On the Gravitational Perturbation for the Dirac Electron

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(Received April 25, 1939)

Abstract

We studied, by means of the generalized wave equations in the tensor forms, a quantum gravitational problem which might be reduced to a perturbation problem for the stationary state of the Dirac electron.

Introduction

In a previous paper,[1] we obtained the generalized wave equations for the Dirac electron using the tensor calculus which was familiar to us in the relativity theory of gravitation. Since there can be no introduced spin matrix operators in the tensor formulations as in the spinor ones, it seemed difficult to describe the spinning electron by the tensor formula. We showed,[2] however, that the spin momentum could be introduced in the tensor formulations as operators of infinitesimal rotations of vectors of the wave field, and concluded therefore, that the spinor formulations are not necessarily required for description of the spinning electron.

One of the advantages of our tensor formulation was that the formulation enabled us to make geometrical consideration for the wave fields of the electron without use of auxiliary spaces; in the previous paper, we developed a scheme of world geometry which described the physical world consisting of space, time and electron.

By the generalized wave equations one may study some behaviours of the electron in the gravitational field, for, according to the relativity theory of gravitation, one can identify the Riemannian metrical tensors with the gravitational potentials. Some authors,[3] using the generalized wave equations in the spinor forms, have already in-

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1. Jap. Jour. Phys. Vol. XI (1936) p. 35.
2. Jap. Jour. Phys. Vol. XII (1938) p. 27.
3. O. Halpern and G. Heller. Phys. Rev. **48** (1935) 435.
   A. H. Taub. Phys. Rev. **51** (1937) 512.
   S. Sambursky. Phys. Rev. **52** (1937) 335.
   Phys. Rev. **53** (1938) 256.

vestigated these quantum gravitational problems from the standpoint of the cosmology.

In the present paper, we have studied by means of our tensor formulations a simple quantum gravitational problem which might be reduced to the perturbation problem for the stationary state of the Dirac electron.

1. Perturbation Theory

Before proceeding to the problem of the electron in the gravitational field, we shall briefly describe the perturbation theory by means of the tensor formulations.

According to our tensor formulations, the generalized wave equations for the Dirac electron are given by

$$\left.\begin{aligned} &\frac{\hbar}{i}J^{\sigma\varkappa}{}_{\cdot\cdot\lambda\nu}(\nabla_\sigma + i\frac{e}{\hbar c}\varphi_\sigma)\psi^{\lambda\nu} + mc\,\psi^{\varkappa} = 0, \\ &\frac{\hbar}{i}J^{\sigma\cdot\lambda\nu}_{\cdot\varkappa}(\nabla_\sigma + i\frac{e}{\hbar c}\varphi_\sigma)\psi^{\varkappa} + mc\,\psi^{\lambda\nu} = 0. \end{aligned}\right\} \tag{1.1}$$

where

$$J_{\sigma\varkappa\lambda\nu} = \frac{1}{2}(g_{\sigma\varkappa}\,g_{\lambda\nu} + g_{\sigma\lambda}\,g_{\varkappa\nu} - g_{\sigma\nu}\,g_{\varkappa\lambda} + E_{\sigma\varkappa\lambda\nu}),$$

$g_{\sigma\varkappa}$: fundamental tensor of metric,

$E_{\sigma\varkappa\lambda\nu} = i\sqrt{-g}\ \varepsilon_{\sigma\varkappa\lambda\nu}$,

g : determinant of $g_{\sigma\varkappa}$,

$E_{\sigma\varkappa\lambda\nu}$: coefficient of determinant,

Δ_σ : symbol for covariant differentiation,

φ_σ : electro-magnetic four-potential.

In the case of the special relativity where the tensor $g_{\sigma\varkappa}$ is given by

$$g_{\sigma\varkappa} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad g = -1, \tag{1.2}$$

we can reduce, as in the previous paper, the equations (1.1) to the forms:

$$\left.\begin{aligned} &P_l^{\cdot m}F^l - \frac{1}{c}(E_0 - E - c\varphi)E_l^m\,G^l = 0, \\ &P_l^{\cdot m}G^l + \frac{1}{c}(E_0 + E + c\varphi)\,E_l^m F^l = 0. \end{aligned}\right\} \tag{1.3}$$

where

$$\left.\begin{aligned} &P_l^{\cdot m} = 2\sum_{s=1}^{3} J^{s\cdot m}_{\cdot l\cdot 4}\,\frac{\hbar}{i}\partial_s, \\ &E_l^m = 2J^{4m}_{\cdot\cdot l4}, \quad E_0 = mc^2 \end{aligned}\right\} \tag{1.4}$$

$$F^l = \varphi^l + 2\, E^l_s \psi^{s4}, \quad G^l = \psi^l - 2\, E^l_s \psi^{s4}. \tag{1.5}$$

