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Kyoto University
Mass formula for Jacobi weight enumerators of type II binary codes and some relationships of it with Jacobi forms

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1 Definitions from binary linear codes

1.1 Binary codes

Let $F_2 = GF(2)$ be the field of 2 elements. Let $V = F_2^n$ be the vector space of dimension $n$ over $F_2$. A linear $[n,k]$ code $C$ is a vector subspace of $V$ of dimension $k$. An element $x$ in $C$ is called a codeword of $C$. The inner product on $V$, which is denoted by $x \cdot y$ for $x,y$ in $V$, is defined as usual. Two codes $C_1$ and $C_2$ are said to be equivalent if and only if after a suitable change of coordinate positions of $C_1$ all the codewords in both codes coincide.

Let $C$ be a binary code of length $n$. An automorphism $\sigma$ of the code $C$ is an element of the permutation group of $n$ letters $S_n$ which leaves $C$ invariant. All automorphisms of the code $C$ form a group and it is denoted by $\text{Aut}(C)$.

The dual code $C^\perp$ of $C$ is defined by

$$C^\perp = \{ u \in V \mid u \cdot v = 0 \ \forall v \in C \}.$$

The code $C$ is called self-orthogonal if it satisfies $C \subseteq C^\perp$, and the code $C$ is called self-dual if it satisfies $C = C^\perp$. Self-dual codes exist only if $n \equiv 0 \pmod{2}$. For even $n$ we let $S_n$ denote the set of all self-dual binary codes of length $n$. Let

$$x = (x_1, x_2, \ldots, x_n)$$

be a vector in $V$, then the Hamming weight $wt(x)$ of the vector $x$ is defined to be the number of 1's such that $x_i \neq 0$. The Hamming distance $d$ on $V$ is also defined by $d(x,y) = wt(x - y)$. Let $C$ be a code, then $d$ of the code $C$ is defined by

$$d = \min_{x,y \in C, x \neq y} d(x,y) = \min_{x \in C, x \neq 0} wt(x).$$

Let $C$ be a self-dual binary code, then the weight $wt(x)$ of each codeword $x$ in $C$ is even. Further, if the weight of each codeword $x$ in $C$ is divisible by 4, then the code is called doubly even. It is known that a doubly even self-dual binary code exists only when the length $n$ of $C$ is a multiple of 8. In short a doubly even self-dual binary code is type II binary code.

Let $C$ be a self-dual doubly even code of length $n$, which are embedded in $F_2^n$. Let $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n)$ be any pair of vectors in $F_2^n$, then the number of common 1's of the corresponding coordinates for $u$ and $v$ is denoted by $u \cdot v$. This is called the intersection number of $u$ and $v$, and $u \cdot u$ is nothing else $wt(u)$.

Let $C$ be a type II binary $[n, \frac{n}{2}]$ code. The homogeneous weight enumerator $W_C(x,y)$ of the code $C$ is defined by

$$W_C(x,y) = \sum_{v \in C} x^{n-wt(v)} y^{wt(v)}$$
Following identity is known as the MacWilliams identity:

\[
W_C(x, y) = \frac{1}{2^2} W_C(x + y, x - y) = W_C(\frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}}),
\]

(1)

Since \( C \) is doubly even, each codeword \( u \) of \( C \) has weight divisible by 4, and we know that

\[
W_C(x, iy) = W_C(x, y).
\]

(2)

Let \( G_1 \) be the group generated by

\[
\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.
\]

The above two equations (1) and (2) show that the homogeneous weight enumerator of a type II binary code is invariant under linear action of the elements of the group \( G_1 \). Let \( \mathbb{C}[x, y] \) be the polynomial ring over the field of complex numbers \( \mathbb{C} \). We let \( \mathbb{C}[x, y]^{G_1} \) to denote the subring of \( \mathbb{C}[x, y] \) consisting of all elements in \( \mathbb{C}[x, y] \) invariant under linear action of \( G_1 \). The following theorem is due to A. Gleason [9]

Theorem 1.1 It holds that

\[
\mathbb{C}[x, y]^{G_1} = \mathbb{C}[W_{e_8}(x, y), W_{gol_{24}}(x, y)],
\]

where \( W_{e_8}(x, y) \) is the weight enumerator of the extended Hamming code of length 8, and \( W_{gol_{24}}(x, y) \) is the weight enumerator of the binary Golay code of length 24.

