Remarks on O. Zariski's Paper

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Let $\sum$ be a field of algebraic functions of $r(>0)$ independent variables over an arbitrary ground field K of characteristic zero.[1] Let V_r be a projective model of $\sum$ in an affine space S_n, whose generic point is $(\xi_1, \ldots, \xi_n)$, and $\mathfrak{o}=K[\xi_1, \ldots, \xi_n]$ the integral domain of all polynomials of $\xi_1, \ldots, \xi_n$ with coefficients from K. An s-dimensional (K-)irreducible subvariety V_s, defined by an s-dimensional prime $\mathfrak{o}$-ideal $\mathfrak{p}_s$, is called simple subvariety of V_r if in the quotient ring $\mathfrak{o}_{\mathfrak{p}_s}$ the ideal $\mathfrak{p}_s$ has just $r-s$ basis elements, i. e. if there exist $r-s$ $\mathfrak{o}$-elements $\eta_1, \ldots, \eta_{r-s}$ such that

$$(1) \qquad \mathfrak{p}_s\mathfrak{o}_{\mathfrak{p}_s}=(\eta_1, \ldots, \eta_{r-s})\mathfrak{o}_{\mathfrak{p}_s};$$

and $\eta_1, \ldots, \eta_{r-s}$ are said uniformizing parameters (u. p. s. in abbreviation) of $\mathfrak{p}_s$ (or of V_s). The condition (1) amounts to that $\mathfrak{p}_s$ is one of the isolated primary components of the $\mathfrak{o}$-ideal $(\eta_1, \ldots, \eta_{r-s})$. O. Zariski has recently proved the following theorem (O. Zariski [2], theorem 15), which is rather fundamental in the arithmetic theory of algebraic varieties:

THEOREM. *Let V_s be an s-dimensional* $(0\leqq s<r)$ *subvariety of V_r, $\mathfrak{p}_s$ the corresponding prime $\mathfrak{o}$-ideal. Let $\eta_1, \ldots, \eta_r$ be r elements of $\mathfrak{o}$, which are algebraically independent over K and such that $\mathfrak{o}$ depends integrally on $K[\eta_1, \ldots, \eta_r]$—s of them, say $\eta_1, \ldots, \eta_s$, being algebraically independent over K mod. $\mathfrak{p}_s$. V_s is a simple subvariety of V_r and $f_1(\eta_1, \ldots, \eta_s; \eta_{s+1})$, $f_2(\eta_1, \ldots, \eta_s; \eta_{s+2})$, $\ldots, f_{r-s}(\eta_1, \ldots, \eta_s; \eta_r)$ are u.p.s. of V_s—$f_i(y_1, \ldots, y_s; z)$ being the irreducible polynomial in $K[y_1, \ldots, y_s, z]$*[2] *such that $f_i(\eta_1, \ldots, \eta_s; \eta_{s+i})\equiv 0$ $(\mathfrak{p}_s)$—if and only if an element ω of $\mathfrak{o}$ exists and $G'_\omega(\eta_1, \ldots, \eta_r; \omega)\not\equiv 0$ $(\mathfrak{p}_s)$, where the polynomial $G(\eta_1, \ldots, \eta_r; z)$ is the norm of $z-\omega$ with respect to the algebraic extension $\sum/K(\eta_1, \ldots, \eta_r)$.*

The proof of the theorem is reduced readily to that of the case $s=0$ (O. Zariski [2], theorem 11′)—we shall formulate this case in

1. The same results will be obtained if the ground field is not of characteristic zero but complete, and has infinitely meny elements.

2. $y_1, \ldots\ldots, y_s$, z are indeterminates.

our theorem II below (p. 108). It seems after all troublesome for us to prove theorem II, in spite of its rather intuitive feature, but Zariski's proof might be somewhat roundabout. We are going to prove it more directly, by a slight change of the method of the ground field extension and following the plan of Zariski's proof which he has carried out in the case of algebraically closed ground field (Zariski [1], I §§ 6.8).