We denote by $F^l(n)$, $G^l(n)$ the wave functions which belong to the energy state $E(n)$. Then we have

$$\left.\begin{aligned} &P^{\cdot m}_{l}\, F^l(n) - \frac{1}{c}(E_0 - E(n) - c\varphi) E^m_l\, G^l(n) = 0, \\ &P^{\cdot m}_{l}\, G^l(n) + \frac{1}{c}(E_0 + E(n) + c\varphi) E^m_l\, F^l(n) = 0, \end{aligned}\right\} \tag{1.6a}$$

Similarly, the conjugate equations for the state $E(n')$ are written

$$\left.\begin{aligned} &-P^{l}_{\cdot m}\, \bar{F}^m(n') - \frac{1}{c}(E_0 - E(n') - c\varphi) E^l_m\, \bar{G}^m(n') = 0, \\ &-P^{l}_{\cdot m}\, \bar{G}^m(n') + \frac{1}{c}(E_0 + E(n') + c\varphi) E^l_m \bar{F}^m(n') = 0. \end{aligned}\right\} \tag{1.6b}$$

Multiplying the first equation of (1.6a) through by $\bar{G}_m(n')$ and the second through by $\bar{F}_m(n')$, and the first equation of (1.6b) through by $G_l(n)$ and then second through by $F_l(n)$ and then adding them all, we get

$$\begin{aligned} &P^{\cdot m}_{l}(\bar{G}_m(n') F^l(n) + \bar{F}_m(n') G^l(n)) \\ &\quad + (E(n) - E(n'))\, (E^m_l\, \bar{G}_m(n')\, G^l(n) \\ &\quad + E^m_l\, \bar{F}_m(n')\, F^l(n)) = 0. \end{aligned} \tag{1.7}$$

With the help of (1.4), we can bring the first term of (1.7) into the following divergent form:

$$2\frac{\hbar}{i}\sum_{s=1}^{3}\partial_s J^{s\cdot m}_{\cdot l\cdot 4}(\bar{G}_m(n') F^l(n) + \bar{F}_m(n') G^l(n))$$

of which volume integral for the whole space vanishes.

Hence, we have

$$\begin{aligned} (E(n) - E(n')) \iiint & (E^m_l) \bar{G}_m(n') G^l(n) \\ & + E^m_l \bar{F}_m(n') F^l(n)) dv = 0. \end{aligned} \tag{1.8}$$

We now write

$$E(n', n) = \iiint E^m_l(\bar{G}_m(n') G^l(n) + \bar{F}_m(n') F^l(n) dv \tag{1.9a}$$

then we have

$$E(n', n) = 0 \qquad \text{when } n \neq n'. \tag{1.9b}$$

In this way we may define the orthogonality of the field functions.

It has been shown in the previous paper,[1] that the tensor defined by

$$J^{\mu\nu} = \bar{\Psi}^{\mu}_{\cdot\lambda}\, \psi^{\lambda\nu} + \frac{1}{2} J^{\rho\sigma\tau\nu}\, \bar{\psi}_\sigma\, \psi_\tau,$$

satisfies the conservation theorem: namely

$$\nabla_\nu J^{\mu\nu} = 0.$$

Using the vectors F^l and G^l defined by (1.5), we have

$$J^4_{\cdot 4}=\frac{1}{8}E^m_l[(\bar{G}_m+\bar{F}_m)(G^l+F^l)+(\bar{G}_m-\bar{F}_m)(G^l-F^l)]$$

which by the relation

$$E^m_l\bar{F}_m G_l=E^m_l\bar{F}^l G_m.$$

becomes

$$=\frac{1}{4}E^m_l(G_m G^l+\bar{F}_m F^l).$$

We thus obtain

$$\frac{1}{4}E(n', n)=\iiint J^4_{\cdot 4}(n',n)dv. \tag{1.10}$$

We shall next consider the perturbed wave equations which may be given by the expression

$$\left.\begin{aligned} &P^{;m}_l F^l-\frac{1}{c}(E_0-E-c\varphi)E^m_l G^l=\gamma S^m(F,G),\\ &P^{;m}_l G^l+\frac{1}{c}(E_0+E_1+c\varphi)E^m_l F^l=\gamma T^m(F,G,),\end{aligned}\right\} \tag{1.11}$$

where $S^m(F, G)$ and $T^m(F, G)$ mean the systems of functions of F^l, G^l and γ is a perturbation parameter which we can assume as a small quantity.

As the usual perturbation theory, we denote by $\mathring{E}$, $\mathring{F}^l$ and $\mathring{G}^l$ unperturbed solutions and put

$$\begin{aligned} &E=\mathring{E}(n)+\gamma\varepsilon,\\ &F^l=\mathring{F}^l(n)+\gamma u^l,\\ &G^l=\mathring{G}^l(n)+\gamma v^l,\end{aligned} \tag{1.12}$$

into the equations of (1.11). Neglecting the terms γ^2 and using the relations of (1.2) for the unperturbed functions, we have

$$\left.\begin{aligned} &P^{;m}_l u^l-\frac{1}{c}(E_0-\mathring{E}(n)-c\varphi)E^m_l v^l\\ &\qquad=\mathring{S}^m(n)-\frac{\varepsilon}{c}E^m_l\mathring{G}^l(n),\\ &P^{;m}_l v^l+\frac{1}{c}(E_0+\mathring{E}(n)+c\varphi)E^m_l u^l\\ &\qquad=\mathring{T}^m(n)-\frac{\varepsilon}{c}E^m_l\mathring{F}^l(n),\end{aligned}\right\} \tag{1.13}$$

where $\mathring{S}^m(n)$ and $\mathring{T}^m(n)$ mean respectively $S^m(\mathring{F}(n), \mathring{G}(n))$ and $T^m(\mathring{F}(n), \mathring{G}(n))$.

