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Mass formula for Jacobi weight enumerators of type II binary codes and some relationships of it with Jacobi forms

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1 Definitions from binary linear codes

1.1 Binary codes

Let $\mathbb{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. A linear $[n, k]$ code $\mathbf{C}$ is a vector subspace of V of dimension k. An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. The inner product on V, which is denoted by $\mathbf{x}\cdot\mathbf{y}$ for $\mathbf{x},\mathbf{y}$ in V, is defined as usual. Two codes $\mathbf{C}_1$ and $\mathbf{C}_2$ are said to be equivalent if and only if after a suitable change of coordinate positions of $\mathbf{C}_1$ all the codewords in both codes coincide.

Let $\mathbf{C}$ be a binary code of length n. An automorphism σ of the code $\mathbf{C}$ is an element of the permutation group of n letters S_n which leaves $\mathbf{C}$ invariant. All automorphisms of the code $\mathbf{C}$ form a group and it is denoted by $Aut(\mathbf{C})$.
The dual code $\mathbf{C}^{\perp}$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^{\perp} = \{\mathbf{u} \in V \mid \mathbf{u}\cdot\mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^{\perp}$, and the code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^{\perp}$. Self-dual codes exist only if $n \equiv 0 \pmod 2$. For even n we let S_n denote the set of all self-dual binary codes of length n. Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

be a vector in V, then the Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance d on V is also defined by $d(\mathbf{x},\mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$. Let $\mathbf{C}$ be a code, then d of the code $\mathbf{C}$ is defined by

$$\begin{aligned} d &= \mathrm{Min}_{\mathbf{x},\mathbf{y}\in\mathbf{C},\mathbf{x}\neq\mathbf{y}} d(\mathbf{x},\mathbf{y}) \\ &= \mathrm{Min}_{\mathbf{x}\in\mathbf{C},\mathbf{x}\neq\mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

Let $\mathbf{C}$ be a self-dual binary code, then the weight $wt(\mathbf{x})$ of each codeword $\mathbf{x}$ in $\mathbf{C}$ is even. Further, if the weight of each codeword $\mathbf{x}$ in $\mathbf{C}$ is divisible by 4, then the code is called doubly even. It is known that a doubly even self-dual binary codes $\mathbf{C}$ exist only when the length n of $\mathbf{C}$ is a multiple of 8. In short a doubly even self-dual binary code is type II binary code.

Let $\mathbf{C}$ be a self-dual doubly even code of length n, which are embedded in $\mathbb{F}_2^n$. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbb{F}_2^n$, then the number of common 1's of the corresponding coordinates for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is nothing else $wt(\mathbf{u})$.

Let $\mathbf{C}$ be a type II binary $[n, \frac{n}{2}]$ code. The homogeneous weight enumerator $W_{\mathbf{C}}(x, y)$ of the code $\mathbf{C}$ is defined by

$$W_{\mathbf{C}}(x,y) = \sum_{\mathbf{v}\in\mathbf{C}} x^{n-wt(\mathbf{v})} y^{wt(\mathbf{v})}$$

Following identity is known as the MacWilliams identity:

$$\begin{aligned} W_{\mathbf{C}}(x,y) &= \frac{1}{2^{\frac{n}{2}}} W_{\mathbf{C}}(x+y, x-y) \\ &= W_{\mathbf{C}}(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}), \end{aligned} \tag{1}$$

Since **C** is doubly even, each codeword **u** of **C** has weight divisible by 4, and we know that

$$W_{\mathbf{C}}(x, iy) = W_{\mathbf{C}}(x,y). \tag{2}$$

Let G_1 be the group generated by

$$\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The above two equations (1) and (2) show that the homogeneous weight enumerator of a type II binary code is invariant under linear action of the elements of the group G_1. Let $\mathbb{C}[x,y]$ be the polynomial ring over the field of complex numbers $\mathbb{C}$. We let $\mathbb{C}[x,y]^{G_1}$ to denote the subring of $\mathbb{C}[x,y]$ consisting of all elements in $\mathbb{C}[x,y]$ invariant under linear action of G_1. The following theorem is due to A. Gleason [9]