1. Before taking up the proof of theorem II, we must give some preliminary considerations concerning our new ground field extension. Let $R=K[x_1, \dots, x_n]$ be the ring of K-polynomials of n independent indeterminates $x_1, \dots, x_n$, $\bar{\mathfrak{p}}_r$ the r-dimensional prime ideal of R which defines V_r; $\xi_1, \dots, \xi_n$ are looked on as the rest-class of $x_1, \dots, x_n$ mod. $\mathfrak{p}_r$: $\mathfrak{o}=K[\xi_1, \dots, \xi_n]=R/\bar{\mathfrak{p}}_r$. Let K^* be an algebraic closure of K and $R^*=K^*[x_1, \dots, x_n]$. The ring $\mathfrak{o}^*=R^*/\bar{\mathfrak{p}}_r R^*$ of rest-classes modulo the extended ideal $\bar{\mathfrak{p}}_r R^*$ is, as usual, looked on as containing $\mathfrak{o}$ and K^*. Since it is well known (W. Krull [2], 16 (§ 3)) that $\bar{\mathfrak{p}}_r R^*$ splits into the intersection of all prime R^*-ideals (finite in number, say λ) which lie over $\bar{\mathfrak{p}}_r$—these ideals being necessarily r-dinensional and mutually K-conjugate—$\mathfrak{o}^*$ may have null-divisors. We call $\mathfrak{o}^*$ the ring obtained from $\mathfrak{o}$ by the ground field extension $K \rightarrow K^*$. $\mathfrak{o}^*$ is nothing but the ring obtained by the direct multiplication (Ringadjunction) of $\mathfrak{o}$ and K^* over K as carried on by E. Noether [1]; $\mathfrak{o}^*$ is contained in the ring $\sum^*=\sum \times K^*$—the direct product of $\sum$ and K^* over K—which would be readily proved to decompose into a direct sum of λ fields[1].

The K^*-irreducible (or absolutely irreducible) subvarieties of V_r correspond one-to-one to the prime $\mathfrak{o}^*$-ideals and by the homomorphic mapping of R^* onto $\mathfrak{o}^*=R^*/\bar{\mathfrak{p}}_r R^*$ these correspond again one-to-one to the prime R^*-ideals which contain $\bar{\mathfrak{p}}_r R^{*}$[2]. From the fact that t-dimensional ($0 \leqq t < r$) prime R-ideal $\bar{\mathfrak{p}}_t (\supset \bar{\mathfrak{p}}_r)$ splits in R^* into the intersection $[\bar{\mathfrak{p}}_{t1}^*, \dots, \bar{\mathfrak{p}}_{t\mu_t}^*]$ of the t-dimensional prime R^*-ideals $\bar{\mathfrak{p}}_{ti}^*$ $(\supset \bar{\mathfrak{p}}_r R^*)$ which lie over $\bar{\mathfrak{p}}_t$ (Krull [2]), going over to restclasses mod. $\bar{\mathfrak{p}}_r R^*$, it follows that the t-dimensional prime $\mathfrak{o}$-ideal $\mathfrak{p}_t$ (corresponding to $\bar{\mathfrak{p}}_t$) splits in $\mathfrak{o}^*$ into the intersection $[\mathfrak{p}_{t1}^*, \dots, \mathfrak{p}_{t\mu}^*]$ of the t-dimen-

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1. We find no use of this decomposition of $\sum^*$ ihto a direct sum below in the course of this paper. We owe the use of our method of ground field extension to Mr. Prof. Y. Akizuki; here the author expresses his hearty thanks for his valuable advises.

2. In the following, ideals and elements of $\mathfrak{o}^*$ are to be denoted by letters with stars, while those of $\mathfrak{o}$ without stars.

sional prime $\mathfrak{o}^*$-ideals $\mathfrak{p}_{ti}^*$ (corresponding to $\bar{\mathfrak{p}}_{ti}^*$)—these prime $\mathfrak{o}^*$-ideals $\mathfrak{p}_{ti}^*$ over $\mathfrak{p}_t$ being necessarily mutually K-conjugate. This decomposition reflects the decomposition of the K-irreducible t-dimensional subvariety V_t (defined by $\mathfrak{p}_t$) into K^*-irreducible t-dimensional subvarieties $V_{t1}^*, \ldots, V_{t\mu}^*$ (defined by $\mathfrak{p}_{t1}^*, \ldots, \mathfrak{p}_{t\mu}^*$)[1].