Multiplying the first and the second equations (1.18) respectively through by $\bar{\mathring{G}}_m(n')$ and $\bar{\mathring{F}}_m(n')$, and then adding, we find

$$\begin{aligned}
&P_l^{\cdot m}\{\bar{G}_m(n')u^l+\bar{F}_m(n')v^l\}\\
&\quad -u^l\{P_l^{\cdot m}\,\bar{G}_m(n')-\frac{1}{c}(E_0+\mathring{E}(n)+c\varphi)E_l^m\,\bar{F}_m(n')\}\\
&\quad -v^l\{P_l^{\cdot m}\,\bar{F}_m(n')+\frac{1}{c}(E_0-\mathring{E}(n)-c\varphi)E_l^m\,\bar{G}_m(n')\}\\
&=\bar{G}_m(n')\mathring{S}^{\cdot m}(n)+\bar{F}_m(n')\mathring{T}^{\cdot m}(n)\\
&\quad -\frac{\varepsilon}{c}E_l^m(\bar{G}_m(n')\mathring{G}^l(n)+\bar{F}_m(n')\mathring{F}^l(n)). \qquad (1.14)
\end{aligned}$$

Since the first term of the left-hand side can be brought to a divergent form, its volume integral vanishes. In dealing with the second and the third terms we use the equations of (1.6_b). We thus arrive at the relation

$$\begin{aligned}
&\frac{1}{c}\iiint(\mathring{E}(n)-\mathring{E}(n')(u^l\cdot E_l^m\,\bar{F}_m(n')+v^l\,E_l^m\,\bar{G}_m(n'))dv\\
&\qquad =W(n',n)-E(n',n), \qquad (1.15)
\end{aligned}$$

in which we denote

$$W(n',n)=\iiint(\bar{G}_m(n')\mathring{S}^{\cdot m}(n)+\bar{F}_m(n')\mathring{T}^{\cdot m}(n))dv. \quad (1.16)$$

When $n'=n$, the left-hand side of (1.15) vanishes and we get

$$\frac{\varepsilon}{c}=\frac{W(n,n)}{E(n,n)}=\frac{\iiint(\bar{G}_m(n)\mathring{S}^{\cdot m}(n)+\bar{F}_m(n)\mathring{T}^{\cdot m}(n)dv}{\iiint E_l^m(\bar{G}_m(n)\mathring{G}^l(n)+\bar{F}_m(n)\mathring{F}^l(n))dv}, \quad (1.17)$$

which is the first order perturbation energy.

We can also determine the perturbation function u^l and v^l by a procedure similar to that of the usual perturbation theory; they are written

$$\left.\begin{aligned}
u^l&=\sum_{n'}\frac{\mathring{F}^l(n')W(n',n)}{(\mathring{E}(n)-\mathring{E}(n'))E(n',n')/c}\\
v^l&=\sum_{n'}\frac{\mathring{G}^l(n')W(n',n)}{(\mathring{E}(n)-\mathring{E}(n'))E(n',n')/c}
\end{aligned}\right\} \quad (1.18)$$

2. Wave Equations in the Gravitational Field

According to the relativity theory of gravitation, the line element for a static gravitational field with the spherical symmetry, is given by the expression

$$ds^2=\frac{1}{1-2\gamma\Phi}((dx^1)^2+(dx^2)^2+(dx^3)^2)+(1-2\gamma\Phi)(dx^4)^2, \quad (2.1)$$

in which Φ is a function of $r=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$ and γ is a parameter which we can arbitrarily introduce.

We now assume that the gravitational field is so weak that the square of γ may be neglected. In such a case, the metrical tensors may be written in the forms

$$\left.\begin{aligned} &g_{11}=g_{22}=g_{33}=-(1+2\gamma\Phi),\ g_{44}=(1-2\gamma\Phi),\\ &g^{11}=g^{22}=g^{33}=-(1-2\gamma\Phi),\ g^{44}=(1+2\gamma\Phi),\\ &\sqrt{-g}=1+2\gamma\Phi. \end{aligned}\right\} \qquad (2.2)$$

which with the help of the quantity E^{μ}_{λ} defined by (1.4), may also be expressed by

$$g_{\mu\nu}=\mathring{g}_{\mu\alpha}(A^{\alpha}_{\nu}-2\gamma\Phi E^{\alpha}_{\nu}),\ g^{\mu\nu}=\mathring{g}^{\mu\alpha}(A^{\nu}_{\alpha}+2\gamma\Phi E^{\nu}_{\alpha}), \qquad (2.3)$$

where $\mathring{g}_{\mu\alpha}$ and $\mathring{g}^{\mu\alpha}$ mean the metrical tensors given by (1.2).