Theorem 1.1 *It holds that*

$$\mathbb{C}[x,y]^{G_1} = \mathbb{C}[W_{e_8}(x,y), W_{gol_{24}}(x,y)],$$

where $W_{e_8}(x,y)$ *is the weight enumerator of the extended Hamming code of length 8, and* $W_{gol_{24}}(x,y)$ *is the weight enumerator of the binary Golay code of length 24.*

Let H_1 be a subgroup of G_1 generated by $\sigma_1\sigma_2\sigma_1$ and σ_1. This subgroup is of index 2 in G_1. Let $\mathbb{C}[x,y]^{H_1}$ be the ring of invariants for H_1. Then it is known that (see for instance [19])

Theorem 1.2 *It holds that*

$$\mathbb{C}[x,y]^{H_1} = \mathbb{C}[W_{e_8}(x,y), E_{12}(x,y)],$$

where $E_{12}(x,y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$.

1.2 Jacobi weight enumerator

Definition: Jacobi polynomials for binary codes
Jacobi polynomial $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ for **C** with respect to $\mathbf{v} \in \mathbf{F}_2^n$ is defined by

$$Jac(\mathbf{C}, \mathbf{v} \mid X, Z) = \sum_{\mathbf{u} \in \mathbf{C}} X^{\mathbf{u}*\mathbf{u}} Z^{\mathbf{u}*\mathbf{v}}.$$

The homogeneous form of $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ is given by

$$Jac(\mathbf{C}, \mathbf{v}; x, y, u, v) = \sum_{\mathbf{t} \in \mathbf{C}} x^{n - wt(\mathbf{v}) - wt(\mathbf{t}) + \mathbf{t}*\mathbf{v}} y^{wt(\mathbf{t}) - \mathbf{t}*\mathbf{v}} u^{wt(\mathbf{v}) - \mathbf{t}*\mathbf{v}} v^{\mathbf{t}*\mathbf{v}}.$$

Theorem 1.3 *Let the notations be as above, then we have*

$$Jac(\mathbf{C}, \mathbf{v}; x', y', u', v') = Jac(\mathbf{C}, \mathbf{v}; x, y, u, v), \tag{3}$$

where

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

It may be remarked here that it holds

$$Jac(\mathbf{C}, \mathbf{v}; x, iy, u, iv) = Jac(\mathbf{C}, \mathbf{v}; x, y, u, v) \tag{4}$$

Let $G_1 \oplus G_1$ be the group generated by $diag(\sigma_1, \sigma_1)$ and $diag(\sigma_2, \sigma_2)$, and $\mathbb{C}[x, y, u, v]$ be the polynomial ring in 4 independent variables over $\mathbb{C}$. We let $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$ to denote the subring of $\mathbb{C}[x, y, u, v]$ invariant under the linear action of each element of $G_1 \oplus G_1$. The above equations (3) and (4) implies that $Jac(\mathbf{C}, \mathbf{v}; x, y, u, v)$ belongs to $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$. We have a Gleason type result for $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$ ([4]).

Let $H_1 \oplus H_1$ be the group generated by $diag(\sigma_1, \sigma_1)$ and $diag(\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1)$, and $R = \mathbb{C}[x, y, u, v]^{H_1 \oplus H_1}$ be the ring of invariants for the group $H_1 \oplus H_1$. We also have a Gleason type result for R. Here we briefly describe the result. When a polynomial $f(x, y, u, v)$ of total degree n belongs to R we call the partial degree of f with respect to the varaibles u and v the index of f. The Molien series for $H_1 \oplus H_1$ is given by

$$\begin{aligned}
&\Phi_{H_1 \bigoplus H_1}(t) \\
&= \sum_{n \geq 0} \dim_{\mathbb{C}}(FJac_n) t^n \\
&= \frac{1 + 8t^8 + 18t^{12} + 21t^{16} + 19t^{20} + 21t^{24} + 7t^{28} + t^{32}}{(1 - t^8)^2(1 - t^{12})^2} \\
&= 1 + 10t^8 + 20t^{12} + 40t^{16} + 75t^{20} + 130t^{24} + 179t^{28} + 283t^{32} + \\
&\quad 383t^{36} + 513t^{40} + 678t^{44} + 883t^{48} + 1078t^{52} + 1372t^{56} + \\
&\quad +1658t^{60} + 1994t^{64} + 2385t^{68} + 2836t^{72} + \cdots .
\end{aligned}$$

We decompose this ring R into a direct sum :

$$R = \bigoplus_{n \geq 0} R_n,$$

where R_n is the n-th homogeneous part of R. Further we decompose R_n as

$$R_n = \bigoplus_{0 \leq m \leq n} R_{n,m},$$

where $R_{n,m}$ is the set of polynomials $f(x, y, u, v) \in R_n$ with partial degree with respect to u and v equal to m. This set $R_{n,m}$ forms a vector subspace of R.