2. Now a point $P^*(c_1^*, \ldots, c_n^*)$ $(c_i^* \in K^*)$ is called simple point of V_r if the corresponding zero-dimensional prime $\mathfrak{o}^*$-ideal $\mathfrak{p}^*=(\xi_1-c_1^*, \ldots, \xi_n-c_n^*)$ has just r basis elements $\eta_1^*, \ldots, \eta_r^*(\in \mathfrak{o}^*)$ in the ring $\mathfrak{o}_{\mathfrak{p}^*}^*$:

$$\mathfrak{p}^*\mathfrak{o}_{\mathfrak{p}^*}^*=(\eta_1^*, \ldots, \eta_r^*)\mathfrak{o}_{\mathfrak{p}^*}^*, \qquad (2)$$

where $\mathfrak{o}_{\mathfrak{p}^*}^*$ is the quotient ring of $\mathfrak{p}^*$, i. e. the ring of all fractions $\sigma^*/\tau^*(\sigma^*, \tau^* \in \mathfrak{o}^*)$ whose denominators τ^* are non-null-divisors of $\mathfrak{o}^*$ not belonging to $\mathfrak{p}^*$. Elements $\eta_1^*, \ldots, \eta_r^*$ are said u. p. s. of $\mathfrak{p}^*$ (or at P^*). This definition of simple point of V_r would be the same as that of Zariski ([1], I § 2) but for the fact that in $\mathfrak{o}^*$ null-divisors may appear.

The zero-dimensional prime $\mathfrak{o}^*$-ideal $\mathfrak{p}^*$ corresponding to the simple point P^* has similar properties as those that are enumerated in §§ 2–4 of part I, Zariski [1]. These properties would be proved, following O. Zariski's plan, and in some parts even more neatly by going over to $\mathfrak{o}_{\mathfrak{p}^*}^*$ and using the definition (2). One of them, of which we find use in the followings, is the power series expansion:

To any element σ^* of $\mathfrak{o}^*$ there corresponds uniquely a formal power series $\psi_0^*+\psi_1^*+\ldots\ldots+\psi_\lambda^*+\ldots$ of u. p. s. $\eta_1^*, \ldots, \eta_r^*$ such that

$$\sigma^*\equiv\psi_0^*+\psi_1^*+\ldots\ldots+\psi_\lambda^* \qquad ((\mathfrak{p}^*)^{\lambda+1}) \ (\lambda=0, 1, 2, \ldots),$$[2]

where ψ_λ^* is a form of degree λ of $\eta_1^*, \ldots, \eta_r^*$ with coefficients from K^*. However, in our case of $\mathfrak{o}^*$, it may happen that two different elements σ_1^*, σ_2^* have an equal expression $\psi_0^*+\psi_1^*\ldots\ldots+\psi_\lambda^*+\ldots\ldots$— this takes place only if $\sigma_1^*-\sigma_2^*$ is a null-divisor. Namely, since

$$\sigma_1^*\equiv\sigma_2^*\equiv\psi_0^*+\psi_1^*+\ldots\ldots+\psi_\lambda^* \qquad ((\mathfrak{p}^*)^{\lambda+1}) \ (\lambda=0, 1, 2, \ldots),$$

$\sigma_1^*-\sigma_2^*\equiv 0$ $((\mathfrak{p}^*)^{\lambda+1})$ $(\lambda=0, 1, 2, \ldots)$. As is well known, the intersection of all $(\mathfrak{p}^*)^{\lambda+1}(\lambda=0, 1, 2, \ldots)$ is an isolated component ideal of the null-ideal (0) (Krull [1]), hence $\sigma_1^*-\sigma_2^*$ must be a null-divisor. We can also prove that the idea of the above defined "simple point P^*"

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1. Similarly we see that a t-dimensional preper primary $\mathfrak{o}$-ideal $\mathfrak{q}_t$ (belonging to $\mathfrak{p}_t$) splits in $\mathfrak{o}^*$ into an intersection of proper primary $\mathfrak{o}^*$-ideals $\mathfrak{q}^*_{t1}, \ldots\ldots, \mathfrak{q}^*_{t\mu}$ (belonging to $\mathfrak{p}^*_{t1}, \ldots\ldots, \mathfrak{p}^*_{tu}$).

2. In the following, elements of K^* shall be denoted, in silence, by a^*, b^*, c^*, d^* with stars, elements of K by a, b etc. without stars.

3. In what follows we treat only zero-dimensional prime ideals, thus the indices to ideals do not indicate their dimensions.

is precisely identical with that of the simple point of V_r in the usual geometric sense, but we deem it unnecessary to prove it here in full detail.