In the case when metrical tensors are those given in (2.3), we can reduce, by procedure similar to that which was used in the previous paper, our equations (1.1) to the forms

$$\left.\begin{aligned} &4\frac{\hbar}{i}J^{\sigma\kappa}_{\cdot\cdot\lambda 4}\Big(\Gamma_{\sigma}+i\frac{e}{\hbar c}\varphi_{\sigma}\Big)\psi^{\lambda 4}+mc\psi^{\kappa}=0,\\ &\frac{\hbar}{i}J^{\sigma\cdot\lambda 4}_{\cdot\kappa}\Big(\Gamma_{\sigma}+i\frac{e}{\hbar c}\varphi_{\sigma}\Big)\psi^{\kappa}+mc\psi^{\lambda 4}=0, \end{aligned}\right\} \qquad (2.4)$$

which, in the case of the statical problem where $\varphi_1=\varphi_2=\varphi_3=0$ and $\varphi_4=\varphi$, can further be brought to

$$\left.\begin{aligned} &4\frac{\hbar}{i}\sum_{\omega=1}^{3}J^{\omega\kappa}_{\cdot\cdot\lambda 4}\partial_{\omega}\psi^{\lambda 4}+2\Big(\frac{E}{c}+\frac{e}{c}\varphi\Big)E^{\kappa}_{\lambda}\psi^{\lambda 4}\\ &\qquad+4\frac{\hbar}{i}J^{\sigma\kappa}_{\cdot\cdot\lambda 4}(\Gamma^{\lambda}_{\sigma\tau}\psi^{\tau 4}+\Gamma^{4}_{\sigma\tau}\psi^{\lambda\tau})+mc\psi^{\kappa}=0,\\ &2\frac{\hbar}{i}\sum_{\omega=1}^{3}J^{\omega\cdot\lambda}_{\cdot\kappa\cdot 4}\partial_{\omega}\psi^{\kappa}+\Big(\frac{E}{c}+\frac{e}{c}\varphi\Big)E^{\lambda}_{\kappa}\psi^{\kappa}\\ &\qquad+2\frac{\hbar}{i}J^{\sigma\cdot\lambda}_{\cdot\kappa\cdot 4}\Gamma^{\kappa}_{\sigma\tau}\psi^{\tau}+2mc\,\psi^{\lambda 4}g_{44}=0. \end{aligned}\right\} \qquad (2.5)$$

in which $\Gamma^{\kappa}_{\sigma\tau}$ may be given by the expression

$$\Gamma^{\kappa}_{\sigma\tau}=-\gamma(\partial_{\sigma}\Phi.\,E^{\kappa}_{\tau}+\partial_{\tau}\Phi.\,E^{\kappa}_{\sigma}-\partial_{\alpha}\Phi.\,\mathring{g}^{\kappa\alpha}\mathring{g}_{\tau\beta}E^{\beta}_{\sigma}). \qquad (2.6)$$

Using (2.6) and neglecting the higher powers of γ, we find the relations

$$\begin{aligned} &J^{\sigma\kappa}_{\cdot\cdot\lambda 4}\Gamma^{\lambda}_{\sigma\tau}\psi^{\tau 4}=-2\gamma\Phi_{\sigma}\mathring{J}^{\kappa\sigma}_{\cdot\cdot\tau 4}\psi^{\tau 4}\\ &J^{\sigma\kappa}_{\cdot\cdot\lambda 4}\Gamma^{4}_{\sigma\tau}\psi^{\lambda\tau}=2J^{\sigma\kappa}_{\cdot\cdot\lambda 4}J^{\lambda\tau}_{\cdot\cdot\omega 4}\Gamma^{4}_{\sigma\tau}\psi^{\omega 4}\\ &\qquad=-\frac{1}{2}\gamma\Phi_{\sigma}\mathring{E}^{\sigma\kappa}_{\cdot\cdot\omega 4}\psi^{\omega 4}, \end{aligned}$$

where $\mathring{J}^{\sigma\kappa}_{\cdot\cdot\tau 4}$ and $\mathring{E}^{\sigma\kappa}_{\cdot\cdot\tau 4}$ mean that they are constructed with $\mathring{g}_{\mu\nu}$'s.

