Divergent Series and Special Applications of Tauberian Theorems

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1. Let us suppose that $f(x)$ is measurable and of integrable square over any finite range. We begin with

Theorem 1.1. *Let*

$$(1) \qquad \frac{1}{T}\int_0^T |f(x)|^2 dx$$

be bounded uniformly in T. Then for every positive ε

$$\int_0^\infty e^{-2\varepsilon x}|f(x)|^2 dx \leqq (2\varepsilon)^{-1}K,$$

K being an absolute constant.

Proof. If we put $\varphi(x)=\int_0^x |f(t)|^2 dt$, we have $\varphi(0)=0$ and $\varphi'(x)=|f(x)|^2$ almost everywhere. Hence by integrating by parts

$$\int_0^T e^{-2\varepsilon x}|f(x)|^2 dx = \int_0^T e^{-2\varepsilon x}\varphi'(x)dx$$

$$= \Big[e^{-2\varepsilon x}\varphi(x)\Big]_0^T + 2\,\varepsilon\int_0^T e^{-2\varepsilon x}\varphi(x)dx$$

$$= Te^{-2\varepsilon T}T^{-1}\int_0^T |f(x)|^2 dx + 2\varepsilon\int_0^T xe^{-2\varepsilon x}dx\Big\{\frac{1}{x}\int_0^x |f(t)|^2 dt\Big\}$$

$$\leqq \Big\{Te^{-2\varepsilon T} + 2\varepsilon\int_0^T xe^{-2\varepsilon x}dx\Big\}\limsup_{0<T<\infty} T^{-1}\int_0^T |f(t)|^2 dt$$

$$\leqq \Big\{Te^{-2\varepsilon T} + (2\varepsilon)^{-1}\Gamma(2)\Big\}K.$$

Let $T\to\infty$. Then we obtain the conclusion. Q. E. D.

From theorem 1.1 we have

Theorem 1.2. *Let*

$$(2) \qquad \phi_{\sigma\tau}(x)=e^{-\sigma x} \quad [x>0]; \quad =e^{\tau x} \quad [x<0].$$

If

$$(3) \qquad \frac{1}{2T}\int_{-T}^T |f(x)|^2 dx$$

is bounded uniformly in T, then for a given positive number ε

$$f(x)\phi_{\sigma\tau}(x) \in L_2$$

over $(-\infty, \infty)$ *uniformly in* σ, τ *such as* σ, $\tau \geqq \varepsilon$.

If we use the Schwarz inequality,

$$\left|\int_0^\infty f(x)e^{-\sigma x}dx\right| \leqq \int_0^\infty |f(x)|\, e^{-\frac{1}{2}\sigma x}e^{-\frac{1}{2}\sigma x}dx$$
$$\leqq \left\{\int_0^\infty |f(x)|^2 e^{-\sigma x}dx \int_0^\infty e^{-\sigma x}dx\right\}^{\frac{1}{2}}$$
$$= \sigma^{-\frac{1}{2}}\left\{\int_0^\infty |f(x)|^2 e^{-\sigma x}dx\right\}^{\frac{1}{2}} < K\sigma^{-\frac{3}{2}}.$$

Thus we have proved:

Theorem 1.3. *Under the hypothesis of* theorem 1.1, *we can conclude that for a given positive number* ε

$$f(x)e^{-\sigma x} \in L_1$$

over $(0, \infty)$ *uniformly in such* σ *as* $\sigma \geqq \varepsilon$.

Now since the integral

$$F(\sigma, t) = (2\pi)^{-\frac{1}{2}}\int_0^\infty f(x)e^{-(\sigma+it)x}dx$$

converges absolutely and uniformly over the range $0 < \varepsilon \leqq \sigma$, thus by a well known theorem from the theory of functions of a complex variable, $F(\sigma, t) = F(\sigma+it)$ will be an analytic function of $\sigma+it$ over the interior of the right half-plane $0 < \sigma < \infty$. Furthermore we find that

$$f(x)e^{-\sigma x}, \quad F(\sigma+it)$$

are Fourier transforms. Therefore by Plancherel's theorem we have

$$\int_{-\infty}^\infty |F(\sigma+it)|^2 dt = \int_0^\infty |f(x)|^2 e^{-2\sigma x}dx.$$

We have thus proved:

Theorem 1.4. *Under the hypothesis of* theorem 1.1, *the Fourier transform*

$$(4) \qquad F(\sigma+it) = (2\pi)^{-\frac{1}{2}}\int_0^\infty f(x)e^{-(\sigma+it)x}dx$$

defines a function $F(\sigma+it)$ *analytic over the right half-plane* $\sigma>0$, *and*

$$(5) \qquad \int_{-\infty}^\infty |F(\sigma+it)|^2 dt = \int_0^\infty |f(x)|^2 e^{-2\sigma x}dx\,;$$

accordingly, for all positive values $\sigma \geqq \varepsilon > 0$,

$$\int_{-\infty}^\infty |F(\sigma+it)|^2 dt$$

is bounded.

2. If we put

$$L(\sigma) = \underset{0\leq|t|<\infty}{\text{Max}} |F(\sigma+it)|$$

and
$$L^*(\sigma) = (2\pi)^{-\frac{1}{2}}\int_0^\infty |f(x)|\, e^{-\sigma x}dx,$$

then $L(\sigma) \leqq L^*(\sigma)$. Let us notice that $L^*(\sigma)$ is decreasing and tends to zero as $\sigma\to\infty$. For, for any two numbers σ_1 and σ_2 such as $\sigma_1<\sigma_2$, $L^*(\sigma_1)>L^*(\sigma_2)$; and for a fixed positive number $\sigma_0\geqq\varepsilon>0$

$$\begin{aligned}\int_0^\infty|f(x)|e^{-\sigma x}dx&=\int_0^\infty|f(x)|e^{-\sigma_0 x}e^{-(\sigma-\sigma_0)x}dx\\&\leqq\left\{\int_0^\infty|f(x)|^2e^{-2\sigma_0 x}dx\int_0^\infty e^{-2(\sigma-\sigma_0)x}dx\right\}^{\frac{1}{2}}\\&\leqq\left\{2(\sigma-\sigma_0)\right\}^{-\frac{1}{2}}\left\{\int_0^\infty|f(x)|^2e^{-2\sigma_0 x}dx\right\}^{\frac{1}{2}}\\&=\text{const}\left\{2(\sigma-\sigma_0)\right\}^{-\frac{1}{2}}.\end{aligned}$$

Hence $L^*(\sigma)\to 0$ as $\sigma\to\infty$, also $L(\sigma)\to 0$.

On the other hand, it is known that $L(\sigma)$ is a convex function of σ, and $\log L(\sigma)$ is also[1]. Consequently either of the limits

$$\lim_{\sigma\to+0}L(\sigma)=\textit{finite}\quad\text{or}\quad\textit{infinite}$$

exists.

Next if we put

$$I(\sigma)=\int_{-\infty}^{\infty}|F(\sigma+it)|^2dt,\qquad \sigma\geqq\varepsilon>0,$$

we can easily prove that the function $\log I(\sigma)$ is a convex function of σ,[2] and $I(\sigma)$ is decreasing steadily to 0 as in $L(\sigma)$; moreover either of the limits

$$\lim_{\sigma\to+0}I(\sigma)=\textit{finite}\quad\text{or}\quad\textit{infinite}.$$

Now from the convexity of $\log I(\sigma)$,

$$I(\sigma)\leqq\{I(\alpha)\}^{(\beta-\sigma)/(\beta-\alpha)}\{I(\beta)\}^{(\sigma-\alpha)/(\beta-\alpha)},$$

where $0<\varepsilon\leqq\alpha<\sigma<\beta$. We shall give a brief proof for these results. Let

$$I_\lambda(\sigma)=\int_0^\lambda|f(x)|^2e^{-2\sigma x}dx,$$

then by means of $\dfrac{d^2}{d\sigma^2}I_\lambda(\sigma)>0$, $I_\lambda(\sigma)$ is a convex function of σ for every finite λ. But for any two numbers λ', λ such as $\lambda'>\lambda$

$$\begin{aligned}I_{\lambda'}(\sigma)-I_\lambda(\sigma)&=\int_\lambda^{\lambda'}|f(x)|^2e^{-2\sigma x}dx\\&\leqq\int_\lambda^{\lambda'}|f(x)|^2e^{-2\varepsilon x}dx\qquad[0<\varepsilon\leqq\sigma].\end{aligned}$$

From (5), $I_\lambda(\sigma)$ converges to $I(\sigma)$ uniformly in σ ($\geqq\varepsilon$) as $\lambda\to\infty$. Thus $I(\sigma)$ will be a convex function of σ. Furthermore by the same

1. G. Doetsch, Ueber die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden, Math. Zeitschrift, Bd. 8, 1920, pp. 237-240.

2. Hardy, Ingham and Polya, Theorems concerning mean values of analytic functions, Proc. of the Royal Soc., A, vol. 113 (1927), pp. 542-569.

argument as in the case of $L(\sigma)$, we can show that $I(\sigma)\to 0$ with $\sigma\to\infty$. Now even if we replace $e^{\alpha\sigma/2}f(x)$ with $f(x)$ of (1), the same reasoning is still valid and we can conclude the convexity of $e^{\alpha\sigma}I(\sigma)$ for arbitrary values of α.[1]

Thus we shall have

Theorem 2.1. *Let*

$$L(\sigma)=\operatorname*{Max}_{-\infty<t<\infty}|F(\sigma+it)| \quad \textit{and} \quad I(\sigma)=\int_{-\infty}^{\infty}|F(\sigma+it)|^2dt.$$

Then $L(\sigma)$, $I(\sigma)$ *decrease steadily to zero as* $\sigma\to\infty$ *and* $\log L(\sigma)$, $\log I(\sigma)$ *are convex functions of* σ.

In the sequel we shall investigate the relations among orders in T and σ of

$$\int_0^T|f(x)|^2dx,\quad \int_0^T x^{2\vartheta}|f(x)|^2dx,\quad L^*(\sigma),\quad I(\sigma).$$

If we now assume

$$\int_0^T|f(x)|^2dx=O(T^\rho), \tag{6}$$

then by putting $\varphi(x)=\int_0^x|f(x)|^2dx$,

$$\begin{aligned}\int_0^T x^{2\vartheta}|f(x)|^2dx&=\int_0^T x^{2\vartheta}\varphi'(x)dx\\ &=\Big[x^{2\vartheta}\varphi(x)\Big]_0^T-2\vartheta\int_0^T x^{2\vartheta-1}\varphi(x)dx\\ &=O(T^{2\vartheta+\rho})+O\Big(\int_0^T x^{2\vartheta-1+\rho}dx\Big)\\ &=O(T^{2\vartheta+\rho})+O(T^{2\vartheta+\rho})\\ &=O(T^{2\vartheta+\rho}).\end{aligned}$$

$$\begin{aligned}I(\sigma)&=\int_0^\infty|f(x)|^2e^{-2\sigma x}dx\\ &=\Big[e^{-2\sigma x}\varphi(x)\Big]_0^\infty+2\sigma\int_0^\infty e^{-2\sigma x}\varphi(x)dx\\ &=O\Big(2\sigma\int_0^\infty x^\rho e^{-2\sigma x}dx\Big).\end{aligned}$$

However, as will be seen,

$$2\sigma\int_0^\infty x^\rho e^{-2\sigma x}dx=(2\sigma)^{-\rho}\Gamma(\rho+1)\,;$$

therefore we have $I(\sigma)=O(\sigma^{-\rho})$ as $\sigma\to 0$.

By the Schwarz inequality, we have

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1. P. Montel, Sur les fonctions convexes et les fonctions sousharmoniques, Jordan Jour. de Math. pure et appl. ix, (1928), pp. 32–33.

$$L^*(\sigma) \leqq I\left(\frac{\sigma}{2}\right)\left\{\int_0^\infty e^{-\sigma x}dx\right\}^{\frac{1}{2}} = I\left(\frac{\sigma}{2}\right)\sigma^{-\frac{1}{2}}.$$

Hence $L^*(\sigma)=O(\sigma^{-\rho-\frac{1}{2}})$ as $\sigma \to 0$. Thus we have proved:

Theorem 2.2. *Let*

$$\int_0^T |f(x)|^2 dx = O(T^\rho).$$

Then we have

$$\int_0^T x^{2\delta} |f(x)|^2 dx = O(T^{2\delta+\rho}),$$

$$I(\sigma)=O(\sigma^{-\rho}),$$

$$L^*(\sigma)=O(\sigma^{-\rho-\frac{1}{2}}).$$

Now let us consider a specific case:

Let $f(s)$ be regular for $\alpha \leqq \sigma \leqq \beta$ and $O(e^{\varepsilon t})$ as $t \to \infty$ for every positive ε. Let

$$\int_0^T |f(\alpha+it)|^2 dt = O(T^a), \quad \int_0^T |f(\beta+it)|^2 dt = O(T^b)$$

as $T \to \infty$, where $0<\alpha<\beta$, $a \geqq 1$, $b \geqq 1$. Then we can prove that

$$\int_0^T |f(\sigma+it)|^2 dt = O\left\{T^{[a(\beta-\sigma)+b(\sigma-\alpha)]/(\beta-\alpha)}\right\}.$$

However Hardy, Ingham and Polya[1] proved that the order in T of

$$\frac{1}{T}\int_0^T |f(\sigma+it)|^p dt$$

is a convex function of σ, but their method is rather complicated. Recently Titchmarsh[2] gives a simple proof for the case $p=2$. Here we give a more simple proof and obtain an analogous result to this theorem.

Let

$$F_\eta(\sigma+it) = e^{i\eta(\sigma+it)} f(\sigma+it) = e^{i\eta s} f(s), \qquad \eta > 0.$$

Then $F_\eta(\sigma+it)$ will be regular over the interior of the strip $\alpha \leqq \sigma \leqq \beta$, and belongs to $L_2(-\infty, \infty)$ for $\sigma=\alpha$, β by theorem 2.2. Furthermore

$$\int_0^\infty |F_\eta(\alpha+it)|^2 dt = O\left(2\eta \int_0^\infty t^a e^{-2\eta t} dt\right)$$
$$= O\{(2\eta)^{-a}(a+1)\Gamma(a)\}$$

for every positive η. Quite similarly

$$\int_0^\infty |F_\eta(\beta+it)|^2 dt = O\left\{(2\eta)^{-b}\Gamma(b+1)\right\}.$$

1. Proc. Royal Soc. (A), 113 (1926), pp. 542-569; Quart. J. of Math. (Oxford), 8 (1937), pp. 255-266.

2. A convexity theorem, J. of the London Math. Soc., Vol. 13, Part 3 (1938), pp. 196-197.

By the order-condition where $t\to\infty$, for all positive values $\eta\geqq\varepsilon$ and for every positive ε, we have

$$|F_\eta(\sigma+it)|=e^{-\eta t}|f(\sigma+it)|=O(e^{-(\eta-\varepsilon)t})<\infty.$$

While, about such a function as $F_\eta(s)$ Paley[1] obtained the following theorem :

$F(\sigma+it)$ belongs uniformly to $L_2(-\infty,\infty)$ over $\alpha\leqq\sigma\leqq\beta$, accordingly there exists a measurable function $g_\eta(x)$, such that

$$\int_{-\infty}^{\infty}|g_\eta(x)|^2e^{2\alpha x}dx<\infty,\quad \int_{-\infty}^{\infty}|g_\eta(x)|^2e^{2\beta x}dx<\infty,$$

and that over the closed interval $\alpha\leqq\sigma\leqq\beta$,

$$F_\eta(\sigma+it)=\underset{A\to\infty}{\mathrm{l.i.m.}}(2\pi)^{-\frac{1}{2}}\int_{-A}^{A}g_\eta(x)e^{x(\sigma+it)}dx.$$

By using Parseval's formula we obtain

$$\int_0^\infty|F_\eta(\sigma+it)|^2dt=\int_{-\infty}^{\infty}|g_\eta(x)|^2e^{2\sigma x}dx.^2$$

Thus by the same argument as in the proof of the Theorem 2.1, if we put

$$I_\eta(\sigma)=\int_0^\infty|F_\eta(\sigma+it)|^2dt,$$

we may conclude that $I_\eta(\sigma)$ and log $I_\eta(\sigma)$ are together convex functions of σ; and moreover Hölder's inequality

$$I_\eta(\sigma)\leqq\{I_\eta(\alpha)\}^{(\beta-\sigma)/(\beta-\alpha)}\{I_\eta(\beta)\}^{(\sigma-\alpha)/(\beta-\alpha)}$$

will hold. On the other side we have already

$$I_\eta(\alpha)=O((2\eta)^{-a}),\quad I_\eta(\beta)=O((2\eta)^{-b}).$$

Therefore by combining this with the above stated inequality,

$$I_\eta(\sigma)=O\left(\left(\frac{1}{\eta}\right)^{a[(\beta-\sigma)/(\beta-\alpha)]+b[(\sigma-\alpha)/(\beta-\alpha)]}\right),$$

where $\eta\geqq\varepsilon$ and ε are arbitrarily chosen to be positive.