Let \( H_1 \) be a subgroup of \( G_1 \) generated by \( \sigma_1 \sigma_2 \sigma_1 \) and \( \sigma_1 \). This subgroup is of index 2 in \( G_1 \). Let \( \mathbb{C}[x, y]^{H_1} \) be the ring of invariants for \( H_1 \). Then it is known that (see for instance [19])

Theorem 1.2 It holds that

\[
\mathbb{C}[x, y]^{H_1} = \mathbb{C}[W_{e_8}(x, y), E_{12}(x, y)],
\]

where \( E_{12}(x, y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12} \).

1.2 Jacobi weight enumerator

Definition: Jacobi polynomials for binary codes

Jacobi polynomial \( Jac(C, v \mid X, Z) \) for \( C \) with respect to \( v \in \mathbb{F}_2^n \) is defined by

\[
Jac(C, v \mid X, Z) = \sum_{u \in C} X^{u \ast u} Z^{u \ast v}.
\]

The homogeneous form of \( Jac(C, v \mid X, Z) \) is given by

\[
Jac(C, v; x, y, u, v) = \sum_{t \in \mathbb{C}} x^{n-\omega t(v)-\omega t(t)+t \ast v} y^{\omega t(t)-t \ast v} u^{\omega t(v)-t \ast v} x^{t \ast v}.
\]

Theorem 1.3 Let the notations be as above, then we have

\[
Jac(C, v; x', y', u', v') = Jac(C, v; x, y, u, v),
\]

(3)

where

\[
\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}.
\]
It may be remarked here that it holds

\[ Jac(C, v; z, iu, u, v) = Jac(C, v; z, y, u, v) \]  

(4)

Let \( G_1 \oplus G_1 \) be the group generated by \( \text{diag}(\sigma_1, \sigma_1) \) and \( \text{diag}(\sigma_2, \sigma_2) \), and \( C[x, y, u, v] \) be the polynomial ring in 4 independent variables over \( C \). We let \( C[x, y, u, v]^{G_1 \oplus G_1} \) to denote the subring of \( C[x, y, u, v] \) invariant under the linear action of each element of \( G_1 \oplus G_1 \). The above equations (3) and (4) implies that \( Jac(C, v; z, y, u, v) \) belongs to \( C[x, y, u, v]^{G_1 \oplus G_1} \). We have a Gleason type result for \( C[x, y, u, v]^{G_1 \oplus G_1} \) (4)).

Let \( H_1 \oplus H_1 \) be the group generated by \( \text{diag}(\sigma_1, \sigma_1) \) and \( \text{diag}(\sigma_1 \sigma_2 \sigma_1, \sigma_1 \sigma_2 \sigma_1) \), and \( R = C[x, y, u, v]^{H_1 \oplus H_1} \) be the ring of invariants for the group \( H_1 \oplus H_1 \). We also have a Gleason type result for \( R \). Here we briefly describe the result. When a polynomial \( f(x, y, u, v) \) of total degree \( n \) belongs to \( R \) we call the partial degree of \( f \) with respect to the variables \( u \) and \( v \) the index of \( f \). The Molien series for \( H_1 \oplus H_1 \) is given by

\[
\Phi_{H_1 \oplus H_1}(t) = \sum_{n \geq 0} \dim \mathcal{C} \left( FJac_n \right) t^n
\]

We decompose this ring \( R \) into a direct sum:

\[ R = \bigoplus_{n \geq 0} R_n, \]

where \( R_n \) is the \( n \)-th homogeneous part of \( R \). Further we decompose \( R_n \) as

\[ R_n = \bigoplus_{0 \leq m \leq n} R_{n, m}, \]

where \( R_{n, m} \) is the set of polynomials \( f(x, y, u, v) \in R_n \) with partial degree with respect to \( u \) and \( v \) equal to \( m \). This set \( R_{n, m} \) forms a vector subspace of \( R \).