2 Jacobi forms

2.1 Definition of Jacobi forms

Let $\mathbb{H}$ be the complex upper half plane and τ be a variable on $\mathbb{H}$. Let $\mathbb{C}$ be the complex plane and z be a variable on $\mathbb{C}$. A complex valued holomorphic function $\phi(\tau, z)$ defined on $\mathbb{H} \times \mathbb{C}$ is called a Jacobi form of weight k and index h with respect to the pair $(SL_2(\mathbb{Z}), \mathbb{Z})$ if it satisfies the conditions (5), (6) and (7) below:

$$\phi(\tau, z) = (c\tau + d)^{-k} e^{2\pi i h\left(\frac{-cz^2}{c\tau + d}\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \text{ holds for } \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{5}$$

$$\phi(\tau, z) = e^{2\pi i h(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu) \text{ for } \lambda, \mu \in \mathbb{Z} \tag{6}$$

$\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq r^2/4h} c(n, r) q^n \zeta^r \tag{7}$$

2.2 Eisenstein Jacobi forms

One major construction method of Jacobi forms is Eisenstein Jacobi forms (c.f. [8],pages 17-18).

$$\begin{aligned} E_{k,m}(\tau,z) &= \\ &= \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\sum_{\lambda\in\mathbb{Z}}(c\tau+d)^{-k}\mathbf{e}^{m}\left(\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}-\frac{cz^2}{c\tau+d}\right) \\ &= \sum_{\substack{n,r\in\mathbb{Z}\\4nm\geq r^2}} e_{k,m}(n,r)q^n\zeta^r \end{aligned}$$

where a,b are chosen so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

3 Massformula for Jacobi weight enumerators

3.1 Mass formula for ordinary weight enumerators

For $1 \leq h < \frac{n}{2}$ let C_0 be a binary self-orthogonal code of length n and dimension h containing all one vector $\mathbf{1}$ in $\mathbb{F}_2^n$. We denote by

$$\nu(n,h) = \#\{C \in \mathcal{D}_n \mid C \supset C_0\}.$$

This is independent of the choice of C_0.
We recall that $\mathcal{S}_n$ is the set of all binary self-dual codes of length n for each even integer n. We denote by

$$\mu(n,h) = \#\{C \in \mathcal{S}_n \mid C \supset C_0\}.$$

We quote a well-known result

Proposition 3.1 *([20]) It holds that*

$$\nu(n,h) = \prod_{j=0}^{\frac{n}{2}-h-1}(2^j+1).$$

Proposition 3.2 *([20]) For h with $1 \leq h < \frac{n}{2}$ it holds that*

$$\mu(n,h) = \prod_{j=1}^{\frac{n}{2}-h}(2^j+1).$$

J.G. Thompson [20] proved that

$$\sum_{C\in\mathcal{D}_n} W_1(x,y;C) = \nu(n,1)(x^n+y^n) + \nu(n,2)\sum_{\substack{0<j<n\\4|j}}\begin{pmatrix} n \\ j \end{pmatrix} x^{n-j}y^j.$$

If we define

$$\begin{aligned} W_1^{(n)}(x,y) &= \sum_{4|j}\begin{pmatrix} n \\ j \end{pmatrix} x^{n-j}y^j \\ &= \frac{1}{4}((x+y)^n+(x-y)^n+(x+iy)^n+(x-iy)^n), \end{aligned}$$

then

$$\sum_{C\in\mathcal{D}_n} W_1(x,y;C) = \nu(n,2)(2^{n/2-2}(x^n+y^n)+W_1^{(n)}(x,y)).$$

Recall that the root system D_4 consists of the 24 roots listed below:

$$\pm\sqrt{2}e_j \quad (j=1,2,3,4),$$
$$\frac{1}{\sqrt{2}}(\pm1,\pm1,\pm1,\pm1).$$