Using these relations, we can bring the equations of (2.5) to the forms

$$\left.\begin{aligned} &4\frac{\hbar}{i}\sum_{\omega=1}^{3} J^{\omega\varkappa}_{\cdot\cdot\lambda 4}\,\partial_\omega\,\psi^{\lambda 4}+2\left(\frac{E}{c}+\frac{e}{c}\varphi\right)E^{\varkappa}_{\lambda}\,\psi^{\lambda 4}\\ &\qquad -4\frac{\hbar}{i}\gamma\Phi_\sigma(2\mathring{J}^{\varkappa\sigma}_{\cdot\cdot\lambda 4}-\mathring{E}^{\varkappa\sigma}_{\cdot\cdot\lambda 4})\psi^{\lambda 4}+mc\,\psi^{\varkappa}=0,\\ &2\frac{\hbar}{i}\sum_{\omega=1}^{3} J^{\omega\cdot\lambda}_{\cdot\varkappa\cdot 4}\,\partial_\omega\,\psi^{\varkappa}+\left(\frac{E}{c}+\frac{e}{c}\varphi\right)E^{\lambda}_{\varkappa}\,\psi^{\varkappa}\\ &\qquad -4\frac{\hbar}{i}\gamma\Phi_\sigma\,\mathring{J}^{\lambda\cdot\sigma}_{\cdot\varkappa\cdot 4}\,\psi^{\varkappa}+2mc(1-2\gamma\Phi)\psi^{\lambda 4}=0. \end{aligned}\right\}\qquad(2.7)$$

For the sake of convenience, we introduce the following quantities :

$$\left.\begin{aligned} \delta_{i\mu}&=g_{i\mu}-\gamma\Phi E_{i\mu}, &\quad \delta_i^{\cdot\mu}&=A_i^{\mu}+\gamma\Phi E_i^{\mu},\\ \delta^i_{\cdot\nu}&=A^i_{\nu}-\gamma\Phi E^i_{\nu}, &\quad \delta^{i\nu}&=g^{i\nu}+\gamma\Phi E^{i\nu}, \end{aligned}\right\}\qquad(2.8)$$

which, if the higher powers of γ are neglected, satisfy the following relations of orthogonality

$$\delta_{i\mu}\delta_{\varkappa}^{\cdot\mu}=g_{i\varkappa},\quad \delta_{i\mu}\delta^i_{\cdot\nu}=g_{\mu\nu}.\qquad(2.9)$$

With the help of these quantities, we can easily find the formulas

$$\begin{aligned} 2\frac{\hbar}{i}\sum_{\omega=1}^{3} J^{\omega\varkappa}_{\cdot\cdot\lambda 4}\,\partial_\omega\,\psi^{\lambda 4}&=(1-2\gamma\Phi)\delta_k^{\cdot\varkappa}P^k_{\cdot l}\delta^l_{\cdot\lambda}\psi^{\lambda 4}\\ &\quad+2\gamma\frac{\hbar}{i}\delta_k^{\cdot\varkappa}\Phi_s\,\mathring{B}^{k\cdot s}_{\cdot l\cdot 4}\,\delta^l_{\cdot\lambda}\,\psi^{\lambda 4}, \end{aligned}$$

where we denote

$$P^k_{\cdot l}=2\frac{\hbar}{i}\sum_{s=1}^{3} J^{sk}_{\cdot\cdot l4}\,\partial_s.$$

Similarly

$$\begin{aligned} 2\frac{\hbar}{i}\sum_{\omega=1}^{3} J^{\omega\cdot\lambda}_{\cdot\varkappa\cdot 4}\,\partial_\omega\,\psi^{\varkappa}&=(1-2\gamma\Phi)\delta_l^{\cdot\lambda}\,P_k^{\cdot l}\,\delta^k_{\cdot\varkappa}\,\psi^{\varkappa}\\ &\quad+2\gamma\frac{\hbar}{i}\delta_l^{\cdot\lambda}\,\Phi_s\,\mathring{B}^{l\cdot s}_{\cdot k\cdot 4}\,\delta^k_{\cdot\varkappa}\,\psi^{\varkappa}. \end{aligned}$$

Applying these formulas to (2.7), we obtain

$$\left.\begin{aligned} &2(1-2\gamma\Phi)\delta_k^{\cdot\varkappa}\,P^k_{\cdot l}\,\delta^l_{\cdot\lambda}\,\psi^{\lambda 4}+2\left(\frac{E}{c}+\frac{e}{c}\varphi\right)E^{\varkappa}_{\lambda}\,\psi^{\lambda 4}\\ &\qquad-4\frac{\hbar}{i}\gamma\Phi_s\,\mathring{B}^{k\cdot s}_{\cdot l\cdot 4}\,\delta_k^{\cdot\varkappa}\,\delta^l_{\cdot\lambda}\,\psi^{\lambda 4}+mc\,\psi^{\varkappa}=0,\\ &(1-2\gamma\Phi)\delta_l^{\cdot\lambda}\,P_k^{\cdot l}\,\delta^k_{\cdot\varkappa}\,\psi^{\varkappa}+\left(\frac{E}{c}+\frac{e}{c}\varphi\right)E^{\lambda}_{\varkappa}\,\psi^{\varkappa}\\ &\qquad-2\frac{\hbar}{i}\gamma\Phi_s\,\mathring{B}^{l\cdot s}_{\cdot k\cdot 4}\,\delta_l^{\cdot\lambda}\,\delta^k_{\cdot\varkappa}\psi^{\varkappa}+2(1-2\gamma\Phi)mc\psi^{\lambda 4}=0. \end{aligned}\right\}\qquad(2.10)$$