2 Jacobi forms

2.1 Definition of Jacobi forms

Let \( \mathbb{H} \) be the complex upper half plane and \( \tau \) be a variable on \( \mathbb{H} \). Let \( \mathbb{C} \) be the complex plane and \( z \) be a variable on \( \mathbb{C} \). A complex valued holomorphic function \( \phi(\tau, z) \) defined on \( \mathbb{H} \times \mathbb{C} \) is called a Jacobi form of weight \( k \) and index \( h \) with respect to the pair \((SL_2(\mathbb{Z}), \mathbb{Z})\) if it satisfies the conditions (5), (6) and (7) below:

\[
\phi(\tau, z) = (c\tau + d)^{-k} \mathcal{E}^{2\pi ih} \left( \frac{z^2}{c\tau + d} \right) \phi \left( \frac{az + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \quad \text{holds for} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \quad \text{(5)}
\]

\[
\phi(\tau, z) = e^{2\pi i h(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z + \lambda \tau + \mu) \quad \text{for} \quad \lambda, \mu \in \mathbb{Z} \quad \text{(6)}
\]

\[
\phi(\tau, z) \text{ has a Fourier expansion of the form}
\]

\[
\phi(\tau, z) = \sum_{n \geq r^2/4h} c(n, \tau) q^n \zeta^r \quad \text{(7)}
\]
2.2 Eisenstein Jacobi forms

One major construction method of Jacobi forms is Eisenstein Jacobi forms (c.f. [8], pages 17-18).

\[ E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{a} \exp \left( \frac{z^{2}}{c\tau + d} + \frac{2\lambda z}{c\tau + d} - \frac{cz^{2}}{c\tau + d} \right) \]

where \( a, b \) are chosen so that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_{2}(\mathbb{Z}) \).

3 Massformula for Jacobi weight enumerators

3.1 Mass formula for ordinary weight enumerators

For \( 1 \leq h < \frac{n}{2} \) let \( C_{0} \) be a binary self-orthogonal code of length \( n \) and dimension \( h \) containing all one vector \( 1 \) in \( \mathbb{F}_{2}^{n} \). We denote by

\[ \nu(n, h) = \# \{ C \in \mathcal{D}_{n} \mid C \supset C_{0} \} . \]

This is independent of the choice of \( C_{0} \).

We recall that \( \mathcal{S}_{n} \) is the set of all binary self-dual codes of length \( n \) for each even integer \( n \). We denote by

\[ \mu(n, h) = \# \{ C \in \mathcal{S}_{n} \mid C \supset C_{0} \} . \]

We quote a well-known result

Proposition 3.1 ([20]) It holds that

\[ \nu(n, h) = \prod_{j=0}^{h-1} (2^{j} + 1) . \]

Proposition 3.2 ([20]) For \( h \) with \( 1 \leq h < \frac{n}{2} \) it holds that

\[ \mu(n, h) = \prod_{j=1}^{h} (2^{j} + 1) . \]

J.G. Thompson [20] proved that

\[ \sum_{C \in \mathcal{D}_{n}} W_{1}(x, y; C) = \nu(n, 1)(x^{n} + y^{n}) + \nu(n, 2) \sum_{0 < j < n, 4|j} \left( \begin{array}{l} nj \\ ij \end{array} \right) x^{n-j} y^{j} . \]

