# On the Relations between the Set and its Distances

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## Introduction

Let $f(x, y)$ be a distance of a set $E$ *i. e.* $f(x, y)$ is a real function which satisfies the following conditions:

(i) $f(x, x)=0$,

(ii) $f(x, y)=f(y, x)>0$ for $x\neq y$,

(iii) $f(x, y)\leqq f(x, z)+f(z, y)$, for any three points $x$, $y$, $z$.

In the first place, it is clear that *any set $E$ has at least one function which defines a distance of it.*

For instance, a function as follows

$$f(x, y)\begin{cases}\equiv 1 & (x\neq y)\\ \equiv 0 & (x=y)\end{cases}$$

satisfies the conditions (i), (ii) and (iii).

Moreover, *when a distance is given we can define the other distances.* When $f(x, y)$ is a distance of $E$, the functions $\varphi_1(x, y)$, $\varphi_2(x, y)$ and $\varphi_3(x, y)$, for example, are distances of $E$:

$$\varphi_1(x, y)\begin{cases}\equiv f(x, y)+\rho & (x\neq y)\\ \equiv 0 & (x=y)\end{cases}$$

$$\varphi_2(x, y)\begin{cases}\equiv \max\ [f(x_0, x), f(x_0, y)] & (x\neq y)\\ \equiv 0 & (x=y)\end{cases}$$

$$\varphi_3(x, y)\equiv a\ f(x, y)$$

where $\rho$ and $a$ are two positive constants and $x_0$ is a fixed point of $E$. For, at least one of $x\neq z$ and $y\neq z$ takes places provided $x\neq y$.

Thus a set $E$ may have many varieties of distances. Moreover we know a noteworthy theorem that "*any set may be a schlicht continuous image of a certain isolated set*", and so it will be interesting to consider the properties of the distance and the relations between the set and its distances. We shall consider, as a special case, the distances of the set of all real numbers.

**I.** Let $E$ be the set of all real numbers and $f(x, y)$ be a function which defines a distance of $E$.

When we consider, for example, rectangular coordinates of Euklidean space of three dimensions, the equation $z=f(x, y)$ is considered as an equation which defines a real surface, and every surface of this kind has the following properties:

(i) *it contains the straight line* $x=y$, $z=0$,

(ii) *it is symmetric with respect to the plane* $x=y$, *and* $z>0$ *for* $x\neq y$.

These follow directly from the first and the second conditions of the distance.

Now, if there is a system of curves $C$ in $xy$-plane which has the following properties:

(a) when any point $(x, y)$ is given, there is at least one curve $C$ which passes through the point $(x, y)$

(b) $f(x, y)$ is constant when $(x, y) \in C$,

we shall call it a *system of constant curves* with respect to the function $f(x, y)$ and call every curve $C$ a *constant curve* passing through $(x, y)$. Of course, the constant curve which passes through the points $(x, x)$ is the straight line $x=y$, in $xy$-plane, and it is sufficient to consider the curves $C$ in the half plane below the straight line $x=y$ in $xy$-plane. Naturally, it is not necessary that a function $f(x, y)$ has only one system of constant curves, for example, the function

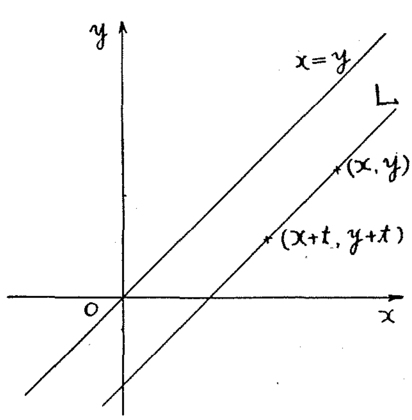
$$f(x, y)\begin{cases}\equiv a & (a>0 \text{ is a constant})\\ \equiv 0\end{cases}$$

has many systems of constant curves.

Example 1. $f(x, y)=|x-y|$

The constant curve which passes through a point $(x, y)$ is a straight line $L$ which is parallel to the straight line $x=y$.

