A Note on Equations for the Meson

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In a previous paper[1] I demonstrated how I arrived at generalized equations for the Dirac electron by means of the usual tensor calculus which is familiar to us in the relativity theory of gravitation. Since there no spin matrix can be introduced in the tensor formulations as is done for the spinor ones, it seems difficult to describe the spinning electron by the tensor formulations. I have, however, further shown[2] that the spin momentum can be introduced in the tensor formulations as a certain operator which can be applied to vectors of the wave-field, and I have drawn the conclusion that spinor formulations are not necessarily needed to describe the spinning electron.

One of the advantages of the tensor formulations is that the formulations enable us to make geometrical considerations for the wave-fields of the electron without use of any auxiliary spaces: in the previous paper[1] I developed a scheme for a world geometory to describe the physical world which consists of space, time, and electron.

In this note I shall briefly describe a method of obtaining a tensor formulation of the so-called U-field of Yukawa's theory[3] by means of the tensor calculation which I used in the previous papers, and shall show a possible scheme of unifying the representations of the Dirac and Yukawa theories.

Let us consider the second order wave equation of Schrödinger's type with respect to a vector $\psi^{\varkappa}$ and a tensor of the second rank $\psi^{\lambda\nu}$; namely

$$\left(-\varDelta+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\mu^2\right)\begin{pmatrix}\psi^{\varkappa}\\ \psi^{\lambda\nu}\end{pmatrix}=0 \tag{1}$$

I showed in the previous paper[1] that these second order equations might be obtained by combining a system of the first order equations such as

$$\left.\begin{aligned} J^{\sigma\varkappa}_{\cdot\cdot\lambda\nu}\nabla_\sigma\psi^{\lambda\nu}+i\mu\psi^{\varkappa}&=0\\ J^{\sigma\cdot\lambda\nu}_{\cdot\varkappa}\nabla_\sigma\psi^{\varkappa}+i\mu\psi^{\lambda\nu}&=0 \end{aligned}\right\} \tag{2}$$

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1. Jap. Jour. Phys. XI (1936), 35.
2. Jap. Jour. Phys. XII (1938), 27.
3. Proc. Phys. Math. Soc. Japan, 20 (1938), 321.

where $J_{\sigma\varkappa\lambda\nu}$ is generally defined by

$$J_{\sigma\varkappa\lambda\nu}=\frac{1}{2}(g_{\sigma\varkappa}g_{\lambda\nu}+g_{\sigma\lambda}g_{\varkappa\nu}-g_{\sigma\nu}g_{\varkappa\lambda}+E_{\sigma\varkappa\lambda\nu}) \qquad (3)$$

$g_{\sigma\varkappa}$: metrical tensor

$E_{\sigma\varkappa\lambda\nu}$: unit antisymmetrical tensor of the fourth rank,

or : $=i\sqrt{-g}\,\varepsilon_{\sigma\varkappa\lambda\nu}$($g$ stands for the determinant of $g_{\sigma\varkappa}$ and $\varepsilon_{\sigma\varkappa\lambda\nu}$ is the coefficient of the determinent.)

∇_σ : Riemannian covariant differentiation.

Since the equations of (1) are special-relativistic ones, we have to consider the case when the values of $g_{\sigma\varkappa}$ are given by

$$g_{\sigma\varkappa}=\begin{pmatrix}-1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1\end{pmatrix},\quad -g=1 \text{ and } \nabla_\sigma=\partial_\sigma\left(\equiv\frac{\partial}{\partial x^\sigma}\right).$$

I showed in the previous paper[1] that in the case of special relativity the system of equations (2) is equivalent to the equation of the Dirac electron.

As $E_{\sigma\varkappa\lambda\nu}$ is a purely imaginary quantity, the tensor $J_{\sigma\varkappa\lambda\nu}$ may be considered as a kind of complex quantity, and we may introduce its conjugate by

$$\bar{J}_{\sigma\varkappa\lambda\nu}=\frac{1}{2}(g_{\sigma\varkappa}g_{\lambda\nu}+g_{\sigma\lambda}g_{\varkappa\nu}-g_{\sigma\nu}g_{\varkappa\lambda}-E_{\sigma\varkappa\lambda\nu}). \qquad (3')$$

Using this conjugate tensor, we may consider the following system of equations:

$$\left.\begin{aligned}\bar{J}^{\sigma\varkappa}_{\cdot\cdot\lambda\nu}\nabla_\sigma\chi^{\lambda\nu}+i\mu\chi^{\varkappa}&=0,\\ \bar{J}^{\sigma\cdot\lambda\nu}_{\cdot\varkappa}\nabla_\sigma\chi^{\varkappa}+i\mu\chi^{\lambda\nu}&=0,\end{aligned}\right\} \qquad (2')$$

which can also be reduced to the second order equations of the form (1). Thus, we have in general two systems of the first order equations which may correspond to the second order wave equations of the form (1).