If we define

\[ W_{1}^{(n)}(x, y) = \sum_{ij} \left( \begin{array}{l} n \\ ij \end{array} \right) x^{n-j} y^{j} = \frac{1}{4} ((x + y)^{n} + (x - y)^{n} + (x + iy)^{n} + (x - iy)^{n}) , \]

then

\[ \sum_{C \in \mathcal{D}_{n}} W_{1}(x, y; C) = \nu(n, 2)(2^{n/2-2}(x^{n} + y^{n}) + W_{1}^{(n)}(x, y)). \]
Recall that the root system $D_4$ consists of the 24 roots listed below:

\[ \pm \sqrt{2}e_j \quad (j = 1, 2, 3, 4), \]
\[ \frac{1}{\sqrt{2}} (\pm 1, \pm 1, \pm 1, \pm 1). \]

We imbed these vectors into $\mathbb{C}^2$ as follows.

\[ i^k \sqrt{2}e_j \quad (j = 1, 2, k = 0, 1, 2, 3), \]
\[ \zeta^j e_1 + \zeta^k e_2 \quad (j, k = 1, 3, 5, 7), \]

where $\zeta = e^{\pi i/4}$. Now, let $D_4$ denote the set of 24 vectors above. If $n \equiv 0 \pmod{4}$, then

\[ \sum_{\alpha \in D_4} (\alpha_1 x + \alpha_2 y)^n = 2^{n/2 + 2}(x^n + y^n) + \sum_{j,k=1,3,5,7} (\zeta^j x + \zeta^k y)^n \]
\[ = 2^{n/2 + 2}(x^n + y^n) + (-1)^{n/4} \sum_{j,k=0,2,4,6} (\zeta^j x + \zeta^k y)^n \]
\[ = 16(2^{n/2 - 2}(x^n + y^n) + (-1)^{n/4} W^{(1)}_n(x, y)). \]

3.2 A Theorem

Using the notation introduced in the previous section, the mass formula for the Jacobi weight enumerator polynomial can easily be established. The Jacobi weight enumerator polynomial for a code $C$ with respect to a reference vector $u$ is defined by

\[ Jac(C, u; x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{v \in C} X(u, v). \]

Denote by $\overline{Jac}_{n,k}$ the sum of the Jacobi weight enumerator polynomial with respect to a fixed reference vector of weight $k$ for all $C \in D_4$. Note that $\overline{Jac}_{n,k}$ is independent of the choice of $u$. We prove

Theorem 3.3 (Munemasa-Ozeki [13])

\[ \overline{Jac}_{n,k} = \frac{1}{16} \nu(n, 2) \sum_{\alpha = (\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k}(\alpha_1 x_{10} + \alpha_2 x_{11})^k. \]

4 An application of the mass formula to the construction of Jacobi forms

4.1 Some instances

If we apply the so called Bannai-Ozeki map (c.f. [2]) to the right hand side of (9), we obtain many important Jacobi forms of weight $n/2$ and index $k$. As the mass formula both hands are meaningful only when $n$ is divisible by 8. However the polynomials in the right hand side are useful even if $n \equiv 4 \pmod{8}$ in constructing Jacobi forms. Here we give few instances of the construction.

To do this we recall Jacobi's theta functions:

\[ \theta_0(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi n^2 \tau + 2n \pi i z}, \]
\[ \theta_2(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi (n+1/2)^2 \tau + 2(n+1) \pi i z}, \]
\[ \theta_3(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi n^2 \tau + 2n \pi i z}. \]
We put $\phi_i(\tau, z) = \theta(2\tau, 2x)$, and $\phi_i(\tau) = \phi_i(\tau, 0) \ (i = 2, 3)$.

When $n = 8$ and $k = 1$ the right hand side of (9) becomes 30 times of

$$7y^4x^3u + 72x^2y^2v + y^7v + x^7u.$$

Substituting $x = \phi_2(\tau), y = \phi_3(\tau), u = \phi_2(\tau, z)$ in this polynomial we get a Jacobi form of weight 4 and index 1:

$$\psi_{4,1} = 1 + (\zeta^2 + 56\zeta + 56\zeta^{-1} + \zeta^{-2} + 126)q + (126\zeta^2 + 576\zeta + 576\zeta^{-1} + 126\zeta^{-2} + 756)q^2 + (56\zeta^3 + 756\zeta^2 + 1512\zeta + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3} + 2072)q^3 + (\zeta^4 + 576\zeta^3 + 2072\zeta^2 + 4032\zeta + 4032\zeta^{-1} + 2072\zeta^{-2} + 576\zeta^{-3} + \zeta^{-4} + 4158)q^4 + (126\zeta^4 + 1512\zeta^3 + 4158\zeta^2 + 5544\zeta + 5544\zeta^{-1} + 4158\zeta^{-2} + 1512\zeta^{-3} + 126\zeta^{-4} + 7560)q^5 + \cdots$$

When $n = 8$ and $k = 2$ the right hand side of (9) becomes 30 times of

$$3x^4y^2v^2 + 8x^3y^3uv + 3x^2y^4u^2 + x^6u^2 + y^6v^2.$$

The last polynomial leads to a Jacobi form of weight 4 and index 2:

$$\psi_{4,2} = 1 + [14(\zeta^2 + \zeta^{-2}) + 64(\zeta + \zeta^{-1}) + 84]q + [4\zeta^2 + \zeta^{-4} + 64(\zeta^3 + \zeta^{-3}) + 280(\zeta^2 + \zeta^{-2}) + 448(\zeta + \zeta^{-1}) + 574]q^2 + (84\zeta^4 + 448\zeta^3 + 840\zeta^2 + 1344\zeta + 1344\zeta^{-1} + 840\zeta^{-2} + 448\zeta^{-3} + 84\zeta^{-4} + 1288)q^3 + [64(\zeta^5 + \zeta^{-5}) + 574(\zeta^4 + \zeta^{-4}) + 1344(\zeta^3 + \zeta^{-3}) + 2368(\zeta^2 + \zeta^{-2}) + 2688(\zeta + \zeta^{-1}) + 3444]q^4 + [14(\zeta^6 + \zeta^{-6}) + 448(\zeta^5 + \zeta^{-5}) + 1288(\zeta^4 + \zeta^{-4}) + 2688(\zeta^3 + \zeta^{-3})] + 3542(\zeta^2 + \zeta^{-2}) + 4928(\zeta + \zeta^{-1}) + 4424]q^5 + \cdots$$

When $n = 12$ and $k = 1$ the right hand side of (9) is a polynomial that is 4050 times of

$$-22x^3y^7v - 11x^8y^3v - 11y^8x^3u - 22y^4x^7u + x^{11}u + y^{11}v.$$ 

This leads to a Jacobi form of weight 6 and index 1

$$\psi_{6,1} = 1 + (\zeta^2 - 88\zeta - 88\zeta^{-1} + \zeta^{-2} - 330)q + (-330\zeta^2 - 4224\zeta - 4224\zeta^{-1} - 330\zeta^{-2} - 7524)q^2 + (-88\zeta^3 - 7524\zeta^2 - 30600\zeta - 30600\zeta^{-1} - 7524\zeta^{-2} - 88\zeta^{-3} - 46552)q^3 + (\zeta^4 - 4224\zeta^3 - 46552\zeta^2 - 130944\zeta^{-1} - 46552\zeta^{-2} - 4224\zeta^{-3} + \zeta^{-4} - 169290)q^4 + (-330\zeta^4 - 30600\zeta^3 - 169290\zeta^2 - 355080\zeta^{-1} - 169290\zeta^{-2} - 30600\zeta^{-3} - 330\zeta^{-4} - 464904)q^5 + \cdots$$

When $n = 12$ and $k = 2$ the right hand side of (9) is 4050 times of the polynomial

$$-14y^4x^4u^2 + 14y^6x^4v^2 - 3y^2x^8v^2 - 3y^8x^2u^2 + x^{10}u^2 + y^{10}v^2 + 16y^3x^7uv - 16y^7x^3uv.$$ 