3. Let $\mathfrak{p}$ be a zero-dimensional prime $\mathfrak{o}$-ideal and $\mathfrak{p}\mathfrak{o}^* = [\mathfrak{p}_1^*, \mathfrak{p}_2^*, \ldots, \mathfrak{p}_\mu^*] = \mathfrak{p}_1^*\mathfrak{p}_2^* \ldots \mathfrak{p}_\mu^*$ where $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_\mu^*$ are the distinct zero-dimensional prime $\mathfrak{o}^*$-ideals which lie over $\mathfrak{p}$—these being mutually K-conjugate[3]. Let P be the "point" (i. e. the K-irreducible zero-dimensional subvariety) befined by $\mathfrak{p}$, and $P_1^*, \ldots, P_\mu^*$ the points, defined by $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_\mu^*$, of which P consists. We propose:

THEOREM (I). *If P is simple, so are $P_1^*, \ldots, P_\mu^*$. Conversely, if $P_1^*, \ldots, P_\mu^*$ are simple, so is P.*

PROOF. a) Let P be simple with u, p. s. $\eta_1, \ldots, \eta_r$: $\mathfrak{p}\mathfrak{o}_\mathfrak{p} = (\eta_1, \ldots, \eta_r)\mathfrak{o}_\mathfrak{p}$. Multiplying K^* we get $\mathfrak{p}\mathfrak{o}_\mathfrak{p}^* = (\eta_1, \ldots, \eta_r)\mathfrak{o}_\mathfrak{p}^*$ where $\mathfrak{o}_\mathfrak{p}^* = \mathfrak{o}_\mathfrak{p} \cdot \mathfrak{o}^*$. Since $\mathfrak{p}\mathfrak{o}_\mathfrak{p}^* = \mathfrak{p}_1^* \ldots \mathfrak{p}_\mu^* \mathfrak{o}_\mathfrak{p}^*$, going over to the quotient ring $\mathfrak{o}_{\mathfrak{p}^*_i}^*$ we arrive at the desired results: $\mathfrak{p}_i^*\mathfrak{o}_{\mathfrak{p}_i^*}^* = (\eta_1, \ldots, \eta_r)\mathfrak{o}_{\mathfrak{p}_i^*}^*$, i. e. P_i^* is a simple point of V_r $(i = 1, \ldots, \mu)$.

b) *The second nalf of our theorem is a fact which can never be realized in Zariski's method of the ground field extension.* Let the coordinates of P_i^* be $(c_{i1}^*, \ldots, c_{in}^*)$ $(i = 1, \ldots, \mu)$, and we get $\mathfrak{p}_i^* = (\xi_1 - c_{i1}^*, \ldots, \xi_n - c_{in}^*)$. By a preliminary linear transformation of coordinates in the space S_n, if necessary, we can assume that $\xi_1 - c_{i1}^*, \ldots, \xi_r - c_{ir}^*$ are u. p. s. of $\mathfrak{p}_i^*$—this taking place at once for every $i (= 1, \ldots, \mu)$. Since $c_{11}^*, c_{21}^*, \ldots, c_{\mu 1}^*$ contains all distinct K-conjugates of c_{11}^*, if we denote by $d_{11}^*, d_{21}^*, \ldots, d_{\nu 1}^*$ all distinct elements among $c_{11}^*, \ldots, c_{\mu 1}^*$, the product $\pi_1(\xi_1) = (\xi_1 - d_{11}^*)(\xi_1 - d_{11}^*) \ldots (\xi_1 - d_{\nu 1}^*)$ belongs to $\mathfrak{o}$. Similarly we get $\pi_2(\xi_2), \ldots, \pi_r(\xi_r)$. Now we see that $\pi_1, \ldots, \pi_r$ are u. p. s. for each $\mathfrak{p}_i^* (i = 1, \ldots, \mu)$: namely,

$$(3) \qquad (\pi_1, \ldots, \pi_r)\mathfrak{o}_{\mathfrak{p}_i^*}^* = (\xi_1 - c_{i1}^*, \ldots, \xi_r - c_{i\nu}^*)\mathfrak{o}_{\mathfrak{p}_i^*}^* = \mathfrak{p}_i^* \mathfrak{o}_{\mathfrak{p}_i^*}^*.$$

Let $\mathfrak{a} = (\pi_1, \ldots, \pi_r)$ be the ideal of $\mathfrak{o}$ generated by $\pi_1, \ldots, \pi_r$. Since we can assume that $\mathfrak{o}$ depends integrally on $\xi_1, \ldots, \xi_r$, $\mathfrak{a}$ is a purely zero-dimensional ideal, i. e. a product of zero-dimensional primary ideals. Taking the remark of the footnote (1) of the preceding page 107 into accocnt, $\mathfrak{a}\mathfrak{o}_\mathfrak{p}$ is certainly identical with $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$, otherwise (3) would not hold. Thus P is a simple point.