Fig. 1




Let $(x', y')$ be any point on $L$, then it may be expressed in the form

$$x'=x+t$$
$$y'=y+t$$

($t$ real parameter)

and $f(x', y')=f(x, y)$.

Inversely, if $f(x', y')=f(x, y)$, $x'-y'=x-y$ or $x'-y'=y-x$, and so, the point $(x', y')$ is respectively on the straight line parallel to the straight line $x=y$ passing through the point $(x, y)$

or $(y, x)$. The points $(x, y)$ and $(y, x)$ are symmetric with respect to the straight line $x=y$.

Example 2. $$f(x, y)\begin{cases}\equiv a|x-y|+\rho & (x\neq y),\\ \equiv 0 & (x=y),\end{cases}$$

where $a$ and $\rho$ are two positive constants.

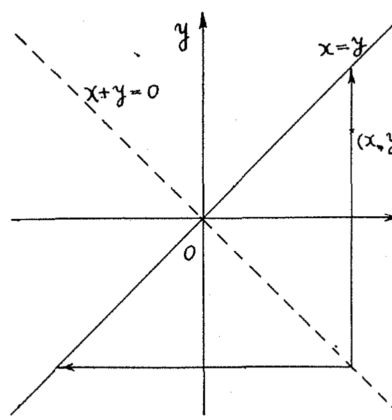
This function also defines a distance of $E$, and the constant curve passing through a point $(x, y)$ is coincident with the case of example 1.

Example 3.

$$f(x, y)\begin{cases}\equiv \max\ [|x|,\ |y|] & (x\neq y),\\ \equiv 0 & (x=y),\end{cases}$$

this function defines a distance of $E$, and the constant curve passing through a point $(x, y)$ is shown in fig. 2.

Fig. 2



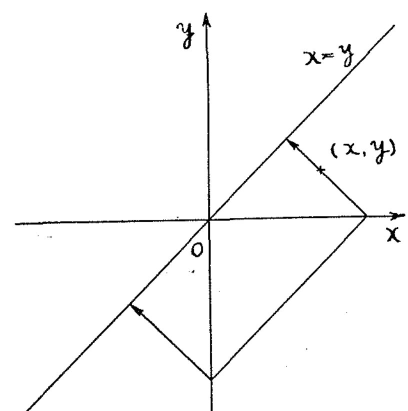

Example 4.

$$f(x, y)\begin{cases}\equiv a(|x|+|y|)+b & (x\neq y)\\ \equiv 0 & (x=y)\end{cases}$$

where $a$ and $b$ are two positive constants.

This function defines a distance of $E$, and the constant curve passing through a point $(x, y)$ is shown in fig. 3.

Fig. 3



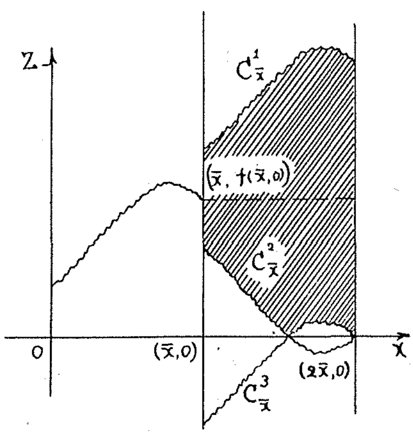

## II. THEOREM 1.

*Given the system of straight lines $C$ in xy-plane parallel to the straight line $x=y$, $z=0$. The necessary and sufficient conditions that a real function $f(x, y)$ should define a distance of $E$ and have the given sytem of lines $C$ as its system of constant curves, are that*

1) *the surface $z=f(x, y)$ contains the straight line $x=y$, $z=0$, and is symmetric with respect to the plane $x=y$ and also $f(x, y)>0$ for all $x\neq y$,*

2) *when we consider the xz-plane, the section of the surface $z=f(x, y)$ cut by the xz-plane satisfies the following properties:*