This leads to a Jacobi form of weight 6 and index 2:
\[ \psi_{6,2} = 1 + (-10\zeta^2 - 128\zeta - 128\zeta^{-1} - 10\zeta^{-2} - 228)q + (\zeta^4 - 128\zeta^3 - 1496\zeta^2 - 3968\zeta - 1496\zeta^{-2} - 128\zeta^{-3} + \zeta^{-4} - 5450)q^2 + (-228\zeta^4 - 3968\zeta^3 - 14088\zeta^2 - 27264\zeta - 1496\zeta^{-1} - 128\zeta^{-2} - 3968\zeta^{-3} - 228\zeta^{-4} - 31880)q^3 + (-128\zeta^5 - 5450\zeta^4 - 27264\zeta^3 - 103680\zeta^2 - 197650\zeta - 124260)q^4 - 10\zeta^6 - 3968\zeta^5 - 31880\zeta^4 - 103680\zeta^3 - 197650\zeta^2 - 10\zeta^{-1} - 316168)q^5 + \cdots \]

In this way we obtain an infinite family of Jacobi forms of various weights and various indeces.

### 4.2 A comparison of two constructions

In [8] only the values \( e_{k,m}(n,r) \) \((k \leq 8, m = 1)\) of the Fourier coefficients of \( E_{k,m}(\tau, z) \) are given explicitly. Here we explain a method to compute \( e_{k,m}(n,r) \) for any even \( k \) and \( m \geq 1 \). For this we start from the formula given in [8] page 22:

\[
e_{k,m}(n,r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H(k-1, \frac{4nm-r^2}{d^2}),
\]

and

\[
e_{k,1}(n,r) = \frac{H(k-1,4n-r^2)}{\zeta(3-2k)},
\]

where \( \zeta(3-2k) \) is the special value of Riemann's zeta function. The quantity \( H(k-1,4n-r^2) \) is described at page 30 in [8]:

\[
H(k-1, N) = \begin{cases} 
L_{-N}(2-k) & \text{if } N > 0 \text{ and } N \equiv 0 \text{ or } 3 \pmod{4}, \\
\zeta(3-2k) & \text{if } N = 0, \\
0 & \text{if } N > 0 \text{ and } N \equiv 1 \text{ or } 2 \pmod{4}.
\end{cases}
\]

When \( -N \equiv 0 \text{ or } 1 \pmod{4} \) we put \( -N = (-N_0)u^2 \) \( u \in \mathbb{N} \) so that \( -N_0 \) is the discriminant of the quadratic number field \( \mathbb{Q}(\sqrt{-N}) \). The number \( L_{-N}(2-k) \) comes from the \( L \)-function \( L_{-N_0}(s) \) by way of

\[
L_{-N_0}(s) = L(s, (_{\overline{d}}^{*_{-N}})) = \sum_{n=1}^{\infty} \frac{(_{\overline{d}}^{-N_0})}{n^s},
\]

and

\[
L_{-N_0}(s) = L(s,\frac{-N_0}{d}) = \sum_{n=1}^{\infty} \frac{(\frac{-N_0}{d})}{n^s}.
\]

To make the value \( e_{k,1}(n,r) \) explicit it is necessary to know the values \( \zeta(3-2k) \) and \( L_{-N_0}(1-m) \). As to the values \( \zeta(3-2k) \) there are many literature available and they tell us that

\[
\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}}{(2k)!} B_{2k} \quad (k \geq 1),
\]

\[
\zeta(1-n) = (-1)^{n-1} \frac{B_{n}}{n} (n = 1, 2, \cdots),
\]

where \( B_n \) is the \( n \)-th Bernoulli number. The beginning few numbers are

\[
B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{-1}{30}, \cdots, B_{odd \geq 3} = 0.
\]
By these one gets
\[
\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}, \ldots
\]
\[
\zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-5) = -\frac{1}{252}, \quad \zeta(-7) = \frac{1}{240}, \quad \zeta(-9) = -\frac{1}{132}, \ldots
\]
It is much complicated to get the values $L_{-N_0}(1-m)$. On reading the book [1] we find a suitable formula to do this task. Note that a similar formula has been given in [11] Chapter XIII in a not straight way.

