<table>
<thead>
<tr>
<th>Title</th>
<th>Principal convergents and mediant convergents associated to $\alpha$-continued fractions (Analytic Number Theory and Surrounding Areas)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Natsui, Rie</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1384: 30-36</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25728">http://hdl.handle.net/2433/25728</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Principal convergents and mediant convergents
associated to $\alpha$-continued fractions

Rie Natsui
Department of Mathematics, Keio University
Hiyoshi, Kohoku-ku, Yokohama 223-8522
Japan
e-mail: r.natsui@math.keio.ac.jp

Abstract
We study some properties of principal and mediant convergents for a class of semi-regular continued fractions, in particular, $\alpha$-continued fractions, $0 < \alpha \leq 1$. We claim that all $\alpha$-principal convergents are the regular convergents if $\frac{1}{2} \leq \alpha < 1$, on the other hand, this is not true in general for $0 \leq \alpha < \frac{1}{2}$. We also show that for every $x$, the set of $\alpha$-principal and $\alpha$-mediant convergents of $x$ are identical with that of the regular principal and the regular mediant convergents of $x$.

1 Regular continued fraction
For an irrational number $x \in (0, 1)$, if a non-zero rational number $\frac{p}{q}$, $(p, q) = 1$, satisfies $|x - \frac{p}{q}| < \frac{1}{2q}$, then it is the $n$th regular principal convergent $\frac{p_n}{q_n}$ for some $n \geq 1$. Here the $n$th regular principal convergents are defined by

\[
\begin{aligned}
\begin{cases}
    p_{-1} = p_{-1}(x) = 1, & \quad p_0 = p_0(x) = 0 \\
    q_{-1} = q_{-1}(x) = 0, & \quad q_0 = q_0(x) = 1
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
    p_n = p_n(x) = a_n \cdot p_{n-1} + p_{n-2} & \quad \text{for } n \geq 1, \\
    q_n = q_n(x) = a_n \cdot q_{n-1} + q_{n-2}
\end{cases}
\end{aligned}
\]

with the regular continued fraction expansion of $x$:

\[
x = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots
\]

It is well-known that

\[
\frac{p_n}{q_n} = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \quad \text{for } n \geq 1.
\]

If $x \in (k, k+1)$ for an integer $k$, we define its $n$th regular principal convergent by $\frac{p_n(x-k)}{q_n(x-k)} + k = \frac{p_n(x-k)+k}{q_n(x-k)}$. On the other hand, for some $x \in (0, 1)$, there exists $\frac{p}{q}$ with $(p, q) = 1$ and $|x - \frac{p}{q}| < \frac{1}{q}$, which is not the $n$th regular principal convergent for any $n \geq 0$. However, we can find such a fraction $\frac{p}{q}$ in the set.
\[
\begin{align*}
\left\{ \frac{p_{n-1}}{q_{n-1}}, \frac{p_{n}+p_{n-1}}{q_{n}+q_{n-1}} : n \geq 1 \right\}. \\
This leads us the notion of the regular mediant convergents of level \( n \), \( \frac{u_{n,t}}{v_{n,t}} \), which is defined by
\[
\begin{align*}
u_{n,t} &= t \cdot p_{n} + p_{n-1}, \\
u_{n,t} &= t \cdot q_{n} + q_{n-1}
\end{align*}
\]
for \( 1 \leq t < a_{n+1}, \ n \geq 0 \).

The regular principal and the regular mediant convergents are obtained by the following maps \( T \) and \( F \) of \([0, 1]\), which are called the Gauss map and the Farey map, respectively, see [2] :
\[
T(x) = \begin{cases} 
\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \in (0, 1) \\
0 & \text{if } x = 0 
\end{cases}
\]
(1.1)
\[
\text{and}
F(x) = \begin{cases} 
\frac{x}{x-1} & \text{if } x \in [0, \frac{1}{2}) \\
\frac{1-x}{x} & \text{if } x \in [\frac{1}{2}, 1],
\end{cases}
\]
where \( \lceil y \rceil = n \) if \( y \in [n, n+1) \). We get the coefficients of the regular continued fraction expansion of \( x \in [0, 1] \) by
\[
a_{n} = a_{n}(x) = \lceil (T^{n-1}(x))^{-1} \rceil, \ n \geq 1.
\]
We refer to Sh.Ito [3] about the relation between \( F \) and the regular mediant convergents.

2 \( \alpha \)-continued fractions and the \( \alpha \)-mediant convergents

We generalize the notion of the mediant convergents to the \( \alpha \)-continued fraction expansions introduced by H.Nakada [5]. The notion of \( \alpha \)-continued fraction expansions is a generalization of the regular continued fraction expansion and the expansions are induced by the following map \( T_{\alpha} \) of \( I_{\alpha} = [\alpha-1, \alpha] \) for \( \frac{1}{2} \leq \alpha \leq 1 \) :
\[
T_{\alpha}(x) = \begin{cases} 
\left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{x} \right\rfloor_{\alpha} & \text{if } x \in I_{\alpha} \setminus \{0\} \\
0 & \text{if } x = 0,
\end{cases}
\]
where \( \left\lfloor y \right\rfloor_{\alpha} = n \) if \( y \in [n-1+\alpha, n+\alpha) \). We note that this definition coincides with (1.1) if \( \alpha = 1 \). For \( n \geq 1 \), we put
\[
\epsilon_{\alpha,n} = \epsilon_{\alpha,n}(x) = \text{sgn} \ T_{\alpha}^{n-1}(x),
\]
\[
c_{\alpha,n} = c_{\alpha,n}(x) = \left\lfloor \frac{1}{T_{\alpha}^{n-1}(x)} \right\rfloor_{\alpha} \quad \text{(or } \epsilon_{\alpha,n} = \infty \text{ if } T_{\alpha}^{n-1}(x) = 0).\]