On the other hand, in separating a real number μ^2, we may introduce a pair of complex conjugate numbers $\varkappa$ and $\bar{\varkappa}$, so that

$$\mu^2=\varkappa\bar{\varkappa}.$$

If we use this separation, it may also be possible to separate the first order equations into the following systems:

$$\left.\begin{aligned}J^{\sigma\varkappa}_{\cdot\cdot\lambda\nu}\nabla_\sigma\psi^{\lambda\nu}+i\varkappa\psi^{\varkappa}&=0\\ J^{\sigma\cdot\lambda\nu}_{\cdot\varkappa}\nabla_\sigma\psi^{\varkappa}+i\bar{\varkappa}\psi^{\lambda\nu}&=0\end{aligned}\right\} \qquad (4)$$

and

$$\left.\begin{aligned}\bar{J}^{\sigma\varkappa}_{\cdot\cdot\lambda\nu}\nabla_\sigma\chi^{\lambda\nu}+i\varkappa\chi^{\varkappa}&=0\\ \bar{J}^{\sigma\cdot\lambda\nu}_{\cdot\varkappa}\nabla_\sigma\chi^{\varkappa}+i\bar{\varkappa}\chi^{\lambda\nu}&=0\end{aligned}\right\} \qquad (4')$$

Now, let $\varkappa_1$ and $\varkappa_2$ be respectively the real and the imaginary part of $\varkappa$, so that

$\varkappa=\varkappa_1+i\varkappa_2$.

If we put $\varkappa_2=0$ and $\varkappa_1=\mu$ the equations (4) becomes (2), and the equations (4′) can be identified with the conjugate equations of (4) by the relations

$$\bar{\psi}^{\varkappa}=i\chi^{\varkappa},\quad \bar{\psi}^{\lambda\nu}=-i\chi^{\lambda\nu}.$$

We may, therefore, identify $\varkappa_1$ with $\frac{mc}{\hbar}$ of the Dirac electron.

We shall next show the possibility of identifying $\varkappa_2$ with $\frac{m_u C}{\hbar}$ of the so-called meson.

Let us suppose $\varkappa_1=0$. We have then the possibility of putting $\psi^{\varkappa}=\chi^{\varkappa}$ as follows:

We write $\psi^{\varkappa}=\chi^{\varkappa}\equiv U^{\varkappa}$,

$$\psi^{\lambda\nu}+\chi^{\lambda\nu}\equiv F^{\lambda\nu},$$

$$\psi^{\lambda\nu}-\chi^{\lambda\nu}\equiv \bar{F}^{\lambda\nu}.$$

Then we can reduce the equations (4) and (4′) to the following forms:

$$g_{\lambda\nu}\nabla_\sigma U^\sigma+2\nabla_{[\lambda}U_{\nu]}=\varkappa_2 F_{\lambda\nu},$$

$$E_{\lambda\nu}^{\cdot\cdot\sigma\tau}\nabla_\sigma U_\tau=\varkappa_2\bar{F}_{\lambda\nu},$$

$$\nabla_\lambda F^{\lambda\nu}+\varkappa_2 U^\nu=0. \qquad (5)$$

Furthermore we assume $\nabla_\sigma U^\sigma=0$, namely, the divergence of vector U^σ vanishes. Then the first equation of (5) means that $F_{\lambda\nu}$ is a rotation (four-dimensional) of the vector U_ν, and the second equation of (5) means that $\bar{F}_{\lambda\nu}$ is the dual tensor of $F_{\lambda\nu}$.

Let us now use the usual vector notation:

$$(F^{14}, F^{24}, F^{34})=\vec{F},\quad (F_{23}, F_{31}, F_{12})=\vec{G},$$

$$(U^1, U^2, U^3)=\vec{U},\quad U_0=U_4.$$

Then, the equations (5) can be written in the forms:

$$\frac{1}{C}\frac{\partial\vec{F}}{\partial t}-\mathrm{rot}\,\vec{G}-\varkappa_2\vec{U}=0,\quad \mathrm{div}\,\vec{F}+\varkappa_2 U_0=0,$$

$$\frac{1}{C}\frac{\partial\vec{U}}{\partial t}+\mathrm{grad}\,U_0+\varkappa_2\vec{F}=0,\quad \mathrm{rot}\,\vec{U}-\varkappa_2\vec{G}=0,$$

$$\frac{1}{C}\frac{\partial\vec{G}}{\partial t}+\mathrm{rot}\,\vec{F}=0,\quad \mathrm{div}\,\vec{G}=0,\quad \frac{1}{C}\frac{\partial U_0}{\partial t}+\mathrm{div}\,\vec{U}=0.$$

These are the equations of the so-called U-field proposed by Yukawa[3]. We may thus arrive at the possibility of identifying $\varkappa_2$ with $\frac{m_u C}{\hbar}$ of the meson.

If we introduce the electro-magnetic field in the usual way, namely by substituting $\left(\partial_\mu + i\frac{e}{\hbar}\varphi_\mu\right)$ for ∂_μ in the equations of (4) and (4′), we may establish some connection between the fields ψ and χ.

In conclusion I wish to express my cordial thanks to Professor Niels Bohr through whose kindness I was given the opportunity of discussing this problem with prominent physicists of the University of Copenhagen.

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