We imbed these vectors into $\mathbb{C}^2$ as follows.

$$i^k\sqrt{2}e_j \quad (j=1,2,\ k=0,1,2,3),$$
$$\zeta^j e_1+\zeta^k e_2 \quad (j,k=1,3,5,7),$$

where $\zeta=e^{\pi i/4}$. Now, let D_4 denote the set of 24 vectors above. If $n\equiv 0 \pmod 4$, then

$$\begin{aligned}\sum_{\alpha\in D_4}(\alpha_1x+\alpha_2y)^n &= 2^{n/2+2}(x^n+y^n)+\sum_{j,k=1,3,5,7}(\zeta^jx+\zeta^ky)^n\\ &= 2^{n/2+2}(x^n+y^n)+(-1)^{n/4}\sum_{j,k=0,2,4,6}(\zeta^jx+\zeta^ky)^n\\ &= 16(2^{n/2-2}(x^n+y^n)+(-1)^{n/4}W_n^{(1)}(x,y)). \qquad (8)\end{aligned}$$

3.2 A Theorem

Using the notation introduced in the previous section, the mass formula for the Jacobi weight enumerator polynomial can easily be established. The Jacobi weight enumerator polynomial for a code $\mathbf{C}$ with respect to a reference vector $\mathbf{u}$ is defined by

$$Jac(\mathbf{C},\mathbf{u};x_{00},x_{01},x_{10},x_{11})=\sum_{\mathbf{v}\in\mathbf{C}}X(\mathbf{u},\mathbf{v}).$$

Denote by $\overline{Jac_{n,k}}$ the sum of the Jacobi weight enumerator polynomial with respect to a fixed reference vector of weight k for all $C\in\mathcal{D}_n$. Note that $\overline{Jac_{n,k}}$ is independent of the choice of $\mathbf{u}$. We prove

Theorem 3.3 (Munemasa-Ozeki [13])

$$\overline{Jac_{n,k}}=\frac{1}{16}\nu(n,2)\sum_{\alpha=(\alpha_1,\alpha_2)\in D_4}(\alpha_1x_{00}+\alpha_2x_{01})^{n-k}(\alpha_1x_{10}+\alpha_2x_{11})^k. \qquad (9)$$

4 An application of the mass formula to the construction of Jacobi forms

4.1 Some instances

If we apply the so called Bannai-Ozeki map (c.f. [2]) to the right hand side of (9), we obtain many important Jacobi forms of weight $n/2$ and index k. As the mass formula the both hands are meaningful only when n is divisible by 8. However the polynomials in the right hand side are useful even if $n\equiv 4 \bmod 8$ in constructing Jacobi forms. Here we give few instances of the construction.

To do this we recall Jacobi's theta functions:

$$\begin{aligned}\theta_0(\tau,z) &= \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi n^2\tau+2n\pi iz},\\ \theta_2(\tau,z) &= \sum_{n\in\mathbb{Z}} e^{\pi(n+1/2)^2\tau+(2n+1)\pi iz},\\ \theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} e^{\pi n^2\tau+2n\pi iz}.\end{aligned}$$

We put $\varphi_i(\tau, z) = \theta(2\tau, 2z)$, and $\varphi_i(\tau) = \varphi_i(\tau, 0)$ $(i = 2, 3)$.

When $n = 8$ and $k = 1$ the right hand side of (9) becomes 30 times of

$$7y^4x^3u + 7x^4y^3v + y^7v + x^7u.$$

Substituting $x = \varphi_2(\tau), y = \varphi_3(\tau), u = \varphi_2(\tau, z), v = \varphi_3(\tau, z)$ in this polynomial we get a Jacobi form of weight 4 and index 1:

$$\begin{aligned}
&\psi_{4,1} = \\
&\quad 1 + (\zeta^2 + 56\zeta + 56\zeta^{-1} + \zeta^{-2} + 126)q \\
&\quad +(126\zeta^2 + 576\zeta + 576\zeta^{-1} + 126\zeta^{-2} + 756)q^2 \\
&\quad +(56\zeta^3 + 756\zeta^2 + 1512\zeta + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3} + 2072)q^3 \\
&\quad +(\zeta^4 + 576\zeta^3 + 2072\zeta^2 + 4032\zeta + 4032\zeta^{-1} + 2072\zeta^{-2} + 576\zeta^{-3} + \zeta^{-4} + 4158)q^4 \\
&\quad (126\zeta^4 + 1512\zeta^3 + 4158\zeta^2 + 5544\zeta + 5544\zeta^{-1} + 4158\zeta^{-2} + 1512\zeta^{-3} + 126\zeta^{-4} + 7560)q^5 + \cdots
\end{aligned}$$