4. Now we have in hand the proof of the case $s = 0$ of the theorem at the head of our paper:

THEOREM (II). *Let P be a "point" (or a zero-dimensionel K-irreducible subvariety) of V_r, $\mathfrak{p}$ the corresponding prime $\mathfrak{o}$-ideal. $\eta_1, \ldots, \eta_r$ be r elements of $\mathfrak{o}$, which are algebraically independent over K and are such that $\mathfrak{o}$ depends integrally on $K[\eta_1, \ldots, \eta_r]$. P is a* *simple point of V_r and $f_1(\eta_1), \ldots, f_r(\eta_r)$ are u.p.s. at P—$f_i(z)$ being the irreducible polynomial in $K[z]$ such that $f_i(\eta_i) \equiv 0$ $(\mathfrak{p})$—if and only if an element ω of $\mathfrak{o}$ exists and $G'_\omega(\eta_1, \ldots, \eta_r; \omega) \not\equiv 0$ $(\mathfrak{p})$, where $G(\eta_1, \ldots, \eta_r; z)$ is the norm of $z-\omega$ with respect to the algebraic extension $\sum/K(\eta_1, \ldots, \eta_r)$.*

PROOF. a) The condition is nesessary. Let P be a simple point with u. p. s. $f_1(\eta_1)=\tau_1, \ldots, f_r(\eta_r)=\tau_r$. By the ground field extension $K \to K^*$, let P split into μ simple points $P_1^*, \ldots, P_\mu^*$ as it was the case in the preceding section. $\tau_1, \ldots, \tau_r$ are u. p. s. at any one of these μ points (theorem I). As in § 3 $P_1^*, \ldots, P_\mu^*$ correspond to the prime ideals $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_\mu^*$, and any one of them is denoted simply by $\mathfrak{p}^*$, the corresponding point by P^*—the index being dropped. Following Zariski's proof of § 6, part I of Zariski [1], we can prove the exsistence of an element $\omega(\varepsilon\ \mathfrak{o})$ such that $H'_\omega(\tau_1, \ldots, \tau_r; \omega) \not\equiv 0(\mathfrak{p})$, where $H(\tau_1 \ldots, \tau_r; z)$ is the norm of $z-\omega$ with respect to the extension $\sum/K(\tau_1, \ldots, \tau_r)$. We shall show it below.

Let $K(\tau_1, \ldots, \tau_r)$ be denoted by Ω. Since $\sum = K(\xi_1, \ldots, \xi_n) = \Omega(\xi_1, \ldots, \xi_n)$.

(4) $$\omega = a_1\xi_1 + \ldots + a_n\xi_n \qquad (a_i \in K)$$

is a generator of $\sum$ over $\Omega : \sum = \Omega(\omega)$, if $a_1, \ldots, a_n$ are non-special elements from K. Let the degree over Ω of ω be ν, and $\omega_0(=\omega)$, $\omega_1, \ldots, \omega_{\nu-1}$ be the conjugates of ω over Ω. Let $\bar{\sum}$ be the least Galois extension of $\sum$ over Ω and $T_0, T_1, \ldots, T_{\nu-1}$ the ν isomorphic mappings (of the Galois group of $\bar{\sum}/\Omega$) of $\sum$ onto its Ω-conjugates $\sum_0(=\sum = \Omega(\omega))$, $\sum_1 = \Omega(\omega_1), \ldots, \sum_{\nu-1} = \Omega(\omega_{\nu-1})$: $T_i\omega = \omega_i$, $\sum_i = T_i\sum$ $(i=0, 1, \ldots, v-1)$. T_i determines uniquely an isomorphic mapping of $\sum^* = \sum \times K^*$ onto $\sum_i^* = \sum_i \times K^*$ in which every element of K^* is mapped upon itself—this mapping we continue to denote by the same notation T_i. By T_i $\mathfrak{o}^*$ is mapped on to a ring $(\mathfrak{o}^*)_i = T_i\mathfrak{o}^*$, $\mathfrak{p}^*$ onto an ideal $(\mathfrak{p}^*)_i = T_i\mathfrak{p}^*$ of $(\bar{\mathfrak{o}}^*)_i$. Let $\mathfrak{o}^* = ((\mathfrak{o}^*)_0, (\mathfrak{o}^*)_1, \ldots, (\mathfrak{o}^*)_{\nu-1})$ be the ring composed of the ν conjugate rings $\mathfrak{o}^*, (\mathfrak{o}^*)_1, \ldots, (\mathfrak{o}^*)_{\nu-1}$. We see that $(\mathfrak{p}^*)_i\bar{\mathfrak{o}}^*$ and $(\mathfrak{p}^*)_j\bar{\mathfrak{o}}^*$ $(i \neq j)$ are relatively prime. Namely, since P^* is a simple point with u. p. s. $\tau_1, \ldots \tau_r$, we have the expansion of ω at P^*:

$$\omega = \psi_0^* + \psi_1^* + \ldots + \psi_\lambda^* + \ldots$$

i. e. $$\omega \equiv \psi_0^* + \psi_1^* + \ldots + \psi_\lambda^* \qquad ((\mathfrak{p}^*)^{\lambda+1}) \ (\lambda = 0, 1, 2, \ldots),$$

where ψ_λ^* is a form of degree λ of $\tau_1, \ldots, \tau_r$ with coefficients from K^*. Performing the mappings T_i, T_j, we get

(5) $$\omega \equiv \psi_0^* + \psi_1^* + \ldots + \psi_\lambda^* \qquad (((\mathfrak{p}^*)_i)^{\lambda+1}),$$
$$\omega \equiv \psi_0^* + \psi_1^* + \ldots + \psi_\lambda^* \qquad (((\mathfrak{p}^*)_j)^{\lambda+1}),\ \lambda = 0, 1, 2, \ldots$$

If $(\mathfrak{p}^*)_i\bar{\mathfrak{o}}^*$ and $(\mathfrak{p}^*)_j\bar{\mathfrak{o}}^*$ had any common prime divisor $\bar{\mathfrak{p}}^*$ in $\bar{\mathfrak{o}}^*$, we would have the congruence (5) modulo $(\mathfrak{p}^*)^{\lambda+1}(\lambda=0, 1, 2, \ldots)$, hence $\omega_i-\omega_j\equiv 0\ ((\bar{\mathfrak{p}}^*)^{\lambda+1})\ (\lambda=0, 1, 2, \ldots)$; thus $\omega_i-\omega_j\ (\neq 0)$ would be a null-divisor in $\overline{\sum}^*$. This would contradict the fact that $\omega_i-\omega_j$ is contained in the field $\overline{\sum}$. Let the decomposition of $\mathfrak{p}^*\bar{\mathfrak{o}}^*$ be

$$\mathfrak{p}^*\bar{\mathfrak{o}}^*=\bar{\mathfrak{q}}_1^*\ldots\ldots\bar{\mathfrak{q}}_\alpha^*$$

and accordingly

$$(\mathfrak{p}^*)_i\bar{\mathfrak{o}}^*=(\bar{\mathfrak{q}}_1^*)_i\ldots\ldots(\bar{\mathfrak{q}}_\alpha^*)_i, \qquad (\mathfrak{q}_j^*)_i=T_i\bar{\mathfrak{q}}_j^*.$$

By the above demonstration the $\alpha\nu$ prime zero-dimensional ideals $(\mathfrak{p}_j^*)_i$ $(=T_i\bar{\mathfrak{p}}_j^*)$ which belong to $(\bar{\mathfrak{q}}_j^*)_i\ (i=0, 1, \ldots, \nu-1\ ;\ j=1, \ldots, \alpha)$ are all distinct.

Since η_i depends integrally on $K(\tau_i)$ $(i=1, \ldots, r)$ and consequently $\mathfrak{o}$ depends integrally on $K[\tau_1, \ldots, \tau_\nu]$, the ideal $(\tau_1, \ldots, \tau_r)\mathfrak{o}$ is purely zero-dimensional. $\tau_1, \ldots, \tau_\nu$ being u. p. s. at P, let

$$(\tau_1, \ldots, \tau_\nu)\mathfrak{o}^*=\mathfrak{p}^*\hat{\mathfrak{q}}_1^*\ldots\ldots\hat{\mathfrak{q}}_\beta^*$$