**Theorem 4.1 (Arakawa-Ibukiyama-Kaneko)** Let $\chi$ be a primitive character mod $f$ and $m$ be a positive integer, then

\[
L(1-m, \chi) = -\frac{B_{m,\chi}}{m},
\]
where $B_{m,\chi}$ is the generalized Bernouilli number associated with $\chi$:

\[
B_{m,\chi} = f^{m-1} \sum_{a=1}^{f} \chi(a) B_{m}(\frac{a}{f}),
\]
and $B_{m}(x)$ is the Bernouilli polynomial of degree $m$.

The Bernouilli polynomials are given by

\[
B_{m}(x) = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} B_{j}x^{m-j},
\]

With the above Theorem we compute $e_{k,m}(n,r)$, and we give small tables of them, that are not contained in [8].

<table>
<thead>
<tr>
<th>4n - r²</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>8</th>
<th>11</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{10,1}(n,r)$</td>
<td>1</td>
<td>800776</td>
<td>9547079</td>
<td>119276768</td>
<td>360156896</td>
<td>58844227000</td>
<td>-11304484304</td>
<td>-374799931648</td>
</tr>
<tr>
<td>$e_{12,1}(n,r)$</td>
<td>1</td>
<td>3398848</td>
<td>6971898</td>
<td>2485775648</td>
<td>10096500384</td>
<td>285489846992</td>
<td>716081570896</td>
<td>745878170816</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>8n - r²</th>
<th>0</th>
<th>4</th>
<th>7</th>
<th>8</th>
<th>12</th>
<th>15</th>
<th>16</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{10,1}(n,r)$</td>
<td>1</td>
<td>130737825008110</td>
<td>5607166776120</td>
<td>8682826945328</td>
<td>438257</td>
<td>4273127984</td>
<td>4273127984</td>
<td>4273127984</td>
<td>4273127984</td>
</tr>
<tr>
<td>$e_{12,1}(n,r)$</td>
<td>1</td>
<td>14531135604394</td>
<td>88601830003192</td>
<td>152244273101400</td>
<td>77884</td>
<td>77884</td>
<td>77884</td>
<td>77884</td>
<td>77884</td>
</tr>
</tbody>
</table>

We remark that the functions $\psi_{4,1}, \psi_{6,1}, \psi_{8,1}$ respectively coincide with Eisenstein-Jacobi forms $E_{4,1}, E_{6,1}, E_{8,1}$ respectively of index 1 described in [8] pages 17-23. Explicit Fourier expansions of Eisenstein-Jacobi forms of index ≥ 2 are not given in [8]. We have verified that $\psi_{4,2}, \psi_{6,2}$ also coincide with Jacobi-Eisenstein series of index 2. This is done by using the relation (7) in [8], page 22. Besides these exceptional cases Eisenstein-Jacobi form $E_{k,m}$ differ from $\psi_{k,m}$. One may be interested with a problem to explore the further relations between these two constructions of Jacobi forms.
5 Eisenstein type polynomials in more variables

\[ E_{k_1,k_2}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}) = \]
\[ \frac{1}{16} \nu(n, 2) \sum_{(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2}, \]  
\[ E_{k_1,k_2,k_3}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}) = \]
\[ \frac{1}{16} \nu(n, 2) \sum_{(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2-k_3} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2} (\alpha_1 z_{10} + \alpha_2 z_{11})^{k_3}, \] 

where all exponents are non negative integers.
The right-hand side of (10) belongs to \( \mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}]^{H_1 \oplus H_1 \oplus H_1} \),
and the right-hand side of (11) belongs \( \mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}]^{H_1 \oplus H_1 \oplus H_1} \). As discussed in [2] these polynomials contribute to the construction of Jacobi forms.

References

[10] W. Kohnen, Modular forms of half-integral weight on \( \Gamma_0(4) \), Math. Ann. 248 (1980) 249-266