Let us now write

$$\left.\begin{aligned} F^m &= \delta^m_{\cdot\varkappa}\,\psi^\varkappa + 2E^m_k\,\delta^k_{\cdot\alpha}\,\psi^{\lambda 4}, \\ G^m &= \delta^m_{\cdot\varkappa}\,\psi^\varkappa - 2E^m_k\,\delta^k_{\cdot\lambda}\,\psi^{\lambda 4}. \end{aligned}\right\} \qquad (2.11)$$

After simple calculations, then, we can bring the equations (2.10) to the forms

$$\left.\begin{aligned} P_l^{\cdot m}\,F^l - \frac{1}{c}(E_0 - E - c\varphi)E^m_l\,G^l &= \gamma S^m, \\ P_l^{\cdot m}\,G^l + \frac{1}{c}(E_0 + E + c\varphi)E^m_l F^l &= \gamma T^m, \end{aligned}\right\} \qquad (2.12)$$

where

$$\left.\begin{aligned} S^m &= 2\Phi P_l^{\cdot m}\,F^l + mc\,E^m_l(F^l - G^l) \\ &\quad + \frac{\hbar}{i}\Phi_s[(\mathring{B}^{m\cdot s}_{\cdot\, l\cdot 4} + \bar{\mathring{B}}^{m\cdot s}_{\cdot\, l\cdot 4})F^l + (\mathring{B}^{m\cdot s}_{\cdot\, l\cdot 4} - \bar{\mathring{B}}^{m\cdot s}_{\cdot\, l\cdot 4})G^l] \\ \text{or: } &= 2\Phi P_l^{\cdot m}\,F^l + mc\,E^m_l(F^l - G^l) \\ &\quad + \frac{\hbar}{i}(\Phi^m\,F_4 - \Phi_s\,F^s\,A^m_4 + \mathring{E}^{m\cdot s}_{\cdot\, l\cdot 4}\,\Phi_s\,G^l). \end{aligned}\right\} \qquad (2.13_a)$$

$$\left.\begin{aligned} T^m &= 2\Phi P_l^{\cdot m} G^l + mc\,E^m_l(F^l - G^l) \\ &\quad + \frac{\hbar}{i}\Phi_s[(\mathring{B}^{m\cdot s}_{\cdot\, l\cdot 4} + \bar{\mathring{B}}^{m\cdot s}_{\cdot\, l\cdot 4})G^l + (\mathring{B}^{m\cdot s}_{\cdot\, l\cdot 4} - \bar{\mathring{B}}^{m\cdot s}_{\cdot\, l\cdot 4})F^l] \\ \text{or: } &= 2\Phi P_l^{\cdot m}\,G^l + mc\,E^m_l(F^l - G^l) \\ &\quad + \frac{\hbar}{i}(\Phi^m\,G_4 - \Phi_s\,G^s\,A^m_4 + \mathring{E}^{m\cdot s}_{\cdot\, l\cdot 4}\,\Phi_s\,F^l). \end{aligned}\right\} \qquad (2.13_b)$$

Thus, we can deal with this gravitational problem in the same way as the perturbation problem developed in the previous section.

3. The Gravitational Perturbation

In this section, we shall study the gravitational perturabation for the electron in the central electric field.

Substituting the values of (2.13) for S^m and T^m in the formula for the perturbation energy given by (1.17), we obtain

$$\left.\begin{aligned} \frac{\varepsilon}{c} &= \frac{\iiint 2\Phi[\bar{\mathring{G}}_m P_l^{\cdot m}\mathring{F}^l + \bar{\mathring{F}}_m P_l^{\cdot m}\mathring{G}^l + \frac{E_0}{2c}E^m_l(\bar{\mathring{F}}_m\mathring{F}^l - \bar{\mathring{G}}_m\mathring{G}^l + \bar{\mathring{G}}_m\mathring{F}^l - \bar{\mathring{F}}_m\mathring{G}^l)]\,dv}{\iiint E^m_l(\bar{\mathring{F}}^l\,\mathring{F}_m + \bar{\mathring{G}}^l\,\mathring{G}_m)dv} \\ &\quad + \frac{\frac{\hbar}{i}\iiint\Phi_s[(\mathring{B}^{mls}_{\cdot\cdot\cdot 4} + \bar{\mathring{B}}^{mls}_{\cdot\cdot\cdot 4})(\bar{\mathring{G}}_m\mathring{F}_l + \bar{\mathring{F}}_m\mathring{G}_l) + (\mathring{B}^{mls}_{\cdot\cdot\cdot 4} - \bar{\mathring{B}}^{mls}_{\cdot\cdot\cdot 4})(\bar{\mathring{G}}_m\mathring{G}_l + \bar{\mathring{F}}_m\mathring{F}_l)]\,dv}{\iiint E^m_l(\bar{\mathring{F}}^l\,\mathring{F}_m + \bar{\mathring{G}}_l\,\mathring{G}_m)dv} \end{aligned}\right\} \qquad (3.1)$$

We shall first show that the second term of the perturbation energy is due to the interaction between the spin of the electron and the gravitational force.