be the decomposition of the $\mathfrak{o}^*$-ideal $(\tau_1, \ldots, \tau_r)\mathfrak{o}^*$, where $\hat{\mathfrak{q}}_1^*, \ldots, \hat{\mathfrak{q}}_\beta^*$ are primary $\mathfrak{o}^*$-ideals belonging to distinct zero-dimensional prime $\mathfrak{o}^*$-ideals $\hat{\mathfrak{q}}_1^*, \ldots, \hat{\mathfrak{q}}_\beta^*$—among these there appears every one of $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_\mu^*$ but for the one $\mathfrak{p}^*$. In (4) $a_1, \ldots, a_n$ can be so taken that $\omega\equiv c^*\ (\mathfrak{p}^*)$ and $\omega\not\equiv c^*(\hat{\mathfrak{p}}_i^*)$ $(i=1, 2, \ldots, \beta)$. Consider any one $(\bar{\mathfrak{p}}_\delta^*)_\gamma$ of the $\alpha(\nu-1)$ ideals $(\bar{\mathfrak{p}}_j^*)_i$ $(i=1, \ldots, \nu-1\ ;\ j=1, \ldots, \alpha)$. $(\bar{\mathfrak{p}}_j^*)_\gamma$ devides $(\tau_1, \ldots, \tau_r)\mathfrak{o}^*$; hence $(\bar{\mathfrak{p}}_\delta^*)_\gamma$ must devides one of $\mathfrak{p}^*, \hat{\mathfrak{p}}_1^*, \ldots, \hat{\mathfrak{p}}_\beta^*$. Now $(\bar{\mathfrak{p}}_\delta^*)_\gamma$ must not divide $\mathfrak{p}^*$ as proved above, so $(\bar{\mathfrak{p}}_\delta^*)_\gamma$ is a divisor of at least one of $\hat{\mathfrak{p}}_1^*, \ldots, \hat{\mathfrak{p}}_\beta^*$. Thus $[(\bar{\mathfrak{p}}_\delta^*)_\gamma, \mathfrak{o}^*]=\mathfrak{p}_\varepsilon^*\ (1\leqq\varepsilon\leqq\beta)$. Hence $\omega\not\equiv c^*((\bar{\mathfrak{p}}_\delta^*)_\gamma)$. Accordingly we have $\omega\not\equiv c^*((\mathfrak{p}_j^*)_i)$ $(i=1, \ldots, \nu-1\ ;\ j=1, \ldots, \alpha)$. By T_2 some $(\mathfrak{p}_\delta^*)_\gamma$ $(\gamma>0)$ is brought onto $\mathfrak{p}_\delta^*$. Hence applying T_2 to the incongruence $\omega\not\equiv c^*((\bar{\mathfrak{p}}_\delta^*)_\gamma)$ we have $\omega_2\not\equiv c^*(\mathfrak{p}_\delta^*)$ and consequently $\omega_2\not\equiv c^*(\mathfrak{p}^*)$. Similarly we get $\omega_j\not\equiv c^*\ (\mathfrak{p}^*)$ $(j=3, \ldots, \nu-1)$. This proves the incongruence $H'_\omega(\tau_1, \ldots, \tau_r\ ;\ \omega)\not\equiv 0(\mathfrak{p}^*)$ and from this we get $H'_\omega\not\equiv 0\ (\mathfrak{p})$.

The rest of the proof is almost plain. Namely, since $\sum\supset K(\eta_1, \ldots, \eta_r)\geqq\Omega$, H is divisible by G: $H(\tau_1, \ldots, \tau_r\ ;\ z)=G(\eta_1, \ldots, \eta_r\ ;\ z)\times L(z)$, we see that $H'_\omega(\tau_1, \ldots, \tau_r\ ;\ \omega)=G'_\omega(\eta_1, \ldots, \eta_r\ ;\ \omega)\times L(\omega)\not\equiv 0\ (\mathfrak{p})$ and that $G'_\omega(\eta_1, \ldots, \eta_r\ ;\ \omega)\not\equiv 0(\mathfrak{p})$.

b) The condition is sufficient. Let $\omega(\varepsilon\ \mathfrak{o})$ exist and $G'_\omega(\eta_1, \ldots, \eta_r\ ;\ \omega)\not\equiv 0\ (\mathfrak{p})$. Hence $G'_\omega\not\equiv 0\ (\mathfrak{p}^*)\mathfrak{p}^*$ where $\mathfrak{p}^*$ is as in a) any one of $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_\mu^*$ of which $\mathfrak{p}\mathfrak{o}^*$ is the product. Following Zariski's proof in § 7, part I of Zariski [1] we can prove that P^* is a simple point of V, and that $\eta_1-b_1^*, \ldots, \eta_r-b_r^*$ are u. p. s. at P^* if $\eta_i\equiv b_i^*(\mathfrak{p}^*)$ $(i=1, \ldots, r)$. For the sake of completeness we shall sketch the proof just later on. Now, assuming this proposition, we see at once by our theorem I that P^* is simple with u. p. s. $f_1(\eta_r), \ldots, f_r(\eta_i)$—$f_i(\eta_i)$ are obtained just as $\pi_i(\xi_i)$ was obtained in the second half b) of the proof of theorem I.

