\frac{1}{4A^2}\int_{x-A}^{x+A}|K(\xi)|^2(1+\xi^2)d\xi\leqq\overline{\lim_{A\to\infty}}\frac{1}{4A^2}\int_{-A}^{A}|K(\xi)|^2(1+\xi^2)d\xi<\infty.$$

Hence we have

$$\overline{\lim_{A\to\infty}}\frac{1}{2A}\int_{-A}^{A}|K(x-\xi)f(\xi)|d\xi\leqq\text{const.}\,x+\text{const.}\,.$$

Here, using the previously stated lemma,

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1. Wiener, Loc. cit., p. 155—Lemma 26_1.

$$\lim_{A\to\infty}\int_{-\infty}^{\infty}K_A^*(x-\xi)f(\xi)d\xi=\lim_{A\to\infty}\frac{1}{2A}\int_{x-A}^{x+A}K(x-\xi)f(\xi)d\xi$$
$$=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}K(x-\xi)f(\xi)d\xi.$$

Now let us put

$$g(x, A)=\frac{1}{2A}\int_{-A+x}^{A+x}K(x-\xi)f(\xi)d\xi,$$
$$g(x)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}K(x-\xi)f(\xi)d\xi.$$

Then for any x in every finite interval and for all A, $g(x, A)$ exists boundedly, and moreover

$$g(x)=\lim_{A\to\infty}g(x, A).$$

By seeing of

$$g(x, A)=\int_{-\infty}^{\infty}K_A^*(x-\xi)f(\xi)d\xi$$

and taking K_A^* in the place of K of theorem A, if we put

$$t_1(u, A)=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_1^B+\int_{-B}^{-1}\right]\frac{g(x, A)e^{-iux}}{-ix}dx,$$
$$t_2(u, A)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}g(x, A)\frac{e^{-iux}-1}{-ix}dx,$$
$$t(u, A)=t_1(u, A)+t_2(u, A),$$

then

$$(3)\qquad t(u+\varepsilon, A)-t(u-\varepsilon, A)$$
$$=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{2A}\int_{-A}^{A}K(\xi)d\xi\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(x-\xi)\frac{2\sin\varepsilon x}{x}e^{-iux}dx,$$

$$(4)\qquad t(u+\varepsilon, A)-t(u-\varepsilon, A)-\left\{s(u+\varepsilon)-s(u-\varepsilon)\right\}\frac{1}{2A}\int_{-A}^{A}K(\xi)e^{-iu\xi}d\xi$$
$$=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{2A}\int_{-A}^{A}K(\xi)d\xi\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}2f(x-\xi)$$
$$\times\left[\frac{\sin\varepsilon x}{x}-\frac{\sin\varepsilon(x-\xi)}{x-\xi}\right]e^{-iux}dx.$$

Now, by Lebesgue's bounded convergence theorem,

$$\lim_{A\to\infty}\int_{-1}^{1}g(x, A)\frac{e^{-iux}-1}{-ix}dx=\int_{-1}^{1}g(x)\frac{e^{-iux}-1}{-ix}dx.$$

Let us consider

$$g^*(x, A)=\frac{g(x,A)}{x}\quad[|x|>1];\qquad =0\quad[|x|\leqq 1].$$

This belongs to $L_2(-\infty, \infty)$; furthermore if we put

$$\lim_{A\to\infty} g^*(x, A) = g^*(x),$$

$$g^*(x)=0 \quad [|x|\leqq 1]; \quad =\frac{g(x)}{x} \quad [|x|>1].$$

Since, for an arbitrary given positive quantity ε,

$$\int_{-\infty}^{\infty}|g^*(x, A)-g^*(x)|^2\,dx \leqq \varepsilon^2\left[\int_{-\infty}^{-1}+\int_1^{\infty}\right]\frac{dx}{x^2}\to 0,$$

$$(5) \qquad \underset{A\to\infty}{\text{l.i.m}}\, g^*(x,A)=g^*(x).$$

While, it is evident that, under a Fourier transform, mean convergence may be invariant. Therefore

$$\underset{A\to\infty}{\text{l.i.m}}\, t_1(u,A)=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_1^B+\int_{-B}^{-1}\right]\frac{g(x)e^{-iux}}{-ix}dx.$$