Also
$$I_\eta(\sigma)=\int_0^\infty e^{-2\eta t}|f(\sigma+it)|^2dt\geqq\int_{\frac{1}{\eta}}^{\frac{2}{\eta}}e^{-2\eta t}|f(\sigma+it)|^2dt$$
$$\geqq e^{-4}\int_{\frac{1}{\eta}}^{\frac{2}{\eta}}|f(\sigma+it)|^2dt.$$

Hence, putting $\eta=2/T$, we have

$$\int_{T/2}^{T}|f(\sigma+it)|^2dt=O(T^{[a(\beta-\sigma)+b(\sigma-\alpha)]/(\beta-\alpha)}).$$

If we replace T by $\frac{1}{2}T$, $\frac{1}{4}T$, ... and add them, then the result follows. Thus we have proved :

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1. Fourier Transforms in the complex domain, Amer. Colloq. (1934) pp. 3-14.
2. It is noticed that $F(\sigma+it)$ belongs uniformly in σ to L_2 over t (0, ∞) for $\alpha\leqq\sigma\leqq\beta$.

Theorem 2.3. *Let $f(\sigma+it)$ be regular over the interior of the strip $\alpha\leqq\sigma\leqq\beta$ and $O(e^{\varepsilon t})$ as $t\to\infty$ for every positive* ε. *Let*

$$\int_0^T|f(\alpha+it)|^2dt=O(T^a),\quad \int_0^T|f(\beta+it)|^2dt=O(T^b)$$

as $T\to\infty$, where $0\leqq\alpha<\beta$, $a\geqq 1$, $b\geqq 1$. Then the order in T of $\int_0^T|f(\sigma+it)|^2dt$ *is*

$$O(T^{\{a(\beta-\sigma)+b(\sigma-\alpha)\}(\beta-\alpha)});$$

and besides the analytic function $f(s)$ is expressed by

$$(7)\qquad f(s)=\lim_{\eta\to 0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_\eta(\beta+iy)}{\beta+iy-s}dy-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_\eta(\alpha+iy)}{\alpha+iy-s}dy\right\},$$

where $\quad F_\eta(s)=e^{i\eta s}f(s)$.

The expression (7) may be obtained as follows: from the same argument that Paley used[1] in his book

$$F_\eta(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_\eta(\beta+iy)}{\beta+iy-s}dy-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_\eta(\alpha+iy)}{\alpha+iy-s}dy$$

is derived directly, there let $\eta\to 0$, then the expression (7) follows.

Now in view of the above process of proof of theorem 2.3 we see that the order-condition $O(e^{\varepsilon t})$ as $t\to\infty$ is not always essential for the conclusion of the theorem. Hence again from Paley's argument we have the following theorem of the Phragmen-Lindelöf type:

Theorem 2.4. *Even if we take the order-condition $O(e^{e^{\rho t}})$, $\rho<\alpha+\beta$, of $f(\sigma+it)$ as $t\to\infty$ instead of $O(e^{\varepsilon t})$ in* theorem 2.3, *it will be true.*

3. The present section and succeeding sections will be devoted to the application of Tauberian theorems[2] to the discussion about the limit of $I(\sigma)$ as $\sigma\to 0$. To see this, we shall begin with

Theorem 3.1. *If $\phi(x)$ is non-negative for $0\leqq x<\infty$, and integrable over any finite interval, then we have*

$$(8)\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T\phi(x)dx=\lim_{\sigma\to 0}\sigma\int_0^\infty e^{-\sigma x}\phi(x)dx$$

in the sense that if either side of (8) *exists, the other side exists and assumes the same value.*

Let us now put

$$(T)\qquad \sigma=e^{-\eta};\quad x=e^{\xi},\ T=e^{\eta};\quad \phi(x)=\phi(e^{\xi})\equiv\psi(\xi).$$

The expression

$$(9)\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T\phi(x)dx$$

becomes

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1. Loc. cit., p. 9. 2. N. Wiener, The Fourier integral, Camb. (1933), pp. 73-75.

(9′) $$\lim_{\eta\to\infty}\int_{-\infty}^{\eta} e^{-(\eta-\xi)}\psi(\xi)d\xi\,;$$

and the expression

(10) $$\lim_{\sigma\to 0}\sigma\int_0^{\infty} e^{-\sigma x}\phi(x)dx$$

becomes

(10′) $$\lim_{\eta\to\infty}\int_{-\infty}^{\infty} e^{-(\eta-\xi)}e^{-e^{-(\eta-\xi)}}\psi(\xi)d\xi.$$

Hence in order to establish theorem 3.1, it is sufficient to show that the two propositions

(11) $$\lim_{\eta\to\infty}\int_{-\infty}^{\infty} K_1(\eta-\xi)dg(\xi)=A\int_{-\infty}^{\infty}K_1(\xi)d\xi$$

and

(12) $$\lim_{\eta\to\infty}\int_{-\infty}^{\infty} K_2(\eta-\xi)dg(\xi)=A\int_{-\infty}^{\infty}K_2(\xi)d\xi$$

are equivalent, where

$$K_1(\xi)=0\quad(-\infty<\xi<0)\,;\qquad K_1(\xi)=e^{-\xi}\quad(0<\xi<\infty)$$

and $$K_2(\xi)=e^{-\xi}e^{-e^{-\xi}}\quad(-\infty<\xi<\infty),$$

and

(13) $$g(\xi)=\int_0^{\xi}\psi(\xi)d\xi.$$

By a similar method to the one Wiener used in his book, especially in § 20, equivalence between (11) and (12) will be established.

In the sequel, for $I(\sigma)$ defined in the section 2 we shall give a function-theoretic meaning.

By using $\phi(x)$ in above theorem 3.1, we define

(14) $$G(\sigma+it)=\int_0^{\infty}\phi(x)e^{-(\sigma+it)x}dx,$$

and $$G(\sigma)=\int_0^{\infty}\phi(x)e^{-\sigma x}dx.$$

Since $G(\sigma)$ holds the same property with $I(\sigma)$, $G(\sigma+it)$ is analytic in the interior of the right half-plane $R(s)>0$. Particularily, if $G(\sigma)$ is boundedly convergent to some finite limit as $\sigma\to 0$, then $G(s)$, $s=\sigma+it$, is analytically continuable on to the imaginary axis $R(s)=0$, and there are no singularities. But in general, from theorem 2.2,

$$G(\sigma)=O(\sigma^{-1})\ \text{i. e.}\ \overline{\lim_{\sigma\to+0}}\,\sigma G(\sigma)\doteq\text{finite}\,;$$

in other words, *if $G(\sigma+it)$ is assumed to be analytically continuable on to $R(s)=0$, and there free from singularities, except for $s=0$, then the point $s=0$ will be possible a simple pole with principal part A/s.* Moreover it will be also seen that if $\phi(x)$ satisfies the more general order-condition such as

$$\int_0^x \phi(x)dx = O(x^\rho),$$

then by theorem 2.2 $\overline{\lim_{\sigma\to+0}}\,\sigma^\rho G(\sigma)=a_p$.[1]

Let $G(\sigma+it)$ *be regular on* $R(s)=0$, *except for a pole of order* p *at the origin* $s=0$. *Then it should be necessary* $\rho=p$, *and* $\lim_{s\to 0} s^p G(s)=a_p$. For, $\sigma^p G(\sigma)=\sigma^{p-\rho}\sigma^\rho G(\sigma)$.

Consequently *the principal part of* $G(s)$ *may be written in the following form*:

$$(15)\qquad \frac{a_p}{s^p}+\frac{a_{p-1}}{s^{p-1}}+\dots\dots\dots+\frac{a_1}{s}.$$

Now take the simple case where $\rho=p=1$.

Let

$$(16)\qquad G(s)-A/s=g(s)$$

be regular on the imaginary axis $R(s)=0$. Then

$$A=\lim_{\sigma\to+0}\sigma G(\sigma)=\lim_{\sigma\to+0}\sigma\int_0^\infty \phi(x)e^{-\sigma x}dx.$$

Hence in view of (8) we have

Theorem 3.3. *Under hypotheses of* theorem 3.1, *the limit of*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \phi(x)dx$$

is equal to the residue of $G(s)$ *at* $s=0$, *provided* $G(s)$ *is given by* (14) *and* (16).

4. We shall now proceed to discuss the preceding integral formulae in their Stieltjes' form.

Let $\alpha(x)$ be defined over $(0,\infty)$ and a monotone increasing function, and let

$$(17)\qquad \int_0^x d\alpha(t)=O(x^\rho)\qquad (\rho\geqq 0),$$

as $x\to\infty$. Here let us assume that $\alpha(+0)=0$.[2]

Then if we integrate by parts,

$$\int_0^\infty e^{-\sigma x}d\alpha(x)=\Big[e^{-\sigma x}\alpha(x)\Big]_{x=\infty}+\Big[\sigma\int_0^x e^{-\sigma x}\alpha(x)dx\Big]_{x=\infty}$$
$$=O\Big(\sigma\int_0^\infty e^{-\sigma x}x^\rho dx\Big)$$
$$=O(\sigma^{-\rho})\qquad(\sigma\to+0).$$

Now if we put

1. Unless $\limsup_{0<T\to\infty}\frac{1}{T^\rho}\int_0^T \phi(x)dx$ vanishes, a_p should not vanish; hence we assume here $a_p\neq 0$. 2. We notice that this assumption is of no essential restriction.

$$(18) \qquad f(s)=\int_0^\infty e^{-sx}d\alpha(x),$$

where $s=\sigma+it$, $f(s)$ will be an analytic function of s for $R(s)\geqq\varepsilon>0$, on account of the absolute and uniform convergence of the integral of the second member in (18). Thus we obtain

Theorem 4.1. *Under the hypothesis of* (17), *if* $\sigma\geqq\varepsilon>0$, ε *being arbitrarily given,*

$$f(\sigma)=\int_0^\infty e^{-\sigma x}d\alpha(x)$$

converges absolutely and uniformly, and

$$(19) \qquad f(\sigma)=O(\sigma^{-\rho}), \qquad \sigma\to+0.$$

Furthermore if $f(s)$, *defined by the expression* (18), *is analytically continuable on to the imaginary axis, then the origin* $s=0$ *becomes at most an isolated essential singular point. Especially, if* $f(s)$ *has a pole of order* p *at* $s=0$, *then it will be necessary that* $\rho=p$.

For the simple case when $\rho=1$, there may exist the following theorem:

Theorem 4.2. *Let* $\alpha(x)$ *be positive and increasing for* $0<x<\infty$. *Then*

$$(20) \qquad \lim_{N\to\infty}\frac{\alpha(N)}{N}=\lim_{\sigma\to+0}\sigma\int_0^\infty e^{-\sigma x}d\alpha(x)$$

in the sense that if either side of (20) *exists, the other side exists and assumes the same value.*

To prove this, let us use the transformation (T). Moreover if we put

$$\gamma(\xi)=\alpha(e^\xi)e^{-\xi}+\int_0^\xi e^{-t}\alpha(e^t)dt,$$
$$d\gamma(\xi)=e^{-\xi}d\alpha(e^\xi),$$

then the expression (20) may be written

$$(20') \qquad \lim_{\eta\to\infty}\int_{-\infty}^\infty K_1(\eta-\xi)d\gamma(\xi)=\lim_{\eta\to\infty}\int_{-\infty}^\infty K_2(\eta-\xi)d\gamma(\xi),$$

where $K_1(\xi)=0 \quad (-\infty<\xi<0)$; $\quad K_1(\xi)=e^{-\xi} \quad (0<\xi<\infty)$,
$K_2(\xi)=e^{-\xi}e^{-e^{-\xi}} \quad (-\infty<\xi<\infty)$.

Now if either side of (20′) exists, it follows easily that

$$\limsup_{(\eta)}\int_\eta^{\eta+1}d\gamma(\xi)$$

is bounded for the reason that $\gamma(\xi)$ is monotone increasing.

Therefore the general Tauberian theorem may be applied to the proof for equality of (20′), quite likely as that of (8) in theorem 3.1. Thus we are able to establish our present theorem.

Remark. If we put

$$f_\lambda(\sigma)=\int_0^\lambda e^{-\sigma x}d\alpha(x)$$

for any finite λ and for every $\sigma \geqq \varepsilon > 0$, it will be seen that the positive function of σ, $f_\lambda(\sigma)$, tends to zero as $\sigma \to \infty$ and is a convex function; therefore the limit

$$\lim_{\lambda\to\infty} f_\lambda(\sigma)=f(\sigma)$$

is also a convex function of σ and besides $f(\sigma)\to 0$ with $\sigma\to\infty$.

Hence we can conclude that

$$\lim_{\sigma\to+0} f(\sigma)=\text{finite, or infinite.}$$

Now in the sequel we are able to generalize theorem 4.2 as follows:

Theorem 4.3. *Let $\alpha(x)$ be of limited variation over any finite range and let $\alpha^+(x)$, $\alpha^-(x)$ be the positive and the negative variations of $\alpha(x)$ over $(0, x)$. Suppose that either of the order-conditions*

$$\alpha^+(x)=O(x) \quad or \quad \alpha^-(x)=O(x)$$

as $x\to\infty$ be satisfied. Then

$$(20_1) \qquad \lim_{x\to\infty}\frac{\alpha(x)}{x}=\lim_{\sigma\to+0}\sigma\int_0^\infty e^{-\sigma x}d\alpha(x)$$

in the sense that if either side of (20_1) exists, the other side exists and assumes the same value.

Proof. Let formally

$$f(\sigma)=\int_0^\infty e^{-\sigma x}d\alpha(x),$$

$$f^+(\sigma)=\int_0^\infty e^{-\sigma x}d\alpha^+(x), \quad f^-(\sigma)=\int_0^\infty e^{-\sigma x}d\alpha^-(x).$$

Let us assume that the integral

$$\int_0^\infty e^{-\sigma x}d\alpha(x)$$

converges absolutely for $0<\sigma<\delta$, and besides the limit

$$(*) \qquad \lim_{\sigma\to+0}\sigma\int_0^\infty e^{-\sigma x}d\alpha(x)=A \quad \text{(finite determinate)}$$

exists.

