A Remark on Riemann Surface defined by M. S. Stoïlow

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1. In the book, M. S. Stoïlow. Leçons sur les principes topologiques de la théorie des fonctions analytiques (1938), it is writen that how the Riemann surface, defined by M. S. Stoïlow, of any analytic function of complexes variable can be constructed. In this purpose, he considered the set V of all the elements p taking the form

$$\sum_{i=\mu}^{i=+\infty} a_i(Z-z)^{\frac{i}{\nu}}$$

(where μ is a zero or integer positive or negative, and ν a positive integer.)

and said as followes:

《By giving a suitable definition of accumulated point of $A \subset V$, V becomes a topological variety of two dimensions and *un recouvrement de plan* (z) *par* V will be a required Riemann surface of the given function.》

He gave the definition of accumulated point of a set $A \subset V$, using a notion of points appertaining in the circle of convergence of an element p and said:

《The set G of all the elements P appertaining in the circle of convergence of an element p is open in V. Moreover, if p is a weierstrass' element, there is a one-to-one correspondence between the elements P of G and the interior points of the circle of convergence of p, and this correspondence is a homeomorphic. Therefore, the condition that V may be a topological variety of two dimensions is satisfied at that element p. If p is algebraic, it is reduced to the above-stated case, by considering the transformation

$$Z-z=\zeta^{\nu}$$

followed or not by a inversion $\zeta=\frac{1}{\zeta'}$, according as the sum of the serie of p is infinite or not at its centre z. And so G is homeomorph to the euclidean plane, in the case $\nu>1$ too............》

But, we must remark that the set G can not contain the element

p when it is not a weierstrass' element, for, by M. Stoïlow's definition, an element P appertaining in the circle of convergence of p is necessarily to be a weierstrass' element. And so, in this case, the set G can not be a neighbourhood of p.

Therefore, we think, it is necessary to ascertain that there exists at least one neighbourhood of p, which is homeomorph to the euclidean plane, in the case $\nu>1$. In this purpose, we consider a system of neighbourhoods $G_{p}^{*}(\rho)$,[1] then the above-stated fact is proved without difficulty.

2. It is called that *an element $P \varepsilon V$ appertains in the circle of convergence of p*, when the following conditions are satisfied:

(1)
- (i) P is a weierstrass' element,
- (ii) the centre Z of P is contained in the interior of the circle of convergence of p, i. e. $|Z-z|<r(p)$, where $r(p)$ is a radius of convergence of p,
- (iii) and the sum of P and that of the serie of p or that of one determination of the serie of p are equal in the common part of their circles of convergence, according as ν, which corresponds to p, is equal to or grater than 1.

An element p is called *an accumulated point of a set $A \subset V$*, if, corresponding to any arbitrarily chosen positive number $\rho>0$ whatever, there exists at least one element P satisfying the following conditions:

(2)
- (i) $P \varepsilon A$,
- (ii) P appertains in the circle of convergence of p,
- (iii) and $|Z-z|<\rho$, $P \neq p$, where Z and z are the centres of P and p respectively.

It is clear that the conditions (2) are equivalent to the conditions

(2′)
- (i) $P \varepsilon A$,
- (ii) P appertains in the circle of convergence of p,
- (iii) and $0<|Z-z|<\rho$,

for, two elements P and p of a same analytic function, which have the same centre, are equal, when P appertains in the circle of convergence of p. And we shall denote the conditions (2′) by $[\rho, A, p]$, for convenience' sake.

Now, let us consider, for an element $p \varepsilon V$ and any given positive number $\rho>0$, the set of all the elements P which appertain in

1) cf. No. 2.

the circle of convergence of p and have the centres Z such that $|Z-z|>\rho$, and we shall denote it by $G_p(\rho)$.

Then *the sets* $G_p(\rho)$ *and* $G_p^*(\rho)\equiv G_p(\rho)+p$ *are all open in* V *and so* $G_p^*(\rho)$ *is a neighbourhood of* p.

(proof) If $G_p(\rho)$ were not open, there would be an element p_1 such that $p_1 \varepsilon (V-G_p(\rho))'$ but $p_1 \bar{\varepsilon}(V-G_p(\rho))$, where $(V-G_p(\rho))'$ means the derived set of $(V-G_p(\rho))$. Since $p_1 \varepsilon G_p(\rho)$, therefore $|z_1-z|<$ min. $[\rho, r(p)]$, and so there exist a positive number $\rho'>0$ such that

$$0<\rho'<\min.[r(p),\ \rho,\ r(p)-|z_1-z|,\ \rho-|z_1-z|]$$

From the condition $p_1 \varepsilon (V-G_p(\rho))'$ there exists at least one element P satisfying the conditions $[\rho', V-G_p(\rho), p]$. P and p appertain in the circles of convergence of p_1 and p respectively and $|Z-z_1|<\rho'<r(p)-|z_1-z|$, hence P appertains in the circle of convergence of p. And so $p \varepsilon G_p(\rho)$, for $|Z-z|\leqq|Z-z_1|+|z_1-z|<\rho'+|z_1-z|<(\rho-|z_1-z|)+|z_1-z|=\rho$. This contradicts (i) of conditions $[\rho', V-G_p(\rho), p]$.

Therefore the set $G_p(\rho)$ must be open.

Further, it is clear that there exists no elment P satisfying the conditions $[\sigma, V-G_p(\rho), p]$ for a positive number σ such that $0<\sigma<\rho$. Therefore $p\bar{\varepsilon}(V-G_p(\rho))'$, and so $G_p^*(\rho)$ is also open. For, since $(V-G_p(\rho))'\subset(V-G_p(\rho))$ and $(V-G_p(\rho))'\subset V-p$, therefore

$$(V-G_p^*(\rho))'\subset(V-G_p(\rho))'\subset(V-G_p(\rho))(V-p)=V-G_p^*(\rho).$$

Moreover, *the system of neighbourhoods* $\{G_p^*(\rho)\}$ *defines a topological space* V *itself.*

(proof) We shall denote a neighbourhood of p in the space V, by U_p. From the fact that $G_p^*(\rho)$ is open, there exists an U_p such that $U_p\subset G_p^*(\rho)$.

On the other hand, since $(V-U_p)'\subset(V-U_p)$ and $p \varepsilon U_p$, therefore $p \bar{\varepsilon} (V-U_p)'$ for any U_p. This means that there exists a number $\rho>0$ such that there is no element satisfying the conditions $[\rho, V-U_p, p]$, in other words, if an element P appertains in the circle of convergence of p and $0<|Z-z|<\rho$, then $P \varepsilon U_p$. Therefore, it is clear that the elements $P \varepsilon G_p^*(\rho)$ having the centres $Z\neq z$, are contained in U_p, and the elements $P \varepsilon G_p^*(\rho)$ having the centre $Z=z$, is equal io p, for if P were not equal to p, then it would be $P \varepsilon G_p(\rho)$ and so $P=p$ for $Z=z$, this contradicts the proposition $P\neq p$. Hence, $G_p^*(\rho)\subset U_p$.

Therefore two system $\{G_p^*(\rho)\}$ and $\{U_p\}$ define a same space and the second defines the space V itself, for V is a topological space.

By using this system of neighbourhoods $\{G_p^*(\rho)\}$, it is easily proved that $G_p^*(r(p))$ is homeomorph to the circle $|\zeta|\leqq(r(p))^{\frac{1}{\nu}}$, where $Z-z=\zeta^\nu$.

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