When $n = 8$ and $k = 2$ the right hand side of (9) becomes 30 times of

$$3x^4y^2v^2 + 8x^3y^3uv + 3x^2y^4u^2 + x^6u^2 + y^6v^2.$$

The last polynomial leads to a Jacobi form of weight 4 and index 2:

$$\begin{aligned}
&\psi_{4,2} = \\
&\quad 1 + [14(\zeta^2 + \zeta^{-2}) + 64(\zeta + \zeta^{-1}) + 84]q + \\
&\quad +[\zeta^4 + \zeta^{-4} + 64(\zeta^3 + \zeta^{-3}) + 280(\zeta^2 + \zeta^{-2}) + 448(\zeta + \zeta^{-1}) + 574]q^2 \\
&\quad +(84\zeta^4 + 448\zeta^3 + 840\zeta^2 + 1344\zeta + 1344\zeta^{-1} + 840\zeta^{-2} + 448\zeta^{-3} + 84\zeta^{-4} + 1288)q^3 \\
&\quad +[64(\zeta^5 + \zeta^{-5}) + 574(\zeta^4 + \zeta^{-4}) + 1344(\zeta^3 + \zeta^{-3}) + 2368(\zeta^2 + \zeta^{-2}) + 2688(\zeta + \zeta^{-1}) + 3444]q^4 \\
&\quad +[14(\zeta^6 + \zeta^{-6}) + 448(\zeta^5 + \zeta^{-5}) + 1288(\zeta^4 + \zeta^{-4}) + 2688(\zeta^3 + \zeta^{-3}) \\
&\quad +3542(\zeta^2 + \zeta^{-2}) + 4928(\zeta + \zeta^{-1}) + 4424]q^5 + \cdots
\end{aligned}$$

When $n = 12$ and $k = 1$ the right hand side of (9) is a polynomial that is 4050 times of

$$-22x^4y^7v - 11x^8y^3v - 11y^8x^3u - 22y^4x^7u + x^{11}u + y^{11}v.$$

This leads to a Jacobi form of weight 6 and index 1

$$\begin{aligned}
&\psi_{6,1} = \\
&\quad 1 + (\zeta^2 - 88\zeta - 88\zeta^{-1} + \zeta^{-2} - 330)q \\
&\quad +(-330\zeta^2 - 4224\zeta - 4224\zeta^{-1} - 330\zeta^{-2} - 7524)q^2 \\
&\quad +(-88\zeta^3 - 7524\zeta^2 - 30600\zeta - 30600\zeta^{-1} - 7524\zeta^{-2} - 88\zeta^{-3} - 46552)q^3 \\
&\quad +(\zeta^4 - 4224\zeta^3 - 46552\zeta^2 - 130944\zeta \\
&\quad -130944\zeta^{-1} - 46552\zeta^{-2} - 4224\zeta^{-3} + \zeta^{-4} - 169290)q^4 \\
&\quad (-330\zeta^4 - 30600\zeta^3 - 169290\zeta^2 - 355080\zeta \\
&\quad -355080\zeta^{-1} - 169290\zeta^{-2} - 30600\zeta^{-3} - 330\zeta^{-4} - 464904)q^5 + \cdots
\end{aligned}$$

When $n = 12$ and $k = 2$ the right hand side of (9) is 4050 times of the polynomial

$$-14y^4x^6u^2 - 14y^6x^4v^2 - 3y^2x^8v^2 - 3y^8x^2u^2 + x^{10}u^2 + y^{10}v^2 - 16y^3x^7uv - 16y^7x^3uv.$$

This leads to a Jacobi form of weight 6 and index 2:

$\psi_{6,2} =$

$$1 + (-10\zeta^2 - 128\zeta - 128\zeta^{-1} - 10\zeta^{-2} - 228)q$$
$$+(\zeta^4 - 128\zeta^3 - 1496\zeta^2 - 3968\zeta$$
$$-3968\zeta^{-1} - 1496\zeta^{-2} - 128\zeta^{-3} + \zeta^{-4} - 5450)q^2$$
$$+(-228\zeta^4 - 3968\zeta^3 - 14088\zeta^2 - 27264\zeta$$
$$-27264\zeta^{-1} - 14088\zeta^{-2} - 3968\zeta^{-3} - 228\zeta^{-4} - 31880)q^3$$
$$+(-128\zeta^5 - 5450\zeta^4 - 27264\zeta^3 - 67712\zeta^2 - 103680\zeta$$
$$-103680\zeta^{-1} - 67712\zeta^{-2} - 27264\zeta^{-3} - 5450\zeta^{-4} - 128\zeta^{-5} - 124260)q^4$$
$$(-10\zeta^6 - 3968\zeta^5 - 31880\zeta^4 - 103680\zeta^3 - 197650\zeta^2 - 292480\zeta$$
$$-292480\zeta^{-1} - 197650\zeta^{-2} - 103680\zeta^{-3} - 31880\zeta^{-4} - 3968\zeta^{-5} - 10\zeta^{-6} - 316168)q^5 + \cdots$$

In this way we obtain an infinite family of Jacobi forms of various weights and various indeces.

4.2 A comparison of two constructions

In [8] only the values $e_{k,m}(n.r)$ $(k \leq 8, m = 1)$ of the Fourier coefficients of $E_{k,m}(\tau, z)$ are given explicitly.
Here we explain a method to compute $e_{k,m}(n.r)$ for any even k and $m \geq 1$. For this we start from the formula given in [8] page 22:

$$e_{k,m}(n.r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H(k-1, \frac{4nm - r^2}{d^2}),$$

and

$$e_{k,1}(n.r) = \frac{H(k-1, 4n - r^2)}{\zeta(3-2k)},$$

where $\zeta(3-2k)$ is the special value of Riemann's zeta function. The quantity $H(k-1, N)$ is described at page 30 in [8]:

$$H(k-1, N) = \begin{cases} L_{-N}(2-k) & \text{if } N > 0 \text{ and } N \equiv 0 \text{ or } 3 \pmod 4, \\ \zeta(3-2k) & \text{if } N = 0, \\ 0 & \text{if } N > 0 \text{and } N \equiv 1 \text{ or} 2 \pmod 4. \end{cases}$$

When $-N \equiv 0$ or $1 \pmod 4$ we put $-N = (-N_0)u^2$ $u \in \mathbb{N}$ so that $-N_0$ is the discriminant of the quadratic number field $\mathbb{Q}(\sqrt{-N})$. The number $L_{-N}(2-k)$ comes from the L-function $L_{-N_0}(s)$ by way of

$$L_{-N}(s) = L_{-N_0}(s) \sum_{d|u} \mu(d) \left(\frac{-N_0}{d}\right) d^{-s} \sigma_{1-2s}(\frac{u}{d}),$$

and

$$L_{-N_0}(s) = L(s, (\frac{-N_0}{*}) = \sum_{n=1}^{\infty} \frac{(\frac{-N_0}{d})}{n^s}.$$

To make the value $e_{k,1}(n.r)$ explicit it is neccesary to know the values $\zeta(3-2k)$ and $L_{-N_0}(1-m)$. As to the values $\zeta(3-2k)$ there are many literature available and they tell us that

$$\zeta(2k) = \frac{(-1)^{k-1}}{2} \frac{(2\pi)^{2k}}{(2k)!} B_{2k} \quad (k \geq 1)$$
$$\zeta(1-n) = (-1)^{n-1} \frac{B_n}{n} \quad (n = 1, 2, \cdots),$$

where B_n is the n-th Bernouilli number. The beginning few numbers are

$$B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \cdots, B_{odd \geq 3} = 0.$$

By these one gets

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450},\ \zeta(10) = \frac{\pi^{10}}{93555},\cdots$$

$$\zeta(-1) = -\frac{1}{12},\ \zeta(-2) = 0,\ \zeta(-3) = \frac{1}{120},\ \zeta(-5) = -\frac{1}{252},\ \zeta(-7) = \frac{1}{240},\ \zeta(-9) = -\frac{1}{132},\cdots.$$

It is much complicated to get the values $L_{-N_0}(1-m)$. On reading the book [1] we find a suitable formula to do this task. Note that a similar formula has been given in [11] Chapter XIII in a not straight way.