Now if either of the order-conditions when $x\to\infty$ is satisfied, either of these two integrals

$$\int_0^\infty e^{-\sigma x}d\alpha^+(x) \quad \text{or} \quad \int_0^\infty e^{-\sigma x}d\alpha^-(x)$$

may converge absolutely for $0<\sigma<\delta$. Thus it may be written that

$$f(\sigma)=f^+(\sigma)-f^-(\sigma).$$

For the present case, conveniently, let us suppose

$$\alpha^+(x)=O(x).$$

Then it will be seen that for any value x_0

$$-\varepsilon+\liminf_{x_0\leqq x}\frac{\alpha^+(x)}{x}\leqq\sigma\int_0^\infty e^{-\sigma x}d\alpha^+(x)$$
$$\leqq\limsup_{x_0\leqq x}\frac{\alpha^+(x)}{x}+\varepsilon',$$

where ε, $\varepsilon'\to+0$ with $\sigma\to+0$. That is, $\sigma f^+(\sigma)$ remains finite as σ tends to $+0$. Hence

$$\overline{\lim_{\sigma\to+0}}\,\sigma\int_0^\infty e^{-\sigma x}d\alpha^-(x)=\overline{\lim_{\sigma\to+0}}\,\sigma f^-(\sigma)$$
$$\leqq\overline{\lim_{\sigma\to+0}}\,\sigma f^+(\sigma)-\underline{\lim_{\sigma\to+0}}\,\sigma f(\sigma)$$
$$=\overline{\lim_{\sigma\to+0}}\,\sigma f^+(\sigma)-A.$$

Now inasmuch as

$$\alpha(x)=\alpha^+(x)-\alpha^-(x)$$
$$d\alpha(x)=d\alpha^+(x)-d\alpha^-(x),$$

if we put $d\gamma(\xi)=e^{-\xi}d\alpha(e^\xi)$

$$=d\gamma^+(\xi)-d\gamma^-(\xi),$$
$$d\gamma^+(\xi)=e^{-\xi}d\alpha^+(e^\xi),\quad d\gamma^-(\xi)=e^{-\xi}d\alpha^-(e^\xi),$$

then by using the same kernels $K_1(\xi)$, $K_2(\xi)$ as in the proof of theorem 4.2,

$$\sigma f(\sigma)=\int_{-\infty}^\infty K_2(\eta-\xi)d\gamma(\xi),$$
$$\sigma f^+(\sigma)=\int_{-\infty}^\infty K_2(\eta-\xi)d\gamma^+(\xi),$$
$$\sigma f^-(\sigma)=\int_{-\infty}^\infty K_2(\eta-\xi)d\gamma^-(\xi).$$

Therefore

$$\overline{\lim_{\eta\to\infty}}\int_{-\infty}^\infty K_2(\eta-\xi)d\gamma^+(\xi)\geqq\overline{\lim_{\eta\to\infty}}\int_\eta^{\eta+1}K_2(\eta-\xi)d\gamma^+(\xi)$$
$$\geqq\min_{-1\leq u\leq 0}K_2(u)\,\overline{\lim_{\eta\to\infty}}\int_\eta^{\eta+1}d\gamma^+(\xi).$$

Thus we see that for $-\infty<\eta<\infty$

$$\int_\eta^{\eta+1}d\gamma^+(\xi)$$

is bounded. Quite similarly

$$\int_\eta^{\eta+1}d\gamma^-(\xi)$$

is also bounded. Hence

$$\int^{\eta+1}|d\gamma(\xi)|$$

is bounded for $-\infty<\eta<\infty$.

In the sequel, we assume

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}=A$$

instead of (*). However

$$\overline{\lim_{x\to\infty}}\frac{a^-(x)}{x}\leqq\overline{\lim_{x\to\infty}}\frac{a^+(x)}{x}-\underline{\lim_{x\to\infty}}\frac{a(x)}{x}$$
$$=\overline{\lim_{x\to\infty}}\frac{a^+(x)}{x}-A,$$

and
$$\frac{a(x)}{x}=\int_{-\infty}^{\infty}K_1(\eta-\xi)d\gamma(\xi),$$
$$\frac{a^+(x)}{x}=\int_{-\infty}^{\infty}K_1(\eta-\xi)d\gamma^+(\xi),$$
$$\frac{a^-(x)}{x}=\int_{-\infty}^{\infty}K_1(\eta-\xi)d\gamma^-(\xi).$$

Here if we repeat the same argument as above, we may also obtain that

$$\int_{\eta}^{\eta+1}|d\gamma(\xi)|$$

is bounded for $-\infty<\eta<\infty$.

Thus by the Tauberian theorem,

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_1(\eta-\xi)d\gamma(\xi)=\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_2(\eta-\xi)d\gamma(\xi)$$

in the sense that if either side exists, the other side exists; that is,

$$\lim_{x\to\infty}\frac{a(x)}{x}=\lim_{\sigma\to+0}\sigma\int_0^{\infty}e^{-\sigma x}da(x). \qquad \text{Q. E. D.}$$

5. In the present section we shall discuss the extension of the above-stated results to the integral of Mellin's type.

Let $a(x)$ be defined over $(1,\infty)$ and a monotone increasing function, and let

$$(17') \qquad \int_{1+0}^{x}da(x)=O(x^{\rho}), \qquad \rho\geqq 0,$$

as $x\to\infty$. Here let us assume—what is no essential restriction that—$a(1+0)=a_0>0$.

Then if we integrate by parts,

$$\int_{1+0}^{x}t^{-\sigma-1}da(t)=\Big[t^{-\sigma-1}a(t)\Big]_{1+0}^{x}+(\sigma+1)\int_{1+0}^{x}t^{-\sigma-2}a(t)dt$$
$$=O(x^{-\sigma-1+\rho})+O\left(\int_{1+0}^{x}t^{-\sigma-2+\rho}dt\right)+O(1),$$

as $x\to\infty$. Thus we have

$$(21)\qquad \begin{cases}\displaystyle\int_{1+0}^{x}t^{-\sigma-1}da(t)=O(x^{-\sigma-1+\rho})+O(1), & \text{if } -\sigma-1+\rho\neq 0;\\ \displaystyle\int_{1+0}^{x}t^{-\sigma-1}da(t)=O(\log x)+O(1), & \text{if } -\sigma-1+\rho=0.\end{cases}$$

From (21) we can make the following statement:

Theorem 5.1. *Under the hypothesis of* (17′)

(22) $$f(1+\sigma)=\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha(x)$$

converges absolutely and uniformly for $\sigma \geqq \rho-1+\varepsilon$; *and in general*

(23) $$f(1+\sigma)=O\left(\frac{1}{\sigma-\rho+1}\right), \qquad \sigma\to\rho-1+0.$$

Accordingly the function of a complex variable $s=\sigma+it$, $f(s)$, *which is defined by the expression*

(24) $$f(s)=\int_{1+0}^{\infty} x^{-s}d\alpha(x)$$

will be an analytic function of s *for* $R(s)>\rho$.

Thus, in view of (23), we have also a corollary to the above theorem.

Corollary. *From the same assumptions made for* theorem 5.1 *it is shown that if* $f(s)$ *defined by* (24) *is analytically continuable on to* $R(s)=\rho$ *and regular on the line* $R(s)=\rho$, *except for* $s=\rho$, *then there are no singularities possible except a simple pole at* $s=\rho$, *for* $R(s)\geqq\rho$.

We shall set forth our argument only for the special case when $\rho=1$.

Assume that

(25) $$\lim_{\sigma\to+0}\sigma f(1+\sigma)=A.$$

Let us put

(26) $$\gamma(\xi)=\alpha(e^{\xi})e^{-\xi}+\int_{0}^{\xi} e^{-\xi}\alpha(e^{\xi})d\xi$$
$$\gamma(+0)=\alpha_0$$
$$d\gamma(\xi)=e^{-\xi}d\alpha(e^{\xi})$$

and

(27) $$\rho(\xi)=\gamma(e^{\xi})e^{-\xi}+\int_{-\infty}^{\xi} e^{-\xi}\gamma(e^{\xi})d\xi$$
$$d\rho(\xi)=e^{-\xi}d\gamma(e^{\xi}).$$

Then by the substitution (T) and by (26), (27), the expression (25) may be written

(25′) $$A=\lim_{\sigma\to+0}\sigma\int_{0}^{\infty} e^{-\sigma\xi}d\gamma(\xi)$$
$$=\lim_{\eta\to\infty}\int_{-\infty}^{\infty} e^{-(\eta-\xi)}e^{-e^{-(\eta-\xi)}}d\rho(\xi)$$
$$=\lim_{\eta\to\infty}\int_{-\infty}^{\infty} K_2(\eta-\xi)d\rho(\xi),$$

where $K_2(\xi)=e^{-\xi}e^{-e^{-\xi}} \qquad (-\infty<\xi<\infty)$.

Since $\rho(\xi)$ is a monotone increasing function, from (25′)

$$A\geqq\limsup_{(\eta)}\int_{\eta}^{\eta+1} K_2(\eta-\xi)d\rho(\xi)$$

$$\geqq \operatorname*{Min}_{-1<u<0} |K_2(u)| \limsup_{(\eta)} \int_\eta^{\eta+1} d\rho(\xi)$$
$$= ee^e \limsup_{(\eta)} \int_\eta^{\eta+1} d\rho(\xi).$$

From this we may readily conclude that for $-\infty < x < \infty$ there exists a number K such that

$$\int_x^{x+1} d\rho(\xi) < K.$$

Now let us put

$$K_1(\xi) = 0 \quad (-\infty < \xi < 0); \qquad K_1(\xi) = e^{-\xi} \quad (0 < \xi < \infty).$$

Then by using the Tauberian theorem as in the proof of theorem (3.1), and on account of (27)

$$A = \lim_{\eta\to\infty} \int_{-\infty}^{\infty} K_1(\eta-\xi) d\rho(\xi)$$
$$= \lim_{\eta\to\infty} \int_{-\infty}^{\eta} e^{-(\eta-\xi)} e^{-\xi} d\gamma(e^\xi)$$
$$= \lim_{\eta\to\infty} e^{-\eta} \int_{-\infty}^{\eta} d\gamma(e^\xi)$$
$$= \lim_{\eta\to\infty} \frac{\gamma(e^\eta)}{e^\eta}.$$

Thus in conclusion we have proved

Theorem 5.2. *Let* $f(1+\sigma)$ *be defined by* (22), *and let us assume* (25). *Then*

$$A = \lim_{\sigma\to+0} \sigma f(1+\sigma) = \lim_{N\to\infty} \frac{\gamma(N)}{N}, \tag{28}$$

where $\gamma(x)$ *is defined by the expression* (26). *And vice versa.*

In the sequel we shall show the following theorem. It reads

Theorem 5.3. *Under the hypotheses of* theorem 5.2 *we can conclude*

$$\lim_{N\to\infty} \frac{\gamma(N)}{N} = \lim_{N\to\infty} \frac{\alpha(N)}{N} = A. \tag{29}$$

In the first place, we wish to prove that if the limit

$$\lim_{N\to\infty} \frac{\gamma(N)}{N} = A$$

exists, the other limit

$$\lim_{N\to\infty} \frac{\alpha(N)}{N} = A$$

may follow.

Now it is easily proved that

$$\alpha(x) = O(x), \quad \text{as } x \to \infty.$$

Because, if on the contrary

$$\frac{\alpha(x)}{x} \nless K \qquad (K \text{ being some finite constant})$$

for all x over $(0, \infty)$, for some finite point, say x_0, it should be $\alpha(x_0)/x_0 = \infty$, that is, $\alpha(x_0) = \infty$; but this contradicts the monotoneness of $\alpha(x)$; hence it must be $\alpha(x)/x \to \infty$ with $x \to \infty$. In other words, for any given great number $g > 0$, there may exist x_0 such that for all $x \geqq x_0$

$$\frac{\alpha(e^x)}{e^x} > g.$$

This result leads us to the following contradiction: from (26)

$$\frac{\gamma(x)}{x} = \frac{\alpha(e^x)}{xe^x} + \frac{1}{x}\left[\int_0^{x_0} + \int_{x_0}^{x}\right] e^{-t}\alpha(e^t)dt$$

$$\geqq \frac{\alpha(e^x)}{xe^x} + \frac{1}{x}\int_0^{x_0} e^{-t}\alpha(e^t)dt + \frac{x - x_0}{x} g,$$

here let $x \to \infty$, then

$$\lim_{x\to\infty} \frac{\gamma(x)}{x} = \infty. \qquad \text{Q. E. D.}$$

Now let us put $\quad \beta(x) = x^{-1}\alpha(x),$
then we have

$$\beta(x+h) - \beta(x) \geqq -\frac{h}{x}K, \qquad \alpha(x)/x < K;$$

accordingly for any finite h

$$\lim_{x\to\infty} \{\beta(x+h) - \beta(x)\} \geqq 0.$$

This shows that, for any sufficiently large x, $\beta(x)$ is non-decreasing. Thus if we combine this and the above stated fact,

$$\lim_{x\to\infty} \frac{\alpha(x)}{x} = A',$$

A' being a finite determinate constant.

On the other hand,

$$A = \lim_{x\to\infty} \frac{\gamma(x)}{x} = \lim_{x\to\infty} \frac{\alpha(e^x)}{xe^x} + \lim_{x\to\infty} \frac{1}{x}\int_0^x \frac{\alpha(e^t)}{e^t}dt$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^x \frac{\alpha(e^t)}{e^t}dt.$$

However since for every positive number ε, however small, there may exist x_0 such that for all $x \geqq x_0$

$$\left|\frac{\alpha(e^x)}{e^x} - A'\right| < \varepsilon.$$

Hence $\quad \left|\frac{1}{x}\int_0^x \left\{\frac{\alpha(e^t)}{e^t} - A'\right\}dt\right| \leqq \frac{1}{x}\left[\int_0^{x_0} + \int_{x_0}^{x}\right]\left|\frac{\alpha(e^t)}{e^t} - A'\right|dt$

$$\leqq \frac{1}{x}\int_0^{x_0}\left|\frac{\alpha(e^t)}{e^t} - A'\right| \cdot dt + \frac{x - x_0}{x}\varepsilon.$$

Consequently $A=A'^{1}$. Thus we have established theorem 5.3.[2]

Now if we combine theorem 5.2 and 5.3, we obtain the theorem as follows:

Theorem 5.4. *Let* $\alpha(x)$ *be a monotone increasing function. Then if either of limits*

$$\lim_{N\to\infty}\frac{\alpha(N)}{N} \quad or \quad \lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\alpha(x)$$

exists, the other limit exists and assumes the same value.

Furthermore if we combine theorems 4.2, 5.2, 5.3 and 5.4, it will be clearly concluded that

Theorem 5.5. *Let* $\alpha(x)$ *be a monotone increasing function. Then if any one of these three limits*

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}$$

$$\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\alpha(x)$$

$$\lim_{\sigma\to+0}\sigma\int_{0}^{\infty}e^{-\sigma x}d\alpha(x)$$

exists, all these limits exist and have the same value.

Now in the sequel, we are going to generalize the theorem last obtained by using theorem 4.3. It reads as follows:

Theorem 5.6. *Under hypotheses of* theorem 4.3, *the conclusion of* theorem 5.5 *is also true.*

In order to prove this, it is sufficient to see that, under the same hypotheses we gave in theorem 4.3, theorems 5.2, 5.3 are also true.

Let us assume that

$$\int_{1+0}^{\infty}x^{-\sigma-1}d\alpha(x)$$

converges absolutely for $0<\sigma<\delta$; and

$$\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\alpha(x)=A.$$

Now formally let

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1. This is also proved as follows: From reason which (i) $\alpha(x)/x$ is bounded over the infinite range $0\ldots\ldots\infty$; (ii) since $\alpha(x)$ is monotone increasing, $\alpha(x)/x$ has only a denumerable infinite of many discontinuous points at most over the infinite range, the function $\alpha(x)/x$ may be integrable over any finite range in Riemann's sense. Hence after the method of finding the limit of indeterminate forms in ordinary calculus,

$$A=\lim_{x\to\infty}\frac{\gamma(x)}{x}=\lim_{x\to\infty}\frac{1}{x}\int_{0}^{x}\frac{\alpha(e^{t})}{e^{t}}dt=\lim_{x\to\infty}\frac{\alpha(e^{x})}{e^{x}}.$$

2. Conversely if the limit $\lim_{x\to\infty}\alpha(x)/x=A$ exists, from (26) it follows directly that the limit $\lim_{x\to\infty}\gamma(x)/x$ exists and assumes the same value.

$$f(\sigma+1)=\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha(x),$$
$$f^{+}(\sigma+1)=\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha^{+}(x),$$
$$f^{-}(\sigma+1)=\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha^{-}(x).$$

But by hypothesis, we can really write as follows:

$$f(\sigma+1)=f^{+}(\sigma+1)-f^{-}(\sigma+1)$$

for $0<\sigma<\delta$. If we choose either of given two order-conditions, e. g.,

$$\alpha^{+}(x)=O(x),$$

then
$$-\varepsilon+\liminf_{x_0\leqq x}\frac{\alpha^{+}(x)}{x}\leqq\sigma\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha^{+}(x)$$
$$\leqq\limsup_{x_0\leqq x}\frac{\alpha^{+}(x)}{x}+\varepsilon',$$

ε, ε' being any given positive quantities such as $\varepsilon, \varepsilon'\to 0$ with $\sigma\to+0$. From this inequality, it follows

$$\overline{\lim_{\eta\to\infty}}\int_{-\infty}^{\infty} K_2(\eta-\xi)d\rho^{+}(\xi)<\infty,$$
$$\overline{\lim_{\eta\to\infty}}\int_{-\infty}^{\infty} K_2(\eta-\xi)d\rho^{-}(\xi)<\infty,$$

where $\rho^{+}(\xi)$, $\rho^{-}(\xi)$ are defined as follows:

$$d\rho(\xi)=e^{-\xi}d\gamma(e^{\xi})$$
$$=d\rho^{+}(\xi)-d\rho^{-}(\xi),$$
$$d\rho^{+}(\xi)=e^{-\xi}d\gamma^{+}(e^{\xi}),\quad d\rho^{-}(\xi)=e^{-\xi}d\gamma^{-}(e^{\xi}).$$

Thus if we repeat the argument which is stated in the process of proof of theorem 4.3, we conclude that for $-\infty<x<\infty$

$$\int_{x}^{x+1}|d\rho(\xi)|<\infty.$$

In the sequel, conversely if we assume that

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}=A,$$

then from the hypothesis: $\alpha^{+}(x)=O(x)$,

$$\overline{\lim_{x\to\infty}}\frac{\alpha^{-}(x)}{x}<\infty.$$

While, on account of the construction of $\alpha^{+}(x)$, $\alpha^{-}(x)$

$$\frac{\gamma^{\pm}(x)}{x}=\frac{\alpha^{\pm}(e^{x})}{e^{x}}\,\frac{1}{x}+\frac{1}{x}\int_{0}^{x}e^{-x}\alpha^{\pm}(e^{x})dx\,;$$

hence we have

$$\overline{\lim_{x\to\infty}}\frac{\gamma^{\pm}(x)}{x}<\infty.$$

However using the kernel $K_1(\xi)$,

$$\frac{\gamma^+(x)}{x}=\int_{-\infty}^{\infty}K_1(\eta-\xi)d\rho^+(\xi).$$

$$\frac{\gamma^-(x)}{x}=\int_{-\infty}^{\infty}K_1(\eta-\xi)d\rho^-(\xi).$$

Thus we have

$$\int_x^{x+1}|d\rho(\xi)|<\infty$$

for $-\infty<x<\infty$.