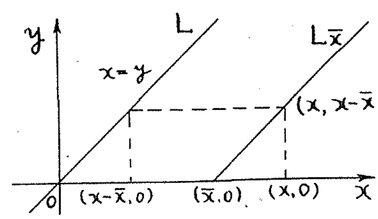
Fig. 4




*for any point* $(\bar{x}, 0)$ *in xz-plane all the points* $(x, f(x, 0))$ *lie on the lower side of the curve* $C^1_{\bar{x}}$ *and lie on the upper sides of the curves* $C^2_{\bar{x}}$ *and* $C^3_{\bar{x}}$ *provided* $2\bar{x} \geqq x \geqq \bar{x} \geqq 0$, *where* $C^1_{\bar{x}}$ *is the locus of the points in xz-plane defined by the equation* $z = f(\bar{x}, 0) + f(x-\bar{x}, 0)$, $\bar{x} \leqq x \leqq 2\bar{x}$, $C^2_{\bar{x}}$ *is the symmetry of* $C^1_{\bar{x}}$ *with respect to the straight line* $z = f(\bar{x}, 0)$, $C^3_{\bar{x}}$ *is the symmetry of* $C^2_{\bar{x}}$ *with respect to the x-axis.*

Demonstration. Let us consider the $xy$-plane. Let $L_{\bar{x}}$ be the straight line in the $xy$-plane which passes through the point $(\bar{x}, 0)$ and is parallel to the straight line $L$ whose equation is given by $x = y$. The intersection of $L_{\bar{x}}$ and the straight line which passes through the point $(x, 0)$ and is parallel to the $y$-axis, where $\bar{x} \leqq x \leqq 2\bar{x}$, is clearly $(x, x-\bar{x})$. Since the function $f(x, y)$ defines a distance,

Fig. 5




$$f(x, 0) \leqq f(x, x-\bar{x}) + f(x-\bar{x}, 0),$$
$$f(x, x-\bar{x}) \leqq f(x-\bar{x}, 0) + f(x, 0),$$
$$f(x-\bar{x}, 0) \leqq f(x, 0) + f(x, x-\bar{x}).$$

$$\therefore \quad \begin{cases} f(x, 0) \leqq f(\bar{x}, 0) + f(x-\bar{x}, 0), \\ f(x, 0) \geqq f(\bar{x}, 0) - f(x-\bar{x}, 0), \\ f(x, 0) \geqq f(x-\bar{x}, 0) - f(\bar{x}, 0). \end{cases}$$

Inversely, let $f(x, y)$ be a function which satisfies the conditions of the theorem. Let $\xi_1, \xi_2, \xi_3$ be the arbitrary three real numbers, but they are all different from each other. Suppose, for instance, $\xi_1 > \xi_2 > \xi_3$ and put $t = \xi_1 - \xi_3 > 0$, $t_2 = \xi_2 - \xi_3 > 0$, $t_3 = \xi_1 - \xi_2 > 0$, then $t_1 > t_2$, $t_1 > t_3$ and $t_1 = t_2 + t_3$. When $t_2 \geqq \frac{t_1}{2}$, we may consider $t_2, t_1, t_3$ respectively as $\bar{x}, x, x-\bar{x}$ for $2t_2 \geqq t_1 > t_2 \geqq t_3$ and $t_3 = t_1 - t_2$. Therefore

$$f(t_1) \leqq f(t_2, 0) + f(t_3, 0),$$
$$f(t_1) \geqq f(t_2, 0) - f(t_3, 0),$$
$$f(t_1) \geqq f(t_3, 0) - f(t_2, 0).$$

$$\therefore \quad f(\xi_1-\xi_3, 0) \leqq f(\xi_2-\xi_3, 0)+f(\xi_1-\xi_2, 0),$$
$$f(\xi_1-\xi_3, 0) \geqq f(\xi_2-\xi_3, 0)-f(\xi_1-\xi_2, 0),$$
$$f(\xi_1-\xi_3, 0) \geqq f(\xi_1-\xi_2, 0)-f(\xi_2-\xi_3, 0).$$

Since $f(\xi'-\xi'', 0)=f(\xi', \xi'')$, we have

$$f(\xi_1, \xi_3) \leqq f(\xi_2, \xi_3)+f(\xi_1, \xi_2)$$
$$f(\xi_2, \xi_3) \leqq f(\xi_1, \xi_2)+f(\xi_1, \xi_3)$$
$$f(\xi_1, \xi_2) \leqq f(\xi_1, \xi_3)+f(\xi_2, \xi_3)$$

We may have the same results when $t_2<\frac{t_1}{2}$ also.