Theorem 4.1 *(Arakawa-Ibukiyama-Kaneko) Let χ be a primitive character mod f and m be a positive integer, then*

$$L(1-m,\chi) = -\frac{B_{m,\chi}}{m},$$

where $B_{m,\chi}$ is the generalized Bernouilli number associated with χ:

$$B_{m,\chi} = f^{m-1}\sum_{a=1}^{f}\chi(a)B_m(\frac{a}{f}),$$

and $B_m(x)$ is the Bernouilli polynomial of degree m.

The Bernouilli polynomials are given by

$$B_m(x) = \sum_{j=0}^{n}(-1)^j\begin{pmatrix} n \\ j \end{pmatrix}B_j x^{m-j}.$$

With the above Theorem we compute $e_{k,m}(n.r)$, and we give small tables of them, that are not contained in [8].

$4n-r^2$	0	3	4	7	8	11	12	15	16	19	20
$e_{10,1}(n,r)$	1	$-\frac{860776}{43867}$	$-\frac{9947070}{43867}$	$-\frac{1159757568}{43867}$	$-\frac{3601586268}{43867}$	$-\frac{53854227000}{43867}$	$-\frac{113044851304}{43867}$	$-\frac{754799931648}{43867}$	$-\frac{1303792306110}{43867}$	$-\frac{5607166776120}{43867}$	$-\frac{8689286943288}{43867}$
$e_{12,1}(n,r)$	1	$\frac{339848}{77683}$	$\frac{6971898}{77683}$	$\frac{2485779648}{77683}$	$\frac{10096500348}{77683}$	$\frac{285849348696}{77683}$	$\frac{713061257096}{77683}$	$\frac{7428376170816}{77683}$	$\frac{14621136806394}{77683}$	$\frac{88801830903192}{77683}$	$\frac{152244273101400}{77683}$

$8n-r^2$	0	4	7	8	12	15	16	20	23	24	28	31	32
$e_{4,2}(n.r)$	1	14	64	84	280	448	574	840	1344	1288	2368	2688	3444
$e_{6,2}(n.r)$	1	-10	-128	-228	-1496	-3968	-5450	-14088	-27264	-31880	-67712	-103680	-124260
$e_{8,2}(n,r)$	1	$\frac{122}{43}$	$\frac{4872}{43}$	$\frac{11052}{43}$	$\frac{3640}{43}$	$\frac{662464}{43}$	$\frac{1015162}{43}$	$\frac{4266360}{43}$	$\frac{10665792}{43}$	$\frac{13948984}{43}$	$\frac{38576704}{43}$	$\frac{74169984}{43}$	$\frac{91963692}{43}$

We remark that the functions $\psi_{4,1}, \psi_{6,1}, \psi_{8,1}$ respectively coincide with Eisenstein-Jacobi forms $E_{4,1}, E_{6,1}, E_{8,1}$ respectively of index 1 described in [8] pages 17-23. Explicit Fourier expansions of Eisenstein-Jacobi forms of index ≥ 2 are not given in [8]. We have verified that $\psi_{4,2}, \psi_{6,2}$ also coincide with Jacobi-Eisenstein series of index 2. This is done by using the relation (7) in [8],page 22. Besides these exceptional cases Eisenstein-Jacobi form $E_{k,m}$ differ from $\psi_{k,m}$. One may be interested with a problem to explore the further relations between these two constructions of Jacobi forms.