Therefore again the Tauberian theorem may be applied; that is, if either of the limits

$$\lim_{x\to\infty}\frac{\gamma(x)}{x} \quad \text{or} \quad \lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\alpha(x)$$

exists, then the other limit exists and assumes the same value.

Now we shall give an application, for which Mr. Karamata has proved by an excellent method, that is, it reads:

Let (i) $\sum_{n=0}^{\infty}a_n x^n$

be convergent for $|x|<1$; (ii) $a_n\geqq 0$ $(n=0, 1, 2, \ldots\ldots)$;

(iii) $\sum_{n=0}^{\infty}a_n x^n\sim\frac{1}{1-x}$

as x tends $1-0$ along the real axis. Then

$a_0+a_1+a_2+\ldots\ldots+a_n\sim n.$ (*Hardy-Littlewood Theorem*)

To prove this, we shall take the following definitions:

$$\begin{cases} \alpha(x+0)-\alpha(x-0)=\begin{cases} a_n, & [x=n] \\ 0, & [x\neq n] \end{cases} \\ \alpha(+0)\equiv a_0, \\ x=e^{-\sigma} \quad [0<\sigma<\infty]. \end{cases}$$

Then the series $\sum_{n=0}^{\infty}a_n x^n$ may be written

$$\int_{+0}^{\infty}e^{-\sigma x}d\alpha(x);$$

and the condition (iii) may be replaced by

$$\lim_{\sigma\to+0}\sigma\int_{+0}^{\infty}e^{-\sigma x}d\alpha(x)=1.$$

Hence from theorem 5.3 our required conclusion directly follows.

6. Suppose that $\alpha(x)$ may satisfy (17′) and $f(s)$ may be defined by (24), furthermore analytic for $R(s)>\rho$. Before setting forth our argument we must notice that $|f(s)|$ is bounded as $R(s)\to\infty$.

For inasmuch as

$$(\sigma+1)\int_{1+0}^{x_0} x^{-\sigma-2}\alpha(x)dx < (\sigma+1)\max_{1<x\leqq x_0}\alpha(x)\int_{1+0}^{x_0} x^{-\sigma-2}dx$$

$$=\alpha(x_0)\left\{\frac{\sigma+1}{\sigma+1}-\frac{\sigma+1}{\sigma+1}\frac{1}{x_0^{\sigma+1}}\right\}\longrightarrow\alpha(x_0)$$

$$(\sigma+1)\int_{x_0}^{\infty} x^{-\sigma-2}\alpha(x)dx < (\sigma+1)K\int_{x_0}^{\infty} x^{-\sigma-2+\rho}dx$$

$$=K\frac{(\sigma+1)}{\sigma+1-\rho}x_0^{-\sigma-1+\rho}\longrightarrow 0,$$

as $\sigma\to\infty$, we have

$$\lim_{\sigma\to\infty}\int_{1+0}^{\infty} x^{-\sigma-1}d\alpha(x)\leqq-\alpha(1+0)+\alpha(x_0).$$

Consequently our purpose will be reached:

$$\lim_{R(s)\to\infty}|f(s)|\leqq\alpha(x_0)-\alpha(1+0).$$

Now we give here a weak hypothesis concerning the order of magnitude of $f(s)$:

(30) $\qquad f(s)=O(e^{C|t|})$

for some finite C, uniformly for $R(s)>\rho$.

But by means of a simple modification it is easily shown that the order-condition (30) may be replaced by what

(30′) $\qquad f(s)=O(e^{\varepsilon|t|})$

as $t\to\infty$ for every positive ε.

To simplify our argument, let us suppose that $f(s)$ is real for real s. Now if we put

(31) $\qquad F_\varepsilon(s)=e^{i\varepsilon s}f(s),$

(32) $\qquad \Omega_\varepsilon(\sigma)=\max_{0\leq t<\infty}|F_\varepsilon(\sigma+it)|,$

(33) $\qquad \Omega(\sigma)=\max_{0\leq t<\infty}|f(\sigma+it)|,$

it will be clearly seen that $F_\varepsilon(s)$ is analytic for $R(s)>\rho$ and for every positive ε; furthermore on account of the order-condition (30′)

$$F_\varepsilon(\sigma+it)=O(1)$$

as $t\to\infty$, uniformly for $R(s)>\rho$. Hence $\Omega_\varepsilon(\sigma)$ is bounded uniformly for $\sigma>\rho$, accordingly $\log\Omega_\varepsilon(\sigma)$ is a convex function of σ;[1] while from (32), (35) it may be derived that

$$\lim_{\varepsilon\to 0}\Omega_\varepsilon(\sigma)=\Omega(\sigma).$$

Therefore $\log\Omega(\sigma)$ will be also a convex function of σ.[2] Thus from this fact and boundedness at $\sigma\to\infty$, stated at first of the present section, the function $\Omega(\sigma)$ increases as σ approaches to $\rho+0$.

Thus in view of (23) we have incidentally proved:

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1. Doetch, Loc. cit.. 2. Montel, Loc. cit..

Theorem 6.1. *If $a(x)$ satisfies (17′), and $f(s)$ defined by (24) is analytic for $R(s)>\rho$ and moreover bears the order-condition of magnitude*

$$f(s)=O(e^{C|t|})$$

as $t\to\infty$ for some finite C, uniformly for $R(s)>\rho$, then

$$\lim_{R(s)\to\rho+0} f(s)=\infty.$$

And besides if $f(s)$ is analytically continuable on to $R(s)=\rho$, and regular on the line $R(s)=\rho$, except $s=\rho$, it has a pole of order one at $s=\rho$.

Now let us return to the simple case $\rho=1$.

As (25) always holds for this case, provided $f(s)$ is supposed to satisfy conditions of theorem 6.1, it yields us the following theorem:

Theorem 6.2. *If we take $\rho=1$ in* theorem 6.1, *we have*

$$(34)\qquad \lim_{x\to\infty}\frac{a(x)}{x}=\lim_{\sigma\to+0}\sigma f(1+\sigma).$$

7. Now we are able to generalize from the preceding results as follows:

Here we notice that the condition concerning the order of magnitude of $a(x)$ as $x\to\infty$ (17′) may be reduced to the simple case $\rho=1$. Instead of $a(x)$, take

$$(35)\qquad a_1(x)\equiv a(x^{\frac{1}{\rho}})\qquad(\rho\geqq 0).$$

Then evidently $a_1(x)=O(x)$. Furthermore

$$\int_{1+0}^{\infty}x^{-s}da_1(x)=\int_{1+0}^{\infty}x^{-\rho s}da(x)=f(\rho s).$$

Hence if we put

$$(36)\qquad f_1(s)=\int_{1+0}^{\infty}x^{-s}da_1(x),$$

we have

$$(36')\qquad f_1(s)=f(\rho s)\quad\text{or}\quad f(s)=f_1(s/\rho).$$

Thus if $\lim_{\sigma+1\to\rho+0}(\sigma+1-\rho)f(1+\sigma)=A$,

from (36′) we obtain

$$A=\lim_{\sigma+1\to\rho+0}(\sigma+1-\rho)f(1+\sigma)$$
$$=\lim_{\rho(\sigma+1)\to\rho+0}\{\rho(\sigma+1)-\rho\}f(\rho(1+\sigma))$$
$$=\lim_{\sigma\to+0}\rho\sigma f_1(1+\sigma)=\rho A_1,$$

where $\lim_{\sigma\to+0}\sigma f_1(1+\sigma)=A_1$.

Thus with the help of theorem 5.5 and (35) we have established the following theorem:

Theorem 7.1. *Let $a(x)$ be positive, non-decreasing and satisfy* (17′). *Then*

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}=A/\rho.$$

Here we must add a few of theorems closely allied to this theorem.

Theorem 7.1$_1$. *Let $\alpha(x)$ be a monotone increasing function. Assume that*

$$\lim_{\sigma\to\rho-1+0}(\sigma-\rho+1)f(1+\sigma)=A,$$

where $f(s)=\int_{1+0}^{\infty}x^{-s}d\alpha(x)$. *Then the limit*

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}$$

exists, and the limit is equal to A/ρ.

For if we take $\alpha_1(x)$ as like (35), $f_1(s)$ defined by (36) may be expressed by (36$'$). Hence we have

$$\lim_{\sigma\to+0}\sigma f_1(1+\sigma)=A/\rho\,;$$

however theorem 5.3 asserts that

$$\frac{A}{\rho}=\lim_{x\to\infty}\frac{\alpha_1(x)}{x}=\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}.$$

Combining theorems 7.1 and 7.1$_1$, we may establish:

Theorem 7.1$_2$. *In order that the limit*

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}$$

may exist, it is necessary and sufficient that the limit

$$\lim_{\sigma\to\rho-1+0}(\sigma-\rho+1)f(1+\sigma)$$

may exist; furthermore there exists

$$\rho\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}=\lim_{\sigma\to\rho-1+0}(\sigma-\rho+1)f(1+\sigma).$$

In the sequel we suppose that $\alpha(x)$ is not always a monotone increasing function, but of limited variation over any finite interval.

Let the function (which is defined by (24))

$$f(s)=\int_{1+0}^{\infty}x^{-s}d\alpha(x)$$

converge absolutely for $R(s)>\rho$. Now if we assume that $\alpha(1+0)=0$ and $\alpha^+(x)$, $\alpha^-(x)$ be respectively the positive and negative variations of $\alpha(x)$ in $(1, x)$, then

$$\alpha(x)=\alpha^+(x)-\alpha^-(x),$$
$$d\alpha(x)=d\alpha^+(x)-d\alpha^-(x),$$

where $\alpha^+(x)$, $\alpha^-(x)$ are monotone increasing functions.

Let us put

$$f^+(s)=\int_{1+0}^{\infty} x^{-s}d\alpha^+(x), \quad f^-(s)=\int_{1+0}^{\infty} x^{-s}d\alpha^-(x),$$

and assume that

$$f(s)=f^+(s)-f^-(s).$$

If any two of three limits

$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f(\sigma+1),$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^+(\sigma+1),$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^-(\sigma+1)$$

exist, then another limit exists. Hence assume

$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f(\sigma+1)=A,$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^+(\gamma+1)=A^+,$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^-(\sigma+1)=A^-,$$

then $\quad A=A^+-A^-.$

However from theorem 7.1

$$\lim_{x\to\infty}\frac{\alpha^+(a)}{x^\rho}=\frac{A^+}{\rho}, \quad \lim_{x\to\infty}\frac{\alpha^-(x)}{x^\rho}=\frac{A^-}{\rho}.$$

Therefore we have

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}=\lim_{x\to\infty}\frac{\alpha^+(x)}{x^\rho}-\lim_{x\to\infty}\frac{\alpha^-(x)}{x^\rho}$$
$$=A^+/\rho-A^-/\rho=A/\rho.$$

We state this result in the following:

Theorem 7.13. *Let $\alpha(x)$ be of limited variation over any finite interval. Let us assume*

$$f(s)=f^+(s)-f^-(s)$$

for $R(s)>\rho$, and that any two of these three limits

$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f(\sigma+1),$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^+(\sigma+1),$$
$$\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f^-(\sigma+1)$$

exist. Then

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}=\frac{1}{\rho}\lim_{\sigma\to\rho-1+0} (\sigma+1-\rho)f(\sigma+1).$$

With a slight modification of detail, we have proved:

Theorem 7.14. *Let $\alpha(x)$ be of limited variation over any finite range. Let either of order-conditions*

$$\alpha^+(x)=O(x^\rho) \quad \text{or} \quad \alpha^-(x)=O(x^\rho)$$

as $x\to\infty$ be satisfied. Then

$$\lim_{x\to\infty}\frac{a(x)}{x^\rho}=\frac{1}{\rho}\lim_{\sigma\to\rho-1+0}(\sigma+1-\rho)f(\sigma+1)$$

in the sense that if either side exists, the other side exists and assumes the same value.

Because, it will be sufficient to prove this, if we take $a_1(x)$ of (35) in the place of $a(x)$ and apply theorem 5.6.

Now we again consider the function $\varphi(s)$ defined by (18):

$$\varphi(s)=\int_{+0}^{\infty}e^{-sx}da(x).$$

Let us put

$$f(s)=\varphi(s-1)\quad \text{i. e.}\quad \varphi(s)=f(s+1).$$

Then we have

$$\begin{aligned}f(s)&=\int_{+0}^{\infty}e^{-sx}e^{x}da(x)\\&=\int_{1+0}^{\infty}x^{-s}xda(\log x)\\&=\int_{1+0}^{\infty}x^{-s}d\beta(x),\end{aligned}$$

where $\beta(x)$ is defined by

$$\beta(x)=\int_{1+0}^{x}xda(\log x).$$

Now if the limit $\lim\limits_{\sigma\to+0}\sigma\varphi(\sigma)=A$ exists, then we have

$$A=\lim_{\sigma\to+0}\sigma\varphi(\sigma)=\lim_{\sigma\to+0}\sigma f(1+\sigma).$$

Inasmuch as $\beta(x)$ is a monotone increasing function, if we combine the last obtained result and theorem 5.5,

$$A=\lim_{N\to\infty}\frac{a(N)}{N}=\lim_{N\to\infty}\frac{\beta(N)}{N}.$$

Thus we have established the following:

Theorem 7.2. *If the limit*

$$\lim_{\sigma\to+0}\sigma\varphi(\sigma)=A$$

exists, then

$$\lim_{x\to\infty}\frac{a(x)}{x}=\lim_{x\to\infty}\frac{1}{x}\int_{1+0}^{x}xda(\log x)=A.$$

Now in the sequel, if we use $a_1(x)$ of (35) instead of $\beta(x)$,

$$da_1(x)=xda_2(\log x),\quad da(x)=x_\rho da_2(\rho\log x),$$

and from (36)

$$f_1(s)=\varphi_1(s-1)\quad\text{or}\quad \varphi_1(s)=f_1(s+1),$$

$$\varphi_1(s)=\int_{+0}^{\infty}e^{-sx}da_2(x).$$

Hence from above theorem 7.2

$$\lim_{x\to\infty}\frac{a_1(x)}{x}=\lim_{x\to\infty}\frac{a(x)}{x^\rho}=\lim_{x\to\infty}\frac{a_2(x)}{x}.$$

Thus we have proved:

Theorem 7.3. *If the limit*

$$\lim_{x\to\infty}\frac{a(x)}{x^\rho}$$

exists, then the limit

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x e^{-x}da(e^{\frac{x}{\rho}})$$

exists, and assumes the same value.

Now from a little different view-point we shall set forth our argument.