We are going to prove the proposition assumed just above. The $\mathfrak{o}^*$-ideal $\mathfrak{a}^* = (\eta_1 - b_1^*, \ldots, \eta_r - b_r^*)$ $(\leqq \mathfrak{p}^*)$ is purely zero-dimensional. Let its decomposition be $\mathfrak{a}^* = \mathfrak{q}^*\mathfrak{q}_1^*\ldots\mathfrak{q}_\lambda^*$, where $\mathfrak{q}^*$ is a primary ideal belonging to $\mathfrak{p}^*$, none of $\mathfrak{q}_i^*$ $(i = 1, \ldots, \lambda)$ belonging to $\mathfrak{p}^*$. We have to prove that $\mathfrak{q}^* = \mathfrak{p}^*$. Let $\omega \equiv d^*(\mathfrak{p}^*)$ and we have

$$G(b_1^*, \ldots, b_r^*\,;\ \omega) = (\omega - d^*)(\omega - d_1^*)\ldots(\omega - d_{\nu-1}^*),$$

since $0 = G(\eta_1, \ldots, \eta_r; \omega) \equiv G(b_1^*, \ldots, b_r^*; \omega) \equiv 0(\mathfrak{p}^*)$. By hypothesis $G'_\omega \not\equiv 0$ $(\mathfrak{p}^*)$, hence $d_i^* \neq d^*(i = 1, \ldots, \nu - 1)$. We have also $0 = G(\eta_1, \ldots, \eta_r\,;\ \omega)$ $\equiv G(b_1^*, \ldots, b_r^*\,;\ \omega)$ $(\mathfrak{a}^*)$, accordingly

$$(\omega - d^*)(\omega - d_1^*)\ldots(\omega - d_{\nu-1}^*) \equiv 0(\mathfrak{a}^*).$$

Since $\omega - d_i^* \not\equiv 0(\mathfrak{p}^*)$ $(i = 1, \ldots, \nu - 1)$, it follows that $\omega - d^* \equiv 0$ $(\mathfrak{q}^*)$. Let now π be any element of $\mathfrak{p}$, and put $\sigma = t\pi + \omega$ where t is a parameter. The norm of $z - \sigma$ with respect to $\sum/K(\eta_1, \ldots, \eta_r)$ is a polynomial $F(\eta_1, \ldots, \eta_r\,;\ t\,;\ z)$ which for $t = 0$ becomes $G(\eta_1, \ldots, \eta_r\,;\ z)$. Since $F'_\sigma(\eta_1, \ldots, \eta_r\,;\ t\,;\ \sigma)$ is a polynomial in t with coefficients from $\mathfrak{o}$ and for $t = 0$ it becomes $G'_\omega(\eta_1, \ldots, \eta_r\,;\ \omega)$ $(\not\equiv 0(\mathfrak{p}^*))$, we can substitute t by an element a of K such that $F'_\zeta(\eta_1, \ldots, \eta_r\,;\ a\,;\ \zeta) \not\equiv 0(\mathfrak{p}^*)$, where $\zeta = a\pi + \omega$. Plainly $F(\eta_1, \ldots, \eta_r\,;\ a\,;\ z)$ is the norm of $z - \zeta$. Since $\zeta \equiv d^*(\mathfrak{p}^*)$, discussing as above we get $\zeta - d^* \equiv 0$ $(\mathfrak{q}^*)$ i. e. $a\pi + \omega \equiv d^*(\mathfrak{q}^*)$. We have proved $\omega \equiv d^*(\mathfrak{q}^*)$, hence $a\pi \equiv 0$ $(\mathfrak{q}^*)$ and consequently $\pi \equiv 0(\mathfrak{q}^*)$. Thus we see that $\mathfrak{p} \equiv 0$ $(\mathfrak{q}^*)$, i. e. $\mathfrak{p}^* \equiv 0$ $(\mathfrak{q}^*)$. Hence $\mathfrak{p}^* = \mathfrak{q}^*$. q. e. d.

5. The aim of our paper being achieved we add some remarks that will make plain the situation of Zariski's theorem in the arithmetic theory of algebraic varieties. The theorem seems fit to prove that the idea of the "simple subvariety V_s" defined by the condition (1). is precisely identical with that of the simple subvariety in the usual geometric sense: V_s contains at least one and consequenlty infinitily many simple points of V_r. Also we can deduce from the theorem that the quotient ring $Q(V_s) = \mathfrak{o}_{\mathfrak{p}_s}$ of the simple subvariety V_s is integrally closed in its quotient field $\sum$. By the way it is known that $Q(V_s)$ is integrally closed if and only if there is no $(r-1)$-dimensional singular subvariety of V_r containing V_s.

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