5 Eisenstein type polynomials in more variables

$$E_{k_1,k_2}(x_{00},x_{01},x_{10},x_{11},y_{10},y_{11}) = \frac{1}{16}\nu(n,2)\sum_{\alpha=(\alpha_1,\alpha_2)\in D_4}(\alpha_1 x_{00}+\alpha_2 x_{01})^{n-k_1-k_2}(\alpha_1 x_{10}+\alpha_2 x_{11})^{k_1}(\alpha_1 y_{10}+\alpha_2 y_{11})^{k_2}, \tag{10}$$

$$E_{k_1,k_2,k_3}(x_{00},x_{01},x_{10},x_{11},y_{10},y_{11},z_{10},z_{11}) = \frac{1}{16}\nu(n,2)\sum_{\alpha=(\alpha_1,\alpha_2)\in D_4}(\alpha_1 x_{00}+\alpha_2 x_{01})^{n-k_1-k_2-k_3}(\alpha_1 x_{10}+\alpha_2 x_{11})^{k_1}(\alpha_1 y_{10}+\alpha_2 y_{11})^{k_2}(\alpha_1 z_{10}+\alpha_2 z_{11})^{k_3} \tag{11}$$

$$\vdots,$$

where all exponents are non negative integers.
The right-hand side of (10) belongs to $\mathbb{C}[x_{00},x_{01},x_{10},x_{11},y_{10},y_{11}]^{H_1\oplus H_1\oplus H_1}$, and the right-hand side of (11) belongs $\mathbb{C}[x_{00},x_{01},x_{10},x_{11},y_{10},y_{11},z_{10},z_{11}]^{H_1\oplus H_1\oplus H_1\oplus H_1}$. As discussed in [2] these polynomials contribute to the construction of Jacobi forms.

References

[1] T. Arakawa, T. Ibukiyama, and M. Kaneko, Bernouilli numbers and zeta functions (In Japanese) Makino Shoten (a publisher) (2000)

[2] E. Bannai,and M. Ozeki, Construction of Jacobi forms from certain combinatorial polynomials, Proc. Japan Academy Ser.A Vol.72 (1996),12 – 15

[3] E. Bannai, E. Bannai, M. Ozeki and S. Teranishi, Rings of simultaneous invariants for the MacWilliams-Gleason group, European J. Comb. **20** (1999), 619 – 627

[4] E. Bannai, M. Ozeki and K. Tanabe, A Gleason type theorem for the simultaneous invariants for the MacWilliams-Gleason group, in preparation

[5] M. Broué et M. Enguehard, Polynôme des poids de certains codes et fonction théta de certains réseaux, Ann. scien. Éc. Norm. Sup. 4^e série, t.5 (1972), 157-181.

[6] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, Math. Ann. 217 (1975) 271-285

[7] J.H. Conway and N.J.A. Sloane, Sphere Packings, Lattices and Groups, Springer-Verlag 1988. 3rd ed. 1998.

[8] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Birkhäuser, 1985.

[9] A. M. Gleason, Weight polynomials of self-dual codes and the MacWilliams identities, in Actes, Congrés International de Mathématiques, Gauthier-Villars, Paris, **3** (1970), 211-215

[10] W. Kohnen, Modular forms of half-integral weight on $\Gamma_0(4)$, Math. Ann. 248 (1980) 249-266

[11] S. Lang, Introduction to Modular Forms, Springer-Verlag 1976.

[12] F.J. MacWilliams and N.J.A.Sloane, "The Theory of Error-Correcting Codes," North-Holland, Amsterdam, 1977.

[13] A. Munemasa and M. Ozeki, Mass formula for various generalized weight enumerators of binary self-dual codes, Preprint 2003

[14] G. Nebe, E.M. Rains and N.J.A. Sloane, The invariants of the Clifford Groups, Designs Codes and Cryptography **24** (2001), 99 – 122

[15] M. Ozeki, On the notion of Jacobi polynomilas for codes, Math. Proc. Cambridge Philos. Soc. **121** (1997), 15 – 30.

[16] V. Pless and N.J.A. Sloane, On the claasification and enumeration of self-dual codes, J. Comb. Th. Ser. A **18** (1975), 313 – 335.

[17] E.M. Rains and N.J.A.Sloane, Self-Dual Codes, in Handbook of Coding Theory V.S. Pless and W.C. Huffman (Eds.), North-Holland 1998.

[18] B. Runge, Codes and Siegel modular forms, Disc. Math. **148** (1995), 175 – 205.

[19] G.C. Shephard and J.A. Todd, Finite unitary reflection groups , Canad J. Math. **5** (1953), 364 – 383.

[20] J.G. Thompson, Weighted averages associated to some codes, Scripta Math. **29** (1973), 449 – 452.