Let $\gamma(x)$ be a monotone increasing function, and let

(37) $$\int_{1+0}^{\infty}x^{-\sigma-1}d\gamma(x)=\phi(\sigma+1)$$

converge for $\sigma>0$. Furthermore let

(38) $$-\int_{1+0}^{\infty}x^{-\sigma-1}\log x d\gamma(x)=f(\sigma+1)$$

converge for $\sigma>0$ and let

(39) $$\lim_{\sigma\to+0}\sigma f(\sigma+1)=-A$$

hold. If we put

$$a(x)=\int_{1+0}^{x}\log\xi d\gamma(\xi)\quad da(x)=\log x d\gamma(x),$$

then (38) may be written

$$-f(\sigma+1)=\int_{1+0}^{\infty}x^{-\sigma-1}da(x),$$

where $a(x)$ is a monotone increasing function.

Inasmuch as $f(\sigma+1)$ satisfies the hypothesis (25), by means of theorem 5.3

$$-\lim_{\sigma\to+0}\sigma f(\sigma+1)=A$$

$$=\lim_{x\to\infty}\frac{a(x)}{x}=\lim_{N\to\infty}\frac{1}{N}\int_{1+0}^{N}\log x d\gamma(x).$$

While, as will be seen

$$f(\sigma+1)=-\phi'(\sigma+1),$$

if we put

$$-\sigma f(1+\sigma)-A=(\sigma+1-1)g(1+\sigma)=\sigma g(1+\sigma)$$
$$\lim_{\sigma\to+0}g(1+\sigma)=g(1+0),$$

we have

$$F(1+\sigma)=e^{\psi(1+\sigma)}\sigma^{A},$$

where $F'(1+\sigma)/F(1+\sigma)=-g(1+\sigma)$ and $\lim_{\sigma\to+0} F(1+\sigma)=F(1+0)\neq 0$.

Thus we have proved:

Theorem 7.4. *Let $\phi(\sigma+1)$ and $f(\sigma+1)$ be defined as in* (37) *and* (38) *and converge for $\sigma>0$. Let* (39) *hold. Then*

(40) $$F(\sigma+1)=e^{\phi(\sigma+1)}\sigma^A$$

is a positive and analytic function of a real variable σ for $\sigma>0$ and $\lim_{\sigma\to+0} F(\sigma+1)=F(1+0)\neq 0$. *Furthermore*

(41) $$A=\lim_{N\to\infty}\frac{1}{N}\int_{1+0}^{N}\log x\,d\gamma(x).$$

Now if (37) is convergent for $\sigma>0$,

(37′) $$\int_{1+0}^{\infty} x^{-s}d\gamma(x)=\phi(s)$$

will be an analytic function of a complex variable $s=\sigma+it$ over the interior of the right-half plane to the line $R(s)=1$. On account of the theory of functions of a complex variable

$$-\phi'(s)=\int_{1+0}^{\infty} x^{-s}\log x\,d\gamma(x)$$

for $R(s)>1$, accordingly $-\phi'(1+\sigma)$ exists always for $\sigma>0$. Thus a theorem closely allied to the above theorem 7.4 follows:

Theorem 7.5. *Let $\phi(\sigma+1)$ converge for $\sigma>0$. Let $F(\sigma+1)$, defined by* (40), *be analytic for $\sigma\geqq 0$ and let $F(1+0)\neq 0$. Then* (41) *is also true.*[1]

8. Now we are in the position to devote the present section to theorems concerning the prime number theorem.

Landau's Lemma. Let

$$F(x)=\sum_{n=1}^{\infty} a_n n^{-x} \qquad [R(x)>1],$$

and let $a_n\geqq 0, \qquad [n=1, 2, 3,\ldots\ldots].$

Let $F(x)$ be analytically continuable on to $R(x)=1$, and let it there be free from singularities, except for a pole of order one at $x=1$, with principal part $A/(x-1)$. Let there be some finite a for which

(0) $$F(x)=O(|x|^a)$$

in the right half-plane $R(x)\geqq 1$. Then

$$A=\lim_{n\to\infty}\frac{1}{n}\sum_{1}^{n} a_k.$$

The Hardy-Littlewood Theorem. Let λ_n be an increasing real

1. Compare this result with Wiener's theorem **19** which is stated at § 19, pp. 135–136 in his book. There A must be limited as $0<A<2\frac{1}{2}$; while our theorem is true without any such restriction concerning A.

sequence. Let (i) the series $\sum a_n\lambda_n^{-s}$ be absolutely convergent for $R(s)>\sigma_0>0$; (ii) the function $F(s)$ defined by the series be regular for $R(s)>c$ where $0<c\leqq\sigma_0$ and continuous for $R(s)\geqq c$, except for a simple pole with residue g at $s=c$;

(iii) $\qquad F(s)=O(e^{C|t|})$

for some finite C, uniformly for $\sigma\geqq c$;

(iv) $\qquad \lambda_n/\lambda_{n-1}\to 1$;

(v) a_n be real, and satisfy one of the inequalities

$$a_n>-K\lambda_n^{c-1}(\lambda_n-\lambda_{n-1})\,;\quad a_n<K\lambda_n^{c-1}(\lambda_n-\lambda_{n-1}),$$

or complex, and of the form

$$O\{\lambda_n^{c-1}(\lambda_n-\lambda_{n-1})\}.$$

Then $\qquad A_n=a_1+a_2\ldots\ldots+a_n\sim g\lambda_n^c/c.$

Ikehara's theorem. Let $\alpha(x)$ be a monotone increasing function, and let

$$\int_{1+0}^{\infty}x^{-u}d\alpha(x)=f(u)$$

converge for $R(u)>1$. Let

$$f(u)-\frac{A}{u-1}=g(u)$$

converge over any finite interval of the line $R(u)=1$ uniformly to a finite limit as $R(u)\to 1+0$. Then

$$\alpha(N)\sim N.A$$

as $N\to\infty$.

From the point of that which Ikehara's theorem is true without any hypothesis concerning the order of magnitude of $F(s)$, as like (o) and (iii), Wiener makes the following statement in his book (The Fourier Integral, Camb. 1933) "Landau's lemma cannot be regarded as in a satisfactory state until either (o) is shown to be the weakest restriction on the order of $F(x)$ that is sufficient to guarantee the truth of the lemma, or else the true restriction is found."

In generalizing the Hardy-Littlewood theorem Wiener replaced (iv) by (iv′), $\lambda_n-\lambda_{n-1}$ is bounded. It reads as follows:

Wiener's Extension. In the Hardy-Littlewood theorem, conditions (i), (ii), (iv′), and (v), without (iii), are sufficient for the conclusion.

Now in view of theorem 5.6, the hypothesis in Ikehara's theorem will be involved in it as a special case. For, our theorem is true, merely if it is assumed that the limit

$$\lim_{R(u)\to 1+0} uf(u)$$

exists as u tends to $1+0$ along the real axis.

From theorem 7.1_4, the Hardy-Littlewood theorem can be directly generalized better than Wiener's can, as will be evident from the following:

Theorem 8.1. *Let* λ_n *be an increasing sequence. Let* (I) *the series* $\sum a_n\lambda_n^{-s}$ *be absolutely convergent for* $R(s)>c$; (II) *along the real axis,* $\lim_{R(s)\to c}(s-c)F(s)=g$, $F(s)$ *being defined by the series*; (III) a_n *be real, and satisfy one of the inequalities*

$$a_n>-K\lambda_n^{c-1}(\lambda_n-\lambda_{n-1})\,;\quad a_n<K\lambda_n^{c-1}(\lambda_n-\lambda_{n-1}),$$

or complex, and of the form

$$O\{\lambda_n^{c-1}(\lambda_n-\lambda_{n-1})\}.$$

Then $\quad A_n=a_1+a_2+\ldots\ldots+a_n\sim g\lambda_n^c/c.$

Let us put

$$\alpha(x+0)-\alpha(x-0)=\begin{cases}a_n, & x=\lambda_n\\ 0, & x\neq\lambda_n.\end{cases}$$

Then $\sum a_n\lambda_n^{-s}=\int_{1+0}^{\infty}x^{-s}d\alpha(x)$, assuming $\lambda_1\geqq 1$. Hence we have

$$F(s)=\int_{1+0}^{\infty}x^{-s}d\alpha(x),\qquad R(s)>c.$$

Now we prefer

$$a_n<K\lambda_n^{c-1}(\lambda_n-\lambda_{n-1})$$

from the propositions (III),

$$\frac{1}{\lambda_n^c}\int_{1+0}^{\lambda_n}d\alpha^+(x)<K\sum_{\nu=1}^{n}\left(\frac{\lambda_\nu}{\lambda_n}\right)^{c-1}\frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_n}$$
$$\leqq K\frac{1}{\lambda_n}\sum_{\nu=1}^{n}(\lambda_\nu-\lambda_{\nu-1})$$
$$=K\left(1-\frac{\lambda_1}{\lambda_n}\right)<K.$$

Hence theorem 7.1_4 may be applied. However since the proposition (II) asserts that

$$\lim_{\sigma\to c-1+0}(\sigma+1-c)\int_{1+0}^{\infty}x^{-\sigma-1}\alpha(x)=g,$$

we can conclude

$$\frac{1}{\lambda_n^c}\int_{1+0}^{\lambda_n}d\alpha(x)\sim g/c.$$

Thus we have completed the proof of the present theorem.

Here we should notice that the Wiener's Extension Theorem of the Hardy-Littlewood Theorem is true even without such weakened hypothesis (iv). While we should also notice that on account of theorem 6.1, the following theorem may be established:

Theorem 8.2. *When in* theorem 8.1 *the function $F(s)$ defined by the series analytically continuable on to $R(s)=c$ and regular on the line $R(s)=c$, except $s=c$, even if we replace* (II) *by*

$$F(s)=O(e^{C|t|}), \quad s=\sigma+it,$$

for some finite C, uniformly for $\sigma>c$, the conclusion is also true.

9. In the present section, we shall discuss the positive and monotone increasing function $\alpha(x)$ such as

(42) $$\alpha(x)=O(1)$$

as $x\to\infty$.

Now in order to reduce this case to the former, we introduce such a function $\beta(x)$ as follows:

$$\beta(x)=x\alpha(x),$$
$$d\beta(x)=xd\alpha(x)+\alpha(x)dx.$$

Inasmuch as $\beta(x)$ is clearly positive and monotone increasing and

$$\beta(x)=O(x), \quad x\to\infty,$$

all these integrals

$$\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x), \qquad \int_{1+0}^{\infty}x^{-\sigma-1}\alpha(x)dx, \qquad \int_{1+0}^{\infty}x^{-\sigma-1}d\beta(x)$$

may exist, and converge absolutely and uniformly for $\sigma>0$. Hence we have

$$\int_{1+0}^{\infty}x^{-\sigma-1}d\beta(x)=\int_{1+0}^{\infty}x^{-\sigma-1}\{xd\alpha(x)\}+\int_{1+0}^{\infty}x^{-\sigma-1}\alpha(x)dx.$$

Inasmuch as from the hypothesis (42) we can integrate by parts,

$$\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x)=\sigma\int_{1+0}^{\infty}x^{-\sigma-1}\alpha(x)dx=f(\sigma).$$

Therefore the above stated equation may be written as follows:

$$\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\beta(x)=\sigma f(\sigma)+f(\sigma).$$

Now if we assume that

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x)=\lim_{\sigma\to+0}f(\sigma)=A,$$

A being finite, then we obtain

$$\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\beta(x)=\lim_{\sigma\to+0}\sigma f(\sigma)+\lim_{\sigma\to+0}f(\sigma)$$
$$=0+A.$$

Hence by theorem 5.3

$$A=\lim_{x\to\infty}\frac{\beta(x)}{x}=\lim_{x\to\infty}\alpha(x).$$

Conversely if we allow the existence of the limit

$$\lim_{x\to\infty}\alpha(x)=A,$$

then in the first place

$$\lim_{x\to\infty}\frac{\beta(x)}{x}=A;$$

in the second place, by theorem 5.4,

$$\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty}x^{-\sigma-1}d\beta(x)=A;$$

in the third place

$$\lim_{\sigma\to+0}(\sigma+1)f(\sigma)$$

exists, accordingly $\lim\limits_{\sigma\to+0}f(\sigma)$ exists, and moreover

$$A=\lim_{\sigma\to+0}f(\sigma)=\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x).$$

Thus we have proved:

Theorem 9.1. *Let $\alpha(x)$ be a positive and monotone increasing function and let $\alpha(x)=O(1)$, as $x\to\infty$. Let*

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x)=A$$

hold. Then

$$\alpha(x)\sim A.$$

Conversely, let

$$\alpha(x)\sim A$$

hold, as $x\to\infty$. Then we have

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x)=A.$$

Now we shall generalize this theorem. We only assume that $\alpha(x)$ be of limited variation over any finite interval. In the same way as in the section 8, if we introduce the positive and negative variations $\alpha^{+}(x)$ and $\alpha^{-}(x)$ of $\alpha(x)$ in $(1, x)$, then

$$\alpha(x)=\alpha^{+}(x)-\alpha^{-}(x).$$

Now formally let us write

$$f(\sigma)=f^{+}(\sigma)-f^{-}(\sigma),$$

$$f(\sigma)=\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x),$$

$$f^{+}(\sigma)=\int_{1+0}^{\infty}x^{-\sigma}d\alpha^{+}(x),\quad f^{-}(\sigma)=\int_{1+0}^{\infty}x^{-\sigma}d\alpha^{-}(x).$$

However if any two of these three above written integrals exist, the other necessarily exists; besides if any two of these three limits, which follow, exist

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha(x)=f(+0)=A,$$

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty}x^{-\sigma}d\alpha^{+}(x)=f^{+}(+0)=A^{+},$$

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty} x^{-\sigma}d\alpha^{-}(x)=f^{-}(+0)=A^{-},$$

the other also exists; accordingly $A=A^{+}-A^{-}$.

As $\alpha^{+}(x)$, $\alpha^{-}(x)$ are monotone increasing functions, theorem 9.1 may be applied to $f^{+}(\sigma)$, $f^{-}(\sigma)$. Hence

$$\alpha^{+}(x)\sim A^{+},\quad \alpha^{-}(x)\sim A^{-},$$

as $x\to\infty$; accordingly

$$\alpha(x)\sim A^{+}-A^{-}=A.$$

Thus we have concluded that

Theorem 9.2. *If any two of these three integrals*

$$\int_{1+0}^{\infty} x^{-\sigma}d\alpha(x),$$

$$\int_{1+0}^{\infty} x^{-\sigma}d\alpha^{+}(x),$$

$$\int_{1+0}^{\infty} x^{-\sigma}d\alpha^{-}(x)$$

exist, and if any two of these three limits

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty} x^{-\sigma}d\alpha(x),$$

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty} x^{-\sigma}d\alpha^{+}(x),$$

$$\lim_{\sigma\to+0}\int_{1+0}^{\infty} x^{-\sigma}d\alpha^{-}(x)$$

exist, then

$$\alpha(x)\sim A,$$

where $\alpha(x)=\alpha^{+}(x)-\alpha^{-}(x)$, *and*

$$A=\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty} x^{-\sigma}d\alpha(x). \tag{43}$$

With the help of theorem 5.6, we have the following

Theorem 9.3. *Let either*

$$\alpha^{+}(x)=O(1)\quad or\quad \alpha^{-}(x)=O(1)$$

hold. Then

$$\lim_{x\to\infty}\alpha(x)=\lim_{\sigma\to+0}\sigma\int_{1+0}^{\infty} x^{-\sigma}d\alpha(x)$$

in the sense that if either side exists, the other side exists and assumes the same value.

10. We shall again return to section **4.**

Let $\alpha(x)$ be a monotone increasing function, and let

$$\int_{0}^{x} d\alpha(x)=O(x^{\rho}) \tag{17}$$

as $x\to\infty$. By means of (19)

$$f(\sigma)=O(\sigma^{-\rho}), \qquad [\sigma\to 0]$$

here the function $f(\sigma)$ is defined by the integral

$$f(\sigma)=\int_0^\infty e^{-\sigma x}d\alpha(x).$$

Now we are on the point of discussing the limit

$$\lim_{\sigma\to+0}\sigma^\rho f(\sigma),$$

as to whether it may exist or not.