It was shown in the previous paper that the operators defined by

$$\overline{\sum}_{\mu\nu}(F)=F_{\mu}\frac{\partial}{\partial F^{\nu}}-F_{\nu}\frac{\partial}{\partial F^{\mu}}-\frac{1}{2}\mathring{E}_{\mu\nu}^{\;\;\cdot\cdot\sigma\tau}\left(F_{\sigma}\frac{\partial}{\partial F^{\tau}}-F_{\tau}\frac{\partial}{\partial F^{\sigma}}\right)$$

$$=2\bar{\mathring{B}}_{\mu\nu\sigma}^{\;\;\cdot\cdot\cdot\tau}F^{\sigma}\frac{\partial}{\partial F^{\tau}}$$

could be introduced as the spin operators in the tensor formulations. The conjugate operator may be defined by

$$\overline{\sum}_{\mu\nu}(F)=F_{\mu}\frac{\partial}{\partial F^{\nu}}-F_{\nu}\frac{\partial}{\partial F^{\mu}}+\frac{1}{2}\mathring{E}_{\mu\nu}^{\;\;\cdot\cdot\sigma\tau}\left(F_{\sigma}\frac{\partial}{\partial F^{\tau}}-F_{\tau}\frac{\partial}{\partial F^{\sigma}}\right)$$

$$=2\mathring{B}_{\mu\nu\sigma}^{\;\;\cdot\cdot\cdot\tau}F^{\sigma}\frac{\partial}{\partial F^{\tau}}.$$

Since these spin operators are introduced in the case of the special relativity, we can apply these operators to the unperturbed functions. Accordingly, with the help of these operators, we can express the second term of (3.1) in the forms

$$\frac{\frac{\hbar}{2i}\iiint\Phi_s[\mathring{G}_m(\bar{\sigma}^s+\sigma^s)\mathring{F}^m+\bar{\mathring{G}}_m(\bar{\sigma}^s-\sigma^s)\mathring{G}^m+\bar{\mathring{F}}_m(\bar{\sigma}^s-\sigma^s)\mathring{F}^m+\bar{\mathring{F}}^m(\bar{\sigma}^s+\sigma^s)\mathring{G}^m]dv}{\iiint E_l^m(\mathring{G}_m\mathring{G}^m+\bar{\mathring{F}}_m\mathring{F}^l)dv} \tag{3.2}$$

where

$$\sigma_s\equiv\Sigma_{s4},\quad \bar{\sigma}_s=\bar{\Sigma}_{s4},\quad (s=1,2,3),$$

or by the vector notations

$$-(\sigma^1,\sigma^2,\sigma^3)=\boldsymbol{\sigma},\quad -(\bar{\sigma}^1,\bar{\sigma}^2,\bar{\sigma}^3)=\bar{\boldsymbol{\sigma}}$$

$$(\Phi^1,\Phi^2,\Phi^3)=-\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},\frac{\partial\Phi}{\partial z}\right)=\boldsymbol{K},$$

this becomes

$$\frac{\iiint\frac{\hbar}{2i}[\mathring{G}_m\boldsymbol{K}(\bar{\boldsymbol{\sigma}}+\boldsymbol{\sigma})\mathring{F}^m+\bar{\mathring{F}}_m\boldsymbol{K}(\bar{\boldsymbol{\sigma}}+\boldsymbol{\sigma})\mathring{G}^m+\bar{\mathring{G}}\boldsymbol{K}(\bar{\boldsymbol{\sigma}}-\boldsymbol{\sigma})\mathring{G}^m+\bar{\mathring{F}}_m\boldsymbol{K}(\bar{\boldsymbol{\sigma}}-\boldsymbol{\sigma})\mathring{F}^m]dv}{\iiint E_l^m(\mathring{G}^l\,\mathring{G}_m+\bar{\mathring{F}}^l\,\mathring{F}_m)dv} \tag{3.3}$$

From this result we see that, even when the central particle is neutral, the central gravitational field interacts with the spin of the electron.

As the unperturbed functions for the electron in the central electric field, we may use the following values which were obtained in the previous paper:

$$\left.\begin{aligned}
\hat{\boldsymbol{F}}&=i\frac{\chi_1}{r}\left\{(\boldsymbol{r}_0+i\frac{[\boldsymbol{r}_0\boldsymbol{L}]}{k})Y(k\,;\ \theta,\varphi)-\frac{\boldsymbol{L}}{k}Y(-k\,;\ \theta,\varphi)\right\}\\
\hat{\boldsymbol{G}}&=\frac{\chi_2}{r}\left\{(\boldsymbol{r}_0-i\frac{[\boldsymbol{r}_0\boldsymbol{L}]}{k})Y(-k;\ \theta,\varphi)+\frac{\boldsymbol{L}}{k}Y(k\,;\ \theta,\varphi)\right\}\\
\hat{F}^4&=i\frac{\chi_1}{r}Y(-k\,;\ \theta,\varphi),\quad \hat{G}^4=\frac{\chi_2}{r}Y(k\,;\ \theta,\varphi),
\end{aligned}\right\}\quad (3.4)$$

where

$$\hat{\boldsymbol{F}}=(\hat{F}^1,\hat{F}^2,\hat{F}^3),\quad \hat{\boldsymbol{G}}=(\hat{G}^1,\hat{G}^2,\hat{G}^3),$$

$$\boldsymbol{r}_0=\left(\frac{x^1}{r},\frac{x^2}{r},\frac{x^3}{r}\right),\quad \boldsymbol{L}=(L_{14},L_{24},L_{34}).$$