Corollary. *Let $f(x, y)$ be a function which defines a distance of $E$ and have at least one system of constant curves which coincides, in the angular region $x\geqq 0$, $y\geqq 0$, $x\geqq y$* (in the $xy$-plane), *with the system of curves of the theorem 1. Then the section of the surface $z=f(x, y)$ cut by the xz-plane satisfies the conditions of the theorem 1.*

Example 1.

Let $f(x, y)$ be a function as follows:

(a) $f(x, y)=f(y, x)=f(x-y, 0)$,

(b) $f(x, 0)>0$ when $x>0$ and $f(0, 0)=0$,

(c) $f(x, 0)$ is a concave function and is non-decreasing. This function $f(x, y)$ defines a distance of $E$.

Let us consider the locus $z=f(x, 0)$ in the $xz$-plane and let $\bar{x}>0$ be an arbitrary number. The locus $C_{\bar{x}}^2$ or $C_{\bar{x}}^3$ lies respectively on the lower side of the straight line $z=f(\bar{x}, 0)$ or $z=0$, for $f(x, 0)$ is non-decreasing. And so any point $Q(x, f(x, 0))$ is on the upper side of $C_{\bar{x}}^2$ and $C_{\bar{x}}^3$. Moreover, since $f(x, 0)$ is a concave function, $C_{\bar{x}}^1$ lies on the upper side of the straight line $L$ passing through the two points $O$ and $P(\bar{x}, f(\bar{x}, 0))$ and the segment joining $P$ and $Q$ lies on the lower side of the straight line $L$.

Example 2.

It is clear that the following function $f(x, y)$ defines a distance of $E$:

(a) $f(x, y)=f(y, x)=f(x-y, 0)$

(b) $f(0, 0)=0$

$$f(x, 0)\begin{cases} \equiv 2-x & 0<x\leqq 1, \\ \equiv x-2n & 2n-1<x\leqq 2n, \\ \equiv -x+2(n+1) & 2n<x\leqq 2n+1. \end{cases}$$
$$(n=1, 2, 3, \ldots\ldots)$$

THEOREM 2.

*Let $f(x, y)$ be a function which defines a distance of $E$ and have at least one system of constant curves which coincides, in the angular region $x\geqq 0$, $y\geqq 0$, $x\geqq y$* (in $xy$-plane), *with the system of the curves* *of the example* 4 *of* I. *Then the section of the surface* $z=f(x, y)$ *cut by the xz-plane satisfies the conditions of the theorem I.*

Demonstration. Let $\bar{x}\geqq 0$ be an arbitrary number, and $x$ be $2\bar{x}\geqq x\geqq\bar{x}$. Now, let us consider the $xy$-plane, and let $L_x$ be the straight line which is parpendicular to the straight line $x=y$ and passes through the point $(x, 0)$. The intersection of $L_x$ and the straight line $\xi=\bar{x}$ ($\xi$ current coordinate) is clearly $(\bar{x}, x-\bar{x})$. Since the function $f(x, y)$ defines a distance,

$$f(x-\bar{x}, 0)\leqq f(\bar{x}, x-\bar{x})+f(\bar{x}, 0),$$
$$f(\bar{x}, x-\bar{x})\leqq f(\bar{x}, 0)+f(x-\bar{x}, 0),$$
$$f(\bar{x}, 0)\leqq f(x-\bar{x}, 0)+f(\bar{x}, x-\bar{x}).$$

And so
$$f(x, 0)\geqq f(x-\bar{x}, 0)-f(\bar{x}, 0),$$
$$f(x, 0)\leqq f(\bar{x}, 0)+f(x-\bar{x}, 0),$$
$$f(x, 0)\geqq f(\bar{x}, 0)-f(x-\bar{x}, 0),$$
for
$$f(\bar{x}, x-\bar{x})=f(x, 0).$$

THEOREM 3.