To learn this, let us take the transformation (35):

$$\alpha_1(x)=\alpha(x^{\frac{1}{\rho}}), \quad \alpha_1(x)=O(x).$$

Then $f(\sigma)=\int_0^\infty e^{-\sigma x^{\frac{1}{\rho}}}d\alpha(x^{\frac{1}{\rho}})=\int_0^\infty e^{-\sigma x^{\frac{1}{\rho}}}d\alpha_1(x)$

and $\lim_{\sigma\to+0}\sigma^\rho f(\sigma)=\lim_{\sigma\to+0}\sigma f(\sigma^{\frac{1}{\rho}})=\lim_{\sigma\to+0}\sigma\int_0^\infty e^{-\sigma^{\frac{1}{\rho}}x^{\frac{1}{\rho}}}d\alpha_1(x).$

If we put $\sigma^{\frac{1}{\rho}}=e^{-\eta}, \quad x^{\frac{1}{\rho}}=e^{\xi}, \quad T=e^{\eta},$

the above integral expression may be written as follows:

$$(X_1) \qquad \sigma f(\sigma^{\frac{1}{\rho}})=\int_{-\infty}^\infty e^{-\rho(\eta-\xi)}e^{-e^{-(\eta-\xi)}}d\gamma_1(\xi),$$

$$d\gamma_1(\xi)=e^{-\rho\xi}d\alpha_1(e^{\rho\xi}).$$

On the other hand

$$(X_2) \qquad \frac{\alpha(T)}{T^\rho}=\frac{\alpha_1(T^\rho)}{T^\rho}$$

$$=e^{-\rho\eta}\int_{-\infty}^{\eta}d\alpha_1(e^{\rho\xi})$$

$$=\int_{-\infty}^{\eta}e^{-\rho(\eta-\xi)}d\gamma_1(\xi).$$

If we introduce such $K_2^*(\xi)$, $K_1^*(\xi)$, defined as

$$K_2^*(\xi)=e^{-\rho\xi}e^{-e^{-\xi}}, \quad (-\infty<\xi<\infty)$$

$$K_1^*(\xi)=0, \ (-\infty<\xi<0); \quad =e^{-\rho\xi}, \quad (0<\xi<\infty)$$

then (X_1) and (X_2) may be written respectively as follows:

$$\sigma f(\sigma^{1/\rho})=\int_{-\infty}^\infty K_2^*(\eta-\xi)d\gamma_1(\xi),$$

$$\frac{\alpha(T)}{T^\rho}=\int_{-\infty}^\infty K_1^*(\eta-\xi)d\gamma_1(\xi).$$

Fourier transforms of K_2^*, K_1^* are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty K_2^*(\xi)e^{-iu\xi}d\xi=\frac{1}{\sqrt{2\pi}}\int_0^\infty x^{\rho+iu-1}e^{-x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\Gamma(\rho+iu)\neq 0, \quad [\rho>0]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty K_1^*(\xi)e^{-iu\xi}d\xi=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(\rho+iu)\xi}d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho+iu}\neq 0.$$

Since $\gamma_1(\xi)$ is monotone increasing,

$$\overline{\lim_{\eta\to\infty}}\int_{-\infty}^{\infty}K_{1,2}^{*}(\eta-\xi)d\gamma_1(\xi)\geqq \underset{-1<u<0}{\mathrm{Min}}\,|K_{1,2}(u)|\int_{\eta}^{\eta+1}d\gamma_1(\xi).$$

Thus from the Tauberian Theorem it is evident that the two following statements are equivalent:

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_2^*(\eta-\xi)d\gamma_1(\xi)=A\int_{-\infty}^{\infty}K_2^*(\xi)d\xi$$

and

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_1^*(\eta-\xi)d\gamma_1(\xi)=A\int_{-\infty}^{\infty}K_1^*(\xi)d\xi.$$

However we see that

$$\int_{-\infty}^{\infty}K_2^*(\xi)d\xi=\int_0^{\infty}x^{\rho-1}e^{-x}dx=\Gamma(\rho),$$

$$\int_{-\infty}^{\infty}K_1^*(\xi)d\xi=\int_0^{\infty}e^{-\rho\xi}d\xi=\frac{1}{\rho}.$$

Consequently we obtain:

Theorem 10.1. *Let $a(x)$ be monotone increasing, and let the condition*

(17) $\quad a(x)=O(x^{\rho})$

be satisfied. Then the following two statements will be equivalent:

$$\lim_{\sigma\to+0}\sigma^{\rho}f(\sigma)=B$$

and

$$\lim_{x\to\infty}\frac{a(x)}{x^{\rho}}=\frac{B}{\rho\Gamma(\rho)}.$$

Now especially in the place of K_2^*, let us take a K_1, defined as follows:

$$K_1(\xi)=0\ (-\infty<\xi<0);\quad =e^{-\xi}\ (0<\xi<\infty);$$

accordingly there may exist identical relations between

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_1(\eta-\xi)d\gamma_1(\xi)=A\int_{-\infty}^{\infty}K_1(\xi)d\xi=A$$

and

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}K_2^*(\eta-\xi)d\gamma_1(\xi)=B=A\Gamma(\rho).$$

Hence we have

$$B/\Gamma(\rho)=\lim_{\eta\to\infty}\int_{-\infty}^{\eta}e^{-(\eta-\xi)}d\gamma_1(\xi)$$

$$=\lim_{\eta\to\infty}e^{-\eta}\int_0^{e^{\eta}}x^{(1-\rho)}da_1(x^{\rho}).$$

Thus we conclude that

Theorem 10.2. *Under hypotheses of theorem 10.1, the two following statements are identical:*

$$\lim_{\sigma\to+0}\sigma^\rho f(\sigma)=B$$

and
$$\lim_{N\to\infty}\frac{1}{N}\int_0^N x^{(1-\rho)}d\alpha(x)=\frac{B}{\Gamma(\rho)}.$$

11. In the sequel, we shall show the relation between Hölder's sum and Tauberian theorems.

Let $\alpha(x)$ be defined over $(0, \infty)$ and $\alpha(+0)\equiv 0$. Besides let it be a monotone increasing function. Now let us consider the following repeated integral

$$A_n(x)=\frac{1}{x}\int_0^x\frac{dx_n}{x_n}\int_0^{x_n}\frac{dx_{n-1}}{x_{n-1}}\int_0^{x_{n-1}}\cdots\cdots\frac{dx_2}{x_2}\int_0^{x_2}\frac{dx_1}{x_1}\int_0^{x_1}d\alpha(t).$$

By using the Dirichlet's transformation with respect to repeated integrals,

$$A_n(x)=\frac{1}{x}\int_0^x\frac{(\log x-\log t)^n}{n!}d\alpha(t)$$

and
$$A_{n+1}(x)=\frac{1}{x}\int_0^x A_n(t)dt.$$

When we generalize and take any real number λ, such as >-1, in the place of n of the integrand of the above integral, we have

$$H_\lambda(x)=\frac{1}{x}\int_0^x\frac{(\log x-\log t)^\lambda}{\Gamma(\lambda+1)}d\alpha(t). \tag{44}$$

Clearly, if $\lambda=n$, $H_\lambda(x)$ coincides $A_n(x)$. We are now able to write (44) in another form; that is, if we put

$$t=e^\xi,\quad x=e^\eta,$$

$$H^*_\lambda(\eta)=e^{-\eta}\int_{-\infty}^\eta\frac{(\eta-\xi)^\lambda}{\Gamma(\lambda+1)}d\alpha^*(\xi), \tag{45}$$

where $H^*_\lambda(\eta)\equiv H_\lambda(e^\eta)$, $\alpha^*(\xi)\equiv\alpha(e^\xi)$.

Furthermore, let us put

$$M_\lambda(\xi)=0,\ (-\infty<\xi<0);\quad =\frac{1}{\Gamma(\lambda+1)}\xi^\lambda e^{-\xi},\quad (0<\xi<\infty),$$

and
$$\gamma(\xi)=\int_{-\infty}^\xi e^{-\xi}d\alpha(e^\xi),\quad \gamma(-\infty)=0.$$

Then (45) may be written

$$H^*_\lambda(\eta)=\int_{-\infty}^\infty M_\lambda(\eta-\xi)d\gamma(\xi). \tag{46}$$

Now $M_\lambda(\xi)\geqq 0$ for $-\infty<\xi<\infty$, and

$$\int_{-\infty}^\infty M_\lambda(\xi)d\xi=\frac{1}{\Gamma(\lambda+1)}\int_0^\infty\xi^\lambda e^{-\xi}d\xi=1,$$

$$\Gamma(\lambda+1)\int_{-\infty}^\infty M_\lambda(\xi)e^{-iu\xi}d\xi=\int_0^\infty\xi^\lambda e^{-(1+iu)\xi}d\xi$$
$$=\int_0^\infty\frac{t^\lambda}{(1+iu)^{\lambda+1}}e^{-t}dt=\frac{\Gamma(\lambda+1)}{(1+iu)^{\lambda+1}}.$$

The last result may be obtained by Cauchy's theorem as follows:

$$\int_0^\infty \xi^\lambda e^{-(1+iu)\xi}d\xi=\lim_{\substack{\varepsilon\to 0\\ A\to\infty}}\int_\varepsilon^A \xi^\lambda e^{-(1+iu)\xi}d\xi$$

$$=\lim_{\substack{\varepsilon\to 0\\ A\to\infty}}\int_{\varepsilon(1+iu)}^{A(1+iu)}\frac{z^\lambda}{(1+iu)^{\lambda+1}}e^{-z}dz$$

$$=\frac{1}{(1+iu)^{\lambda+1}}\left\{\lim\int_{\varepsilon(1+iu)}^{A+iu\xi} z^\lambda e^{-z}dz+\lim\int_{A+iu\varepsilon}^{A+iuA} z^\lambda e^{-z}dz\right\}$$

$$=\frac{1}{(1+iu)^{\lambda+1}}\lim_{A\to\infty}\int_0^A \xi^\lambda e^{-\xi}d\xi.$$

Hence the Fourier transform of $M_\lambda(\xi)$ does not vanish for any real u. Thus if there exists the limit

$$\lim_{\eta\to\infty}\int_{-\infty}^\infty M_\lambda(\eta-\xi)d\gamma(\xi),$$

if we put $\lim_{\eta\to\infty}\int_{-\infty}^\infty M_\lambda(\eta-\xi)d\gamma(\xi)=A\int_{-\infty}^\infty M_\lambda(\xi)d\xi,$

from the Tauberian theorem, we can replace any function belonging to the class M,[1] into the place of $M_\lambda(\xi)$ of the above equation. For instance, instead of $M_\lambda(\xi)$, we can take the following functions:

(i) $K_1(\xi)=0\ (-\infty<\xi<0);\quad K_1(\xi)=e^{-\xi}\ (0<\xi<\infty),$

(ii) $K_2(\xi)=e^{-\xi}e^{-e^{-\xi}}\quad (-\infty<\xi<\infty).$

Thus we have:

Theorem 11.1. *Let $\alpha(x)$ be a monotone increasing function. Then all these statements are equivalent*:

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x\frac{(\log x-\log t)^\lambda}{\Gamma(\lambda+1)}d\alpha(t)=A,$$

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}=A,$$

$$\lim_{\sigma\to+0}\sigma\int_0^\infty e^{-\sigma x}d\alpha(x)=A.$$

Evidently we have also:

Theorem 11.2. *Under the same hypothesis of* theorem 4.3, *the conclusions of* theorem 11.1 *will be true.*

Now we shall introduce a new definition after Wiener[2]: he says that the mass-distribution determined by $\alpha(x)$ is "bounded below" when

(47) $$\int_y^{y+1}|d\alpha(x)|-\int_y^{y+1}d\alpha(x)\leqq N\qquad(-\infty<y<\infty)$$

and that it is "bounded above" when $-\alpha(x)$ determines a mass-distribution bounded below. Thus on account of this definition, we obtain:

1. Wiener, Loc. cit., p. 73. 2. See, Annals of Math., t. 33 (1932), especially in §6.

Theorem 11.3. *Let $\alpha(x)$ be a function of limited total variation over every finite interval. Let the mass-distribution determined by $\gamma(x)$ be bounded below or bounded above. Then the conclusions of* theorem 11.1 *will also be true.*

Now if we combine theorem 11.3 with the last obtained theorem in § 10, we have:

$$\lim_{x\to\infty}\frac{\alpha(x)}{x^\rho}=\lim_{x\to\infty}\frac{1}{x^\rho}\int_0^x\frac{(\log x-\log t)^\lambda}{\Gamma(\lambda+1)}d\alpha(t)$$

in the sense that if either side exists, the other side exists and assumes the same value. Of course, $\alpha(x)$ should be subject to the hypothesis of theorem 11.3.

Inasmuch as the condition that $\gamma(x)$ is bounded below or bounded above over $(-\infty, \infty)$ may be restated to read

$$\begin{aligned}\int_y^{y+1} e^{-x}\{|d\alpha(e^x)|-d\alpha(e^x)\}\quad &[=0 \text{ for } d\alpha(e^x)\geqq 0]\\ &\leqq 2\int_y^{y+1} e^{-x}|d\alpha(e^x)| \quad [\text{for } d\alpha(e^x)<0]\\ &=-2\int_y^{y+1} e^{-x}d\alpha(e^x)=-2\int_\eta^{e\eta} t^{-1}d\alpha(t) \quad [\eta>0]\\ &\leqq 2\operatorname*{Max}_{\eta\leq t\leq e\eta}\left\{-\alpha(t)\right\}\left(\frac{1}{e\eta}+\frac{1}{\eta}+\frac{1}{\eta}-\frac{1}{e\eta}\right);\end{aligned}$$

therefore $\alpha(\eta)/\eta>-N/4$.

Thus we obtain:

Corollary. *Let $\alpha(x)$ be of limited total variation over every finite interval. Let*

$$\alpha(\eta)/\eta>-N/4, \quad \text{or} \quad \alpha(\eta)/\eta<N/4. \quad [\eta>0]$$

Then the conclusions of theorem 11.1 *will be true.*

In the sequel, we shall say that the series $\sum_1^\infty c_n$ is summable in (H, λ) and the (H, λ)-sum, A, when the limit

$$\lim_{x\to\infty} H_\lambda(x) \quad \text{or} \quad \lim_{\eta\to\infty} H_\lambda^*(\eta)$$

tends to a finite number A. We shall prove:

Theorem 11.4. *Let $\{\nu_n\}$ be a given sequence, and let*

$$\nu_1>0, \quad \nu_n\uparrow\infty.$$

Let the series $\sum c_n$ be summable in (H, λ) and hold the (H, λ)-sum A. In the series $\sum c_n$, if c_n be real, let it be subject to one of the following inequalities

$$c_n<K(\nu_n-\nu_{n-1})\,; \quad c_n>-K(\nu_n-\nu_{n-1}),$$

or complex, let it be subject to that of form

$$c_n = O(\nu_n - \nu_{n-1}).$$

Then $\quad c_1 + c_2 + c_3 + \ldots\ldots + c_n \sim \nu_n A.$

To prove this, it is sufficient to notice that if we put

$$a(x) = \sum_{\nu_n \leq x} c_n,$$

$$a(x+0) - a(x-0) = c_n \quad (x = \nu_n)\,; = 0 \quad (x \neq \nu_n),$$

for the condition, for instance, $c_n < K(\nu_n - \nu_{n-1})$,

$$\frac{1}{\nu_n}\int_0^{\nu_n} da^+(x) \leqq K \sum_{k=1}^{n-1} \frac{\nu_{k+1} - \nu_k}{\nu_n} < K;$$

hence theorem 11.2 may be applied.