Thus if we put

$$t_1(u)=\underset{A\to\infty}{\text{l.i.m}}\, t_1(u,A)$$

$$=\underset{B\to\infty}{\text{l.i.m}}\frac{1}{\sqrt{2\pi}}\left[\int_1^B+\int_{-B}^{-1}\right]\frac{g(x)e^{-iux}}{-ix}dx,$$

$$t_2(u)=\lim_{A\to\infty} t_2(u, A)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} g(x)\frac{e^{-iux}-1}{-ix}dx,$$

then

$$t(u)=t_1(u)+t_2(u)$$

exists almost everywhere.

From Riesz-Fisher theorem and (5), we are able to find a monotone sequence $\{A_n\}$, $A_n\uparrow\infty$, such that for almost all u

$$t_1(u)=\lim_{n\to\infty} t_1(u, A_n).$$

Consequently

$$\lim_{n\to\infty} t(u, A_n)=t(u)$$

almost everywhere. From (4),

$$t(u+\varepsilon)-t(u-\varepsilon)-\Big\{s(u+\varepsilon)-s(u-\varepsilon)\Big\}\cdot a(u)$$

$$=\lim_{n\to\infty}\underset{B\to\infty}{\text{l.i.m}}\frac{1}{2A_n}\int_{-A_n}^{A_n}K(\xi)d\xi\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}2f(x-\xi)$$

$$\times\left[\frac{\sin\varepsilon x}{x}-\frac{\sin\varepsilon(x-\xi)}{x-\xi}\right]e^{-iux}dx.$$

Here the function $a(u)$ is by definition

$$a(u)=\lim_{n\to\infty}\frac{1}{2A_n}\int_{-A_n}^{A_n}K(\xi)e^{-iu\xi}d\xi. \qquad \text{Q. E. D.}$$

Let us suppose

$$\overline{\lim_{A\to\infty}}\frac{1}{2A}\int_{-A}^{A}|\xi K(\xi)|\,d\xi<\infty.$$

Inasmuch as

$$\int_{-\infty}^{\infty}\xi K_A^*(\xi)d\xi=\frac{1}{2A}\int_{-A}^{A}\xi K(\xi)d\xi,$$

we can take again K_A^* in the place of K of theorem B.

Thus theorem B′ follows directly from theorem B.

Now, let us put

$$g(x)=\lambda f(x).$$

Evidently $t(u)=\lambda s(u)$ almost everywhere. Let $s(u)$ be of limited total variation over $(-\infty, \infty)$. However, since all its points of discontinuity are at most innumerably infinite, if we call these points

$$u_1,\ u_2,\ u_3,\ \ldots\ldots\ldots\ldots,\ u_n,\ \ldots\ldots\ldots\ldots,$$

from the conclusion of theorem B′,

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon)-s(u-\varepsilon)|^2|\lambda-a(u)|^2du$$
$$=\sum_{n=1}^{\infty}|s(u_n+0)-s(u_n-0)|^2|\lambda-a(u_n)|^2.$$

Thus we have;

Theorem C. *Under the same hypotheses of* theorem B′, *if* $s(u)$, *defined from* $f(x)$, *is of limited total variation and has discontinuous points, and if* $f(x)$ *satisfies the singular integral equation*

$$\lambda f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(x-\xi)f(\xi)d\xi,$$

then it should be true that

$$\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(\xi)e^{-iu_n\xi}d\xi,\quad n=1,\ 2,\ \ldots\ldots\ldots,\ n,\ \ldots\ldots\ldots\ldots,$$

$\{u_n\}$ *being assumed as the set of all points of discontinuity.*

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