*Given the system of curves C which coincides with the system of constant curves of the example* 3 *of I. The necessary and sufficient conditions that a real function* $f(x, y)$ *should define a distance of E and have the given system of curves as its system of constant curves, are that*

(1) *the surface* $z=f(x, y)$ *contains the straight line* $x=y$, $z=0$, *and is symmetric with respect to the plane* $x=y$ *and also* $f(x, y)>0$ *for all* $x\neq y$,

(2) *when we consider the xz-plane, the section of the surface* $z=f(x, y)$ *cut by the xz-plane satisfies the following conditions:*

$$f(x_1, 0)\leqq 2f(x_2, 0)$$
for
$$0<x_1<x_2.$$

Demonstration. Let us consider the $xy$-plane. Since the function $f(x, y)$ defines a distance, $f(x_1, 0)\leqq f(x_2, 0)+f(x_2, x_1)$ and so $f(x_1, 0)\leqq 2f(x_2, 0)$, for $f(x_2, x_1)=f(x_2, 0)$.

Inversely, let $f(x, y)$ be a function which satisfies the conditions of the theorem. Let $t_1$, $t_2$, $t_3$ be three arbitrary numbers. Suppose, for instance, all the three points $(t_1, t_2)$, $(t_1, t_3)$ and $(t_3, t_2)$ in $xy$-plane, lie on the lower side of the straight line $x=y$, and $t_1>0$, $t_2<0$, $t_3<0$, $t_1+t_3>0$, $t_3+t_2<0$ and $t_1+t_2<0$.

Then
$$f(t_1, t_3)=f(t_1, 0)$$
$$f(t_3, t_2)=f(t_1, t_2)=f(-t_2, t_2)=f(-t_2, 0),$$
and
$$t_1>0, \quad -t_2>0, \quad t_1<-t_2,$$

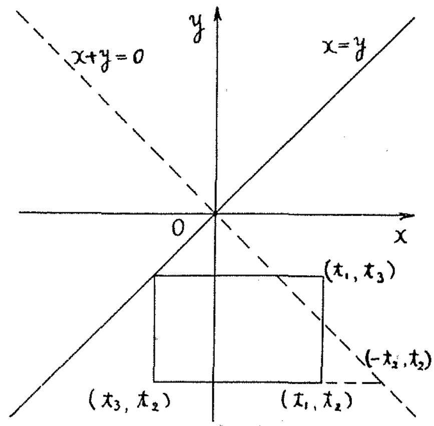
therefore $f(t_1, o) \leqq 2f(-t_2, o)$.

$\therefore$

$$f(t_1, t_3) \leqq f(t_3, t_2) + f(t_1, t_2),$$
$$f(t_3, t_2) \leqq f(t_1, t_2) + f(t_1, t_3),$$
$$f(t_1, t_2) \leqq f(t_1, t_3) + f(t_3, t_2).$$

It is clear that we may obtain the same result in all the other cases.

Fig. 6




### III. Conclusion

Let us again consider the relations between the set and its distances. The properties of the function $f(x, y)$ which defines a distance of a set $E$, *i. e.* the values (heights) of $f(x, y)$ depends, as we have seen in II for instance, upon the properties of the set $E$ or rather upon the three conditions of the distance. Therefore we may say that the properties of $f(x, y)$ are scarcely dependent of the set $E$, and so it will be natural to think that the functions which satisfy the three conditions and the properties of the set itself do not essentially affect each other. And, we think, the facts that "any set has at least one distance" and "any set may be a schlicht continuous image of a certain isolated set", confirm this statement. We have a rather strange impression of the fact that the metric spaces have many common properties with that of Euklidean space, but do not the characters of the metric space $E$ follow from the properties of the set $E$ or rather from the properties of the real function $f(x, y)$ which defines its distance *i. e.* the three conditions of the distance? Therefore, we think, it will be necessary to introduce certain conditions, for instance, conditions with respect to the operations of the distance or a certain condition like the congruent transformations in order that the properties of the set itself and its distance should affect each other.

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