Now we shall consider

$$H^*_{\lambda,\mu}(\eta) = e^{-\eta}\int_{-\infty}^{\eta} \frac{(\eta-\xi)^\mu}{\Gamma(\mu+1)} e^{\xi} H^*_\lambda(\xi)d\xi, \quad [\mu > -1]$$

$$H_{\lambda,\mu}(x) = \frac{1}{x}\int_0^x \frac{(\log x - \log t)^\mu}{\Gamma(\mu+1)} H_\lambda(t)dt.$$

Here it is evident that

$$H^*_{\lambda,\mu}(\eta) = H_{\lambda,\mu}(x), \quad [x = e^\eta].$$

If we put $\quad h_\lambda(\xi) = \int_{-\infty}^{\xi} H^*_\lambda(\xi)d\xi,$

$$dh_\lambda(\xi) = H^*_\lambda(\xi)d\xi, \quad H^*_\lambda(-\infty) = 0.$$

However, since $h_\lambda(\xi)$ will satisfy (47), if

(48) $\qquad H^*_\lambda(y) > -N \quad \text{or} \quad H_\lambda(y) < N \quad (-\infty < y < \infty),$

if we combine this result with theorem 11.3, we are able to conclude:

Theorem 11.5. *Let* $H^*_\lambda(x)$ *be integrable, and let*

$$H^*_\lambda(x) > -N \quad \text{or} \quad H^*_\lambda(x) < N \quad (-\infty < x < \infty).$$

Then the following two statements are equivalent:

(i) $\qquad \lim_{\eta\to\infty} H^*_{\lambda,\mu}(\eta) = B\,;$

(ii) $\qquad \lim_{\eta\to\infty} e^{-\eta}\int_{-\infty}^{\eta} e^{\xi} H^*_\lambda(\xi)d\xi = \lim_{x\to\infty}\frac{1}{x}\int_0^x H_\lambda(t)dt = B.$

Since, if $\lambda > -1$, the function $H_\lambda(t)$ will be continuous for $t > 0$. We have:

Corollary. *Let the function* $H^*_\lambda(\eta)$ *be subject to the hypothesis of* theorem 11.5. *Let* $\lambda > -1$. *Then the following two statements will be coincident*:

(i′) $\qquad \lim_{\eta\to\infty} H^*_{\lambda,\mu}(\eta) = B\,;$

(ii′) $\qquad \lim_{x\to\infty} H_\lambda(x) = B.$

Let us put the expression (45) into the integrand under the integral sign of $H^*_{\lambda,\mu}(\eta)$. Then

$$H^*_{\lambda,\mu}(\eta)=e^{-\eta}\int_{-\infty}^{\eta}\frac{(\eta-\xi)^{\mu}}{\Gamma(\mu+1)}e^{\xi}H^*_{\lambda}(\xi)d\xi$$
$$=\lim_{A\to\infty}e^{-\eta}\int_{-A}^{\eta}\frac{(\eta-\xi)^{\mu}}{\Gamma(\mu+1)}d\xi\int_{-A}^{\xi}\frac{(\xi-\zeta)^{\lambda}}{\Gamma(\lambda+1)}da^*(\zeta)$$
$$=\lim_{A\to\infty}e^{-\eta}\int_{-A}^{\eta}da^*(\zeta)\int_{\zeta}^{\eta}\frac{(\eta-\xi)^{\mu}(\xi-\zeta)^{\lambda}}{\Gamma(\mu+1)\Gamma(\lambda+1)}d\xi$$
$$=e^{-\eta}\int_{-\infty}^{\eta}\frac{(\eta-\zeta)^{\mu+\lambda+1}}{\Gamma(\mu+1)\Gamma(\lambda+1)}da^*(\zeta)\cdot\int_0^1(1-t)^{\mu}t^{\lambda}dt.$$

Here we assume the possibility of these processes. In viewing

$$\int_0^1(1-t)^{\mu}t^{\lambda}dt=\frac{\Gamma(\mu+1)\Gamma(\lambda+1)}{\Gamma(\mu+\lambda+2)},$$

we have $\quad H^*_{\lambda,\mu}(\eta)=e^{-\eta}\int_{-\infty}^{\eta}\frac{(\eta-\zeta)^{\mu+\lambda+1}}{\Gamma(\mu+\lambda+2)}da^*(\zeta),$

that is,

(49) $\qquad H^*_{\lambda,\mu}(\eta)=H^*_{\lambda+\mu+1}(\eta).$

Thus it will be clear that, *for any two* $\lambda'>\lambda$, *if the limit* $\lim_{\eta\to\infty}H^*_{\lambda}(\eta)$ *exists, it follows that the limit* $\lim_{\eta\to\infty}H^*_{\lambda'}(\eta)$ *exists and assumes the same value; and moreover, if either of the inequalities*

$$H^*_{\lambda}(x)>-N \quad\text{or}\quad H^*_{\lambda}(x)<N \quad (-\infty<x<\infty)$$

be satisfied, then the two limits

$$\lim_{\eta\to\infty}H^*_{\lambda'}(\eta),\quad \lim_{\eta\to\infty}H^*_{\lambda}(\eta)$$

exist at the same time and hold the same value.

Now in the above stated argument, since we see that the integral

$$\int_0^{\infty}\xi^{z-1}e^{-\xi}d\xi$$

may express a regular and analytic function of z only except for $z=0, -1, -2, -3,\ldots\ldots$ in the whole z-plane, we can give generally complex values for λ, μ, except for $-1, -2, -3,\ldots\ldots$. Consequently, under a suitable complement, the Tauberian theorem may be applied there also.

In the sequel, we shall show some interesting theorems:

Theorem 11.6. *Let* $\{\nu_n\}$ *be such a given sequence as that of* theorem 11.4. *In the series* $\sum c_n$, *if*

$$c_1+c_2+c_3+\ldots\ldots+c_n\sim\nu_nA,\quad A\neq 0,$$

then the series $\sum_{n=1}^{\infty}\frac{c_n}{\nu_n}$ *should be divergent.*

To prove this, we define

$$a(x)=\sum_{\nu_n\leq x}c_n.$$

On account of the hypothesis, all these limits

$$\lim_{\eta\to\infty} H_\lambda^*(\eta)=\int_{-\infty}^{\eta} \frac{(\eta-\xi)^\lambda}{\Gamma(\lambda+1)} e^{-(\eta-\xi)}d\gamma(\xi)$$

$$\lambda=0,\ 1,\ 2,\ldots\ldots\ldots,\quad n,\ldots\ldots\ldots$$

$$\gamma(\xi)=\int^{\xi} e^{-\xi}da(e^{\xi}),$$

may exist, and assume the same limit A. However, for any η in $(-\infty, \infty)$, the series $\sum_{n=0}^{\infty}\frac{(\eta-\xi)^n}{n\,!}$ is uniformly convergent as to ξ over $(-\infty, \infty)$ and tends to $e^{(\eta-\xi)}$. Therefore

$$\sum_{n=0}^{\infty} H_n(\eta)=\sum_{n=0}^{\infty}\int_{-\infty}^{\eta} \frac{(\eta-\xi)^n}{\Gamma(n+1)} e^{-(\eta-\xi)}d\gamma(\xi)$$

$$=\int_{-\infty}^{\eta}\sum\frac{(\eta-\xi)^n}{n\,!}e^{-(\eta-\xi)}d\gamma(\xi)$$

$$=\int_{-\infty}^{\eta}d\gamma(\xi)=\int_{-\infty}^{\eta}e^{-\xi}da(e^{\xi})$$

$$=\int_{+0}^{e^{\xi}}t^{-1}da(t).$$

Hence $\quad \lim_{\eta\to\infty}\lim_{n\to\infty}\sum_{p=0}^{n}H_p(\eta)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{c_k}{\nu_k}.$

Now let the series in the left-hand side be convergent. Then for any given η_0 and n_0, there may be found a number n_0 such that for all $\eta\geqq\eta_0$ and for all $n\geqq n_0(\eta_0, \varepsilon)$

$$|H_n(\eta)|<\varepsilon.$$

But this contradicts

$$\lim_{\eta\to\infty} H_n(\eta)=A$$

for every n. Thus we have completed the proof.

Theorem 11.7. *Let* $\{\nu_n\}$ *be a sequence defined as that of* theorem 11.4. *Let the series* $\sum c_n$ *be summable in* (H, λ) *to* A. *If* c_n *be real, let it be subject to one of these inequalities*

$$c_n<K\frac{\nu_n-\nu_{n-1}}{\nu_n}\ ;\quad c_n>-K\frac{\nu_n-\nu_{n-1}}{\nu_n},$$

or complex, let it be subject to that form

$$c_n=O\left(\frac{\nu_n-\nu_{n-1}}{\nu_n}\right).$$

Then the series $\sum_{n=1}^{\infty}c_n$ *converges to the sum* A.

Proof. By means of the hypothesis, if we evaluate the positive variation of $a(x)$ over $(0, x)$,

$$[a^+(x)]_{x\leq\nu_n}=a^+(\nu_n)\leqq K\sum_{k=1}^{n-1}\frac{\nu_{k+1}-\nu_k}{\nu_{k+1}}<K.$$

Hence we have

$$e^{-\xi}\int_{-\infty}^{\infty} M_\lambda(\eta-\xi)d\overset{(+)}{\alpha^*}(\xi)=e^{-\eta}\int_{-\infty}^{\eta}\frac{(\eta-\xi)^{\lambda-1}}{\Gamma(\lambda)}\alpha(e^{\xi})e^{\xi}d\xi$$

$$<K\int_0^\infty t^{\lambda-1}e^{-t}dt$$

$$=K\Gamma(\lambda),\quad [\lambda>0].$$

However

$$0<\overline{\lim_{(\eta)}}\int_{-\infty}^{\infty} M_{\lambda-1}(\eta-\xi)\bar{\alpha}^*(\xi)d\xi$$

$$\leqq\overline{\lim_{(\eta)}}\int_{-\infty}^{\infty} M_{\lambda-1}(\eta-\xi)\bar{\alpha}^*(\xi)d\xi+\left|\overline{\lim_{(\eta)}}\int_{-\infty}^{\infty} M_{\lambda-1}(\eta-\xi)\alpha^*(\xi)d\xi\right|$$

$$\leqq K\Gamma(\lambda)+\left|\overline{\lim_{(\eta)}}\, H_\lambda^*(\eta)\right|,$$

where $M_{\lambda-1}(\xi)$ is as defined at the beginning of this section. From this we may readily conclude that, for $-\infty<\eta<\infty$,

$$\int_{\eta-2}^{\eta-1}\bar{\alpha}^*(\xi)d\xi$$

is bounded; hence in view of

$$\int_{\eta-2}^{\eta-1}\bar{\alpha}^*(\xi)d\xi>\bar{\alpha}^*(\eta-2),$$

we see that for $-\infty<\eta<\infty$ there exists an N such that

$$\bar{\alpha}^*(\eta)<N.$$

Consequently, these two limits

$$\lim_{\eta\to\infty}\overset{+}{\alpha}^*(\eta)\quad\text{and}\quad\lim_{\eta\to\infty}\bar{\alpha}^*(\eta)$$

exist, and accordingly the limit

$$\lim_{\eta\to\infty}\alpha^*(\eta)$$

exists.[1] Thus we have completed our theorem. Q. E. D.

Now we shall consider the following integral of a more general form than that of $H_\lambda(x)$:

$$G_\lambda(x)=\frac{1}{\varphi(x)}\int_0^x\frac{[\tau(x)-\tau(t)]^\lambda}{\Gamma(\lambda+1)}d\alpha(t),\tag{50}$$ [2]

where $\varphi(x)$, $\tau(x)$ are defined over $(0, \infty)$ so that they are positive, continuous and $\varphi(x)\uparrow\infty$, $\tau(x)\uparrow\infty$. Furthermore, we suppose that

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1. The conclusion that this limit assumes the value A may be proved as follows: by means of the Tauberian theorem

$$A=\lim_{\eta\to\infty}\eta^{-1}\int_{-\infty}^{\eta}\alpha^*(\xi)d\xi\,;$$

let $\lim_{\eta\to\infty}\alpha^*(\eta)=A'$, then $A=A'$ necessarily.

2. See T. Satô, On Abel's integral equation, Memo. of the Coll. of Sci. Kyoto Imp. Univ. Ser. A, Vol. XVIII, No. 2 (1935), pp. 63-78.

I. $$\lim_{x\to\infty}\frac{[\tau(x)]^{\mu}}{\varphi(x)}=B^{\mu} \quad (\neq 0),$$[1]

or

II. $$\lim_{x\to\infty}\frac{e^{\mu\tau(x)}}{\varphi(x)}=\overline{B}_{\mu} \quad (\neq 0).$$

Now let us take case I. Let us put

(51) $$G_{\lambda}^{(\mu)}(x)=[\tau(x)]^{-\mu}\int_0^x\frac{[\tau(x)-\tau(t)]^{\lambda}}{\Gamma(\lambda+1)}d\alpha(t).$$

Then the expression of (50) may be written

(52) $$G_{\lambda}(x)=\frac{[\tau(x)]^{\mu}}{\varphi(x)}G_{\lambda}^{(\mu)}(x).$$

Moreover, if we put

(T) $\tau(t)=e^{\xi}, \quad \tau(x)=e^{\eta},$

and

(53) $$\beta(\xi)=\int^{\tau^{-1}(e^{\xi})}[\tau(t)]^{\lambda-\mu}d\alpha(t)\,; \quad =0 \quad (\xi<\log\tau(+0)),$$

where $\tau^{-1}(t)$ means the inverse of $\tau(t)$, then (51) will become

(54) $$G_{\lambda}^{*(\mu)}(\eta)=\int_{-\infty}^{\eta}[1-e^{-(\eta-\xi)}]^{\lambda}e^{-(\mu-\lambda)(\eta-\xi)}d\beta(\xi),$$

here evidently $G_{\lambda}^{(\mu)}(x)=G_{\lambda}^{*(\mu)}(\eta)$.

We shall introduce the function

$$M_{\lambda,\mu}(\xi)=0 \quad (\xi<0)\,; \quad =(1-e^{-\xi})^{\lambda}e^{-(\mu-\lambda)\xi} \quad (0<\xi).$$

Clearly $M_{\lambda,\mu}(\xi)>0$ and its Fourier transform is

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^{\infty}(1-e^{-\xi})^{\lambda}e^{-(\mu-\lambda)\xi}e^{-iu\xi}d\xi&=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}(1-e^{-\xi})^{\lambda}e^{-(p+iu)\xi}d\xi\\&=\frac{1}{\sqrt{2\pi}}\int_0^1(1-t)^{\lambda}t^{p+iu-1}dt\\&=\frac{1}{\sqrt{2\pi}}\,\frac{\Gamma(\lambda+1)\Gamma(p+iu)}{\Gamma(p+\lambda+1+iu)}\\&=\frac{1}{\sqrt{2\pi}}\,\frac{\Gamma(\lambda+1)\Gamma(\mu-\lambda+iu)}{\Gamma(\mu+1+iu)},\end{aligned}$$

for all of which there is no real zero, if $\lambda>-1$, $\mu>-1$. Furthermore,

$$\int_{-\infty}^{\infty}M_{\lambda,\mu}(\xi)d\xi=\frac{\Gamma(\lambda+1)\Gamma(\mu-\lambda)}{\Gamma(\mu+1)}.$$

Now let

$$\lim_{x\to\infty}G_{\lambda}(x)=A.$$

Then from (52) and (54), it follows that

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1. For $\tau(x)=x$, $\varphi(x)=x^{\lambda}$, $\mu=\lambda$, this becomes the Riess' sum.

$$\lim_{x\to\infty} G_\lambda^{(\mu)}(x)=\lim_{\eta\to\infty} G_\lambda^{*(\mu)}(\eta)$$
$$=C_{\lambda,\mu}\int_{-\infty}^{\infty} M_{\lambda,\mu}(\xi)d\xi$$
$$=C_{\lambda,\mu}\frac{\Gamma(\lambda+1)\Gamma(\mu-\lambda)}{\Gamma(\mu+1)},$$

where

$$(55)\qquad C_{\lambda,\mu}=\frac{A\cdot\Gamma(\mu+1)}{B_\mu\Gamma(\lambda+1)\Gamma(\mu-\lambda)},\quad [\mu-\lambda>0].$$

Thus since, if $\alpha(x)$ is monotone, $\beta(\xi)$ will be also monotone, from the Tauberian theorem, we are able to conclude that the two following statements become equivalent:

$$\lim_{x\to\infty}\frac{\beta(x)}{x}=\lim_{\xi\to\infty}\frac{1}{\xi}\int_0^{\tau^{-1}(e^\xi)}\frac{d\alpha(t)}{[\tau(t)]^{\mu-\lambda}}=\lim_{\xi\to\infty}\frac{1}{\xi}\int_{\tau(+0)}^{e^\xi}\frac{d\alpha[\tau^{-1}(t)]}{t^{\mu-\lambda}}=C_{\lambda,\mu};$$

$$\lim_{x\to\infty} G_\lambda^{(\mu)}(x)=\lim_{\eta\to\infty} \overset{*}{G}{}_\lambda^{(\mu)}(\eta)=C_{\lambda,\mu}\frac{\Gamma(\lambda+1)\Gamma(\mu-\lambda)}{\Gamma(\mu+1)},\quad [\mu-\lambda>0].$$

More generally, in order to use the notion of the mass-distribution bounded below or above, introduced by N. Wiener, we consider the following evaluation for

$$\int_\eta^{\eta+1}|d\beta(\xi)|-\int_\eta^{\eta+1}d\beta(\xi)$$
$$=\int[\tau(t)]^{\lambda-\mu}\{|d\alpha(t)|-d\alpha(t)\}$$
$$\leqq -2\int[\tau(t)]^{\lambda-\mu}d\alpha(t)\quad [d\alpha(t)<0]$$
$$=-2\int_u^{eu}t^{\lambda-\mu}d\alpha[\tau^{-1}(t)]\quad \begin{bmatrix}\tau(+0)\leqq u<\infty\\ u=e^\eta\end{bmatrix}$$
$$=2\left[t^{\lambda-\mu}\{-\alpha[\tau^{-1}(t)]\}\Big|_u^{eu}-(\lambda-\mu)\int_u^{eu}t^{-\lambda-\mu-1}\{-\alpha[\tau^{-1}(t)]\}dt\right]$$
$$\leqq 2\operatorname*{Max}_{u\leq t\leq eu}\{-\alpha[\tau^{-1}(t)]\}\{(eu)^{\lambda-\mu}+u^{\lambda-\mu}-(eu)^{\lambda-\mu}+u^{\lambda-\mu}\}\quad [\lambda-\mu<0]$$
$$=4\operatorname*{Max}_{u\leq t\leq eu}\{-\alpha[\tau^{-1}(t)]\}u^{\lambda-\mu}.$$

Hence $\beta(\xi)$ will be bounded below, provided

$$\alpha[\tau^{-1}(u)]/u^{\mu-\lambda}=\frac{\alpha(t)}{[\tau(t)]^{\mu-\lambda}}>-N,\quad 0\leqq t<\infty,\quad \tau(+0)\leqq u<\infty.$$

Similarly, it will be seen that if

$$\alpha[\tau^{-1}(u)]/u^{\mu-\lambda}<N,$$

$\beta(\xi)$ will be bounded above.