$Y(k\,;\theta,\varphi)$ or in short $Y(k)$: spherical harmonics of degree k.

$\boldsymbol{L}$: angular momentum operator satisfying the relations :

$\bar{\boldsymbol{L}}=-\boldsymbol{L},\ \boldsymbol{L}^2Y(k\,;\ \theta,\varphi)=k(k+1)Y(k\,;\ \theta,\varphi).$

χ_1 and χ_2 are the functions of r which satisfy the equations :

$$\left.\begin{aligned}
\hbar\left(\frac{d}{dr}-\frac{k}{r}\right)\chi_1&=\frac{1}{c}(E_0-E-c\varphi)\chi_2,\\
\hbar\left(\frac{d}{dr}+\frac{k}{r}\right)\chi_2&=\frac{1}{c}(E_0+E_0+c\varphi)\chi_1.
\end{aligned}\right\}\quad (3.5)$$

It can easily be shown that the function $Y(k\,;\theta,\varphi)$, the spherical harmonics of degree k, satisfies the following relations :

$$\begin{aligned}
&\iint\bar{Y}(k';\ \theta,\varphi)Y(k\,;\ \theta,\varphi)\sin d\theta d\varphi\\
&=\frac{1}{k(k+1)}\iint\bar{\boldsymbol{L}}\bar{Y}k';\ \theta,\varphi)\boldsymbol{L}Y(k\,;\ \theta,\varphi)\sin\theta d\theta d\varphi\\
&=\frac{1}{k(k+1)}\iint[\boldsymbol{r}_1\bar{\boldsymbol{L}}]\bar{Y}(k';\theta,\varphi)[\boldsymbol{r}_0\boldsymbol{L}]Y(k\,;\ \theta,\varphi)\sin d\theta d\varphi\\
&=\frac{k}{2k+1},\quad\text{for } k'=k\\
&=0\quad\text{for } k'\neq k.
\end{aligned}\qquad (3.6)$$

Substituting the values of (3.4) for the unperturbed functions of (3.1) and evaluating the angular parts of the integrals with the help of the relations of (3.6), we obtain

$$\begin{aligned}
\frac{\varepsilon}{c}=&-2\frac{\hbar}{c}\frac{\int\Phi(\chi_1^2+\chi_2^2)dr}{\int(\chi_1^2+\chi_2^2)dr}-\frac{E_0}{c}\frac{\int\Phi(\chi_1^2-\chi_2^2)dr}{\int(\chi_1^2+\chi_2^2)dr}\\
&-\frac{2e}{c}\frac{\int\Phi\varphi(\chi_1^2+\chi_2^2)dr}{\int(\chi_1^2+\chi_2^2)dr}+2\frac{\hbar k}{4k^2-1}\frac{\int\frac{d\Phi}{dr}\chi_1\chi_2dr}{\int(\chi_1^2+\chi_2^2)dr}.
\end{aligned}\qquad (3.7)$$

Thus, if the gravitational potential in the atomic system is known, the energy of the gravitational perturbation may be determined.

We assume, as an example, that the Newtonian law of gravitation holds in the Hydrogen atom, namely

$$\Phi=\frac{1}{r}, \quad \gamma=\frac{fM}{c^2}, \quad \varphi=\frac{Ze}{r},$$

where M is the mass of the nucleus and f is the constant of gravitation. Then, neglecting the relativistic correction in the perturbed terms, we find that the total energy may be given in a rough approximation by the expression

$$E \fallingdotseq \mathring{E}\left(1-\frac{fM}{c^2}\langle r^{-1}\rangle\right) \quad \text{or} \quad \fallingdotseq \mathring{E}\langle g_{44}^{\frac{1}{2}}\rangle$$

where $\langle r^{-1}\rangle$ and $\langle g_{44}^{\frac{1}{2}}\rangle$ mean respectively the mean values of r^{-1} and $g_{44}^{\frac{1}{2}}$.

As the Newtonian force is smaller in the extremendous order of magnitude than the Coulomb force, it may be impossible to check the correctness of the above result by experiments on atomic spectra. However, the fact that the energy is in proportion to the mean value $g_{44}^{\frac{1}{2}}$ seems to be an interesting result, because in the classical theory of the red shifts, the frequency of light is also approximately in proportion to $g_{44}^{\frac{1}{2}}$.

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