Thus we obtain:

Theorem 11.8. *Let $\alpha(x)$ be of bounded variation over every finite interval, and let it be subject to one of these conditions, if real,*

$$\frac{\alpha(t)}{[\tau(t)]^{\mu-\lambda}} > -N, \quad \text{or} \quad < N;$$

or, if complex,

$$\alpha(t) = O([\tau(t)]^{\mu-\lambda}).$$

Then the two following statements are equivalent:

$$\lim_{x\to\infty} G_\lambda(x) = A;$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^{\tau^{-1}(e^x)} \frac{d\alpha(t)}{[\tau(t)]^{\mu-\lambda}} = \frac{A\cdot\Gamma(\mu+1)}{B_\mu\cdot\Gamma(\lambda+1)\Gamma(\mu-\lambda)},$$

provided $\mu > \lambda > -1$, *and*

$$\lim_{x\to\infty} \frac{[\tau(x)]^\mu}{\varphi(x)} = B_\mu.$$

In the sequel, we shall apply our argument to the remaining case where the hypothesis II is taken.

Let us put

$$(51') \qquad \overline{G}_\lambda^{(\mu)}(x) = e^{-\mu\tau(x)}\int_0^x \frac{[\tau(x)-\tau(t)]^\lambda}{\Gamma(\lambda+1)} d\alpha(t).$$

Then for the present case, corresponding to (52), the expression (50) may be written

$$(52') \qquad G_\lambda(x) = \frac{e^{\mu\tau(x)}}{\varphi(x)} \overline{G}_\lambda^{(\mu)}(x).$$

Now if we put

$$(\mathrm{T}') \qquad \tau(t) = \xi, \quad \tau(x) = \eta$$

and

$$(53') \qquad \overline{\beta}(\xi) = \int_0^{\tau^{-1}(\xi)} e^{-\mu\tau(t)} d\alpha(t); \quad = 0 \quad (\xi < \tau(+0)),$$

then (51′) will become

$$(54') \qquad \overset{*}{\overline{G}}{}_\lambda^{(\mu)}(\eta) = \int_{-\infty}^{\eta} \frac{(\eta-\xi)^\lambda e^{-\mu(\eta-\xi)}}{\Gamma(\lambda+1)} d\overline{\beta}(\xi).$$

Here evidently we have

$$\overline{G}_\lambda^{(\mu)}(x) = \overset{*}{\overline{G}}{}_\lambda^{(\mu)}(\eta).$$

Let $\quad \overline{M}_{\lambda,\mu}(\xi) = 0 \quad (\xi < 0); \quad = \dfrac{1}{\Gamma(\lambda+1)}\xi^\lambda e^{-\mu\xi} \quad (\xi > 0).$

The Fourier transform of $\overline{M}_{\lambda,\mu}(\xi)$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{M}_{\lambda,\mu}(\xi) e^{-iu\xi} d\xi = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\Gamma(\lambda+1)}\int_0^\infty \xi^\lambda e^{-(\mu+iu)\xi} d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{(\mu+iu)^{\lambda+1}}.$$

Thus, in viewing

$$\int_{-\infty}^{\infty} \overline{M}_{\lambda,\mu}(\xi) d\xi = \frac{1}{\mu^{\lambda+1}},$$

if we put

(55′) $\overline{C}_{\lambda,\mu}=A.\overline{B}_\mu\cdot\mu^{\lambda+1}$,

we have $\lim\limits_{\eta\to\infty}\int_{-\infty}^{\infty}\overline{M}_{\lambda,\mu}(\eta-\xi)d\overline{\beta}(\xi)=\overline{C}_{\lambda,\mu}\int_{-\infty}^{\infty}\overline{M}_{\lambda,\mu}(\xi)d\xi.$

However
$$\int_\eta^{\eta+1}|d\overline{\beta}(\xi)|-\int_\eta^{\eta+1}d\overline{\beta}(\xi)$$
$$=\int e^{-\mu\tau(t)}\{|da(t)|-da(t)\}$$
$$\leqq -2\int e^{-\mu\tau(t)}da(t)\quad [da(t)<0]$$
$$=-2\int_\eta^{\eta+1}e^{-\mu\xi}da[\tau^{-1}(\xi)]$$
$$=2\left[e^{-\mu\xi}\{-a[\tau^{-1}(\xi)]\}\Big|_\eta^{\eta+1}+\mu\int_\eta^{\eta+1}e^{-\mu\xi}\{-a[\tau^{-1}(\xi)]\}d\xi\right]$$
$$\leqq 2\max_{\eta\leq\xi\leq\eta+1}\{-a[\tau^{-1}(\xi)]\}[e^{-\mu(\eta+1)}+e^{-\mu\eta}+|e^{-\mu\eta}-e^{-\mu(\eta+1)}|]$$
$$=4\max_{\eta\leq\xi\leq\eta+1}\{-a[\tau^{-1}(\xi)]\}e^{-\mu\eta},\quad [\eta,\mu>0].$$

Therefore, it is sufficient that
$$a[\tau^{-1}(x)]/e^{\mu x}>-N$$
in order that $\beta(\xi)$ may be bounded below. Similarly, if
$$a[\tau^{-1}(x)]/e^{\mu x}<N,$$
$\beta(\xi)$ will be bounded above.

Thus we conclude:

Theorem 11.9. *Let $a(x)$ be of bounded variation in every finite interval, and let it satisfy one of these inequalities, if real,*
$$a[\tau^{-1}(t)]/e^{\mu t}>-N,\quad or\quad <N;$$
or, if complex,
$$a[\tau^{-1}(t)]=O(e^{\mu t}).$$
Then there may exist equivalence between the two following statements: if the limit
$$\lim_{x\to\infty}G_\lambda(x)=A,\quad (\lambda>-1),$$
exists, then the limit
$$\lim_{\xi\to\infty}\frac{1}{\xi}\int_0^{\tau^{-1}(\xi)}e^{-\mu\tau(t)}da(t)=\lim_{\xi\to\infty}\frac{1}{\xi}\int_{\tau(+0)}^{\xi}e^{-\mu x}da[\tau^{-1}(x)]=A.\overline{B}_\mu.\mu^{\lambda+1}$$
follows, and vice versa;

here we assume that
$$\lim_{x\to\infty}\frac{e^{\mu\tau(x)}}{\varphi(x)}=\overline{B}_\mu\quad(\mu>0).$$

12. Now again we shall consider theorem 11.8. But we notice that in this theorem, for the special case $\mu=\lambda$, our method is inapplicable, and we adopt the following device.

Integrating by parts, from (51)

$$G_\lambda^{(\lambda)}(x)=[\tau(x)]^{-\lambda}\int_0^x \frac{[\tau(x)-\tau(t)]^\lambda}{\Gamma(\lambda+1)}da(t)$$

$$=\frac{[\tau(x)-\tau(t)]^\lambda}{[\tau(x)]^\lambda\Gamma(\lambda+1)}a(t)\Big|_{t=0}^{t=x}+[\tau(x)]^{-\lambda}\int_0^x \frac{[\tau(x)-\tau(t)]^{\lambda-1}}{\Gamma(\lambda)}\tau'(t)a(t)dt.$$

Since there will be clearly no objection to assuming that $a(+0)=0$, $\tau(+0)=0$, and $\tau'(t)$ exists almost everywhere,

$$\frac{[\tau(x)-\tau(t)]^\lambda}{[\tau(x)]^\lambda\Gamma(\lambda+1)}a(t)\Big|_{t=0}^{t=x}=0, \quad [\lambda>0].$$

Hence we have

$$G_\lambda^{(\lambda)}(x)=[\tau(x)]^{-\lambda}\int_0^x \frac{[\tau(x)-\tau(t)]^{\lambda-1}}{\Gamma(\lambda)}\tau'(t)a(t)dt.$$

Inasmuch as the right-hand side may be written

$$\frac{1}{\Gamma(\lambda)}\int_0^x\left[1-\frac{\tau(t)}{\tau(x)}\right]^{\lambda-1}\frac{\tau(t)}{\tau(x)}\,\frac{\tau'(t)}{\tau(t)}a(t)dt,$$

by means of (T), we have

$$G_\lambda^{(\lambda)}(x)=\overset{*}{G}{}_\lambda^{(\lambda)}(\eta)$$

$$=\frac{1}{\Gamma(\lambda)}\int_{-\infty}^{\eta}(1-e^{-(\eta-\xi)})^{\lambda-1}e^{-(\eta-\xi)}d\beta(\xi),$$

where $\beta(\xi)$ may be defined as

(53″) $$\beta(\xi)=\int^{\tau^{-1}(e^\xi)}\frac{a(t)}{\tau(t)}d\tau(t).$$

Now if we introduce the function

$$R_\lambda(\xi)=0 \quad (\xi<0)\,; \quad =(1-e^{-\xi})^{\lambda-1}e^{-\xi} \quad (0<\xi),$$

we have

(54″) $$\overset{*}{G}{}_\lambda^{(\lambda)}(\eta)=\frac{1}{\Gamma(\lambda)}\int_{-\infty}^{\eta}R_\lambda(\eta-\xi)d\beta(\xi).$$

It is noticed that the Fourier transform of $R_\lambda(\xi)$ is

$$(2\pi)^{-\frac{1}{2}}\int_0^\infty(1-e^{-\xi})^{\lambda-1}e^{-\xi}e^{-iu\xi}d\xi=(2\pi)^{-\frac{1}{2}}\frac{\Gamma(\lambda)\Gamma(1+iu)}{\Gamma(\lambda+1+iu)},$$

and $$\int_{-\infty}^{\infty}R_\lambda(\xi)d\xi=\frac{\Gamma(\lambda)}{\Gamma(\lambda+1)}=\frac{1}{\lambda}, \quad (\lambda>0).$$

Now let

$$\lim_{x\to\infty}G_\lambda(x)=A.$$

From (52) and (54″),

$$\lim_{x\to\infty}G_\lambda^{(\lambda)}(x)=\lim_{\eta\to\infty}\overset{*}{G}{}_\lambda^{(\lambda)}(\eta)$$

$$=C_{\lambda,\lambda}\frac{1}{\Gamma(\lambda)}\int_{-\infty}^{\infty}R_\lambda(\xi)d\xi=\frac{1}{\Gamma(\lambda+1)}C_{\lambda,\lambda},$$

where $C_{\lambda,\lambda}=A.\Gamma(\lambda+1)/\overline{B}_{\lambda}$.

In the sequel, to use the notion of "bounded below or above" defined by Wiener, we consider

$$\int_{\eta}^{\eta+1}|d\beta(\xi)|-\int_{\eta}^{\eta+1}d\beta(\xi)\quad[-\infty<\eta<\infty]$$
$$=\int\{|\alpha(t)|-\alpha(t)\}\frac{d\tau(t)}{\tau(t)}$$
$$\leqq-2\int\alpha(t)\frac{d\tau(t)}{\tau(t)}\qquad[\alpha(t)<0]$$
$$=-2\int_{v}^{ve}\alpha[\tau^{-1}(u)]\frac{du}{u}\qquad[0\leqq v<\infty]$$
$$\leqq 2\operatorname*{Max}_{v\leq u\leq ve}\{-\alpha[\tau^{-1}(u)]\}.$$

Hence from this result we can conclude that if for every positive u

$$\alpha[\tau^{-1}(u)]>-N,$$

the function $\beta(\xi)$ will be bounded below; and similarly, if

$$\alpha[\tau^{-1}(u)]<N,$$

$\beta(\xi)$ will be bounded above.

Thus we have proved:

Theorem 12. *Let $\alpha(x)$ be of bounded variation over every finite interval. Let*

$$\lim_{x\to\infty}\frac{[\tau(x)]^{\lambda}}{\varphi(x)}=\overline{B}_{\lambda}.$$

Let $\alpha(x)$ be real, and satisfy one of the inequalities

$$\alpha[\tau^{-1}(u)]>-N;\quad \alpha[\tau^{-1}(u)]<N,$$

or complex, and of the form

$$\alpha[\tau^{-1}(u)]=O(1).$$

Then the two following statements are equivalent:

$$\lim_{x\to\infty}G_{\lambda}(x)=A,\quad \lambda>0;$$
$$\lim_{x\to\infty}\frac{1}{\log x}\int_0^x\frac{\alpha[\tau^{-1}(u)]}{u}du=A\Gamma(\lambda+1)/\overline{B}_{\lambda}.^{[1]}$$

13. Contribution to a singular integral equation. Let us consider an operator, defined by

$$Kf(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(x-\xi)f(\xi)d\xi.$$

Here we notice that our argument may be stated formally only.

By using the Tauberian theorem (theorem 3.1)

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1. This may be derived as follows:

$$\lim_{x\to\infty}\frac{\beta(x)}{x}=\lim_{x\to\infty}\frac{\beta(\log x)}{\log x}=\lim_{x\to\infty}\frac{1}{\log x}\int_0^{\tau^{-1}(x)}\frac{\alpha(t)}{\tau(t)}d\tau(t)=\lim_{x\to\infty}\frac{1}{\log x}\int_0^x\frac{\alpha[\tau^{-1}(u)]}{u}du.$$

$$Kf(x)=\lim_{\sigma\to 0}\sigma\int_{-\infty}^{\infty}K(x-\xi)f(\xi)\phi_\sigma(\xi)d\xi,$$

where $\phi_\sigma(x)=e^{\sigma x}\quad [x<0]; \quad =e^{-\sigma x}\quad [x>0], \quad \sigma>0.$

Now let us suppose

$$f(x)=\sum_{j=1}^{n}A_j e^{i\lambda_j x}.$$

Since $\displaystyle\int_0^\infty K(x-\xi)e^{-\sigma\xi}e^{i\lambda_j\xi}d\xi=e^{-(\sigma-i\lambda_j)x}\int_{-\infty}^{x}K(t)e^{(\sigma-i\lambda_j)t}dt,$

it will be clear that

$$Ke^{i\lambda_j x}=e^{i\lambda_j x}[\lim_{\sigma\to 0}\sigma e^{-\sigma}\int_{-\infty}^{\infty}K(t)e^{-i\lambda_j t}\phi_\sigma(t)dt]$$

$$=e^{i\lambda_j x}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(t)e^{-i\lambda_j t}dt.$$

Thus we are able to conclude that *in order that the integral equation*

$$\lambda f(x)=Kf(x)$$

may have a solution, expressed by the trigonometrical polynomial, it is necessary that

$$\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}K(t)e^{-i\lambda_j t}dt, \quad j=1,\ 2,\ 3,\ldots\ldots.$$