Then we have the \( \alpha \)-continued fraction expansion of \( x \in I_{\alpha} \) by
\[
x = \frac{\epsilon_{\alpha,1}}{c_{\alpha,1}} + \frac{\epsilon_{\alpha,2}}{c_{\alpha,2}} + \frac{\epsilon_{\alpha,3}}{c_{\alpha,3}} + \cdots, \ c_{\alpha,n} \geq 1.
\]
We define the \( n \)-th \( \alpha \)-principal convergents \( \frac{p_{\alpha,n}}{q_{\alpha,n}}, \ n \geq 1 \), by
\[
\begin{align*}
p_{\alpha,-1} &= 1, \ p_{\alpha,0} = 0, \ q_{\alpha,-1} = 0, \ q_{\alpha,0} = 1 \\
\text{and} \\
p_{\alpha,n} &= c_{\alpha,n} \cdot p_{\alpha,n-1} + \epsilon_{\alpha,n} \cdot p_{\alpha,n-2}, \ q_{\alpha,n} &= c_{\alpha,n} \cdot q_{\alpha,n-1} + \epsilon_{\alpha,n} \cdot q_{\alpha,n-2}.
\end{align*}
\]
We note that the $\{q_{a,n}\}$ is strictly increasing, see [5]. Also we define the $\alpha$-mediant convergents of level $n \geq 0$, $\{u_{a,n,t} = \frac{u_{a,n} + \varepsilon_{a,n+1} \cdot p_{a,n-1}}{v_{a,n,t} = \frac{v_{a,n} + \varepsilon_{a,n+1} \cdot q_{a,n-1}}{1 \leq t < c_{a,n+1}}\}$, by

\begin{align*}
u_{a,n,t} &= t \cdot p_{a,n} + \varepsilon_{a,n+1} \cdot p_{a,n-1} \\
u_{a,n,t} &= t \cdot q_{a,n} + \varepsilon_{a,n+1} \cdot q_{a,n-1}
\end{align*}

Next, we define a map which induces the sequence of the $\alpha$-principal and the $\alpha$-mediant convergents for each $\alpha$, $\frac{1}{2} \leq \alpha \leq 1$. We put $J_\alpha = [\alpha - 1, \frac{1}{\alpha}]$ and define the map $G_\alpha$ of $J_\alpha$ by

$$G_\alpha(x) = \begin{cases}
\frac{1-x}{x} \quad \text{if} \quad x \in [\alpha - 1, 0) := J_{\alpha,1} \\
\frac{1}{z} \quad \text{if} \quad x \in [0, \frac{1}{1+\alpha}] := J_{\alpha,2} \\
\frac{1-x}{z} \quad \text{if} \quad x \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}) := J_{\alpha,3}.
\end{cases}$$

We note that $G_1 = F$. In this sense, $G_\alpha$ is a generalization of the Farey map and is called the $\alpha$-Farey map. In order to get the $\alpha$-principal and the $\alpha$-mediant convergents of $x \in J_\alpha$ by the iterations of $G_\alpha$, it is convenient to use the following matrices:

$$V_- = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

Since

$$\frac{ax+b}{cx+d} = \frac{u}{v} \quad \text{with} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

for any real numbers $x$ and $z \neq 0$, we denote

$$A(x) = \frac{ax+b}{cx+d} \quad \text{and} \quad A(-\infty) = A(\infty) = \frac{a}{c} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Hence, we write

$$G_\alpha(x) = \begin{cases}
V_-^{-1}(x) \quad \text{if} \quad x \in J_{\alpha,1} \\
V_+^{-1}(x) \quad \text{if} \quad x \in J_{\alpha,2} \\
U^{-1}(x) \quad \text{if} \quad x \in J_{\alpha,3}.
\end{cases}$$

We put

$$M_\alpha(x) := \begin{cases}
V_- \quad \text{if} \quad (G_\alpha)^{n-1}(x) \in J_{\alpha,1} \\
V_+ \quad \text{if} \quad (G_\alpha)^{n-1}(x) \in J_{\alpha,2} \\
U \quad \text{if} \quad (G_\alpha)^{n-1}(x) \in J_{\alpha,3}.
\end{cases}$$

Then, we get a sequence of matrices

$$M_1(x), M_2(x), \ldots$$

from the iterations of $G_\alpha$ for each $x \in J_\alpha$. Here, all matrices $M_n$'s are of determinants $\pm 1$. We put

$$k_0(x) := 0 \quad \text{and} \quad k_n(x) := \min\{k > k_{n-1}(x) : (G_\alpha)^{k-1}(x) \in J_{\alpha,3}\}, \quad n \geq 1.$$ 

Then we have the following theorem, which connects the map $G_\alpha$ to the $\alpha$-mediant convergents explicitly.
Theorem 1. For \( x \in \mathbb{I}_\alpha \), we have

(i) If \( l = k_n(x), \ n \geq 1 \),

\[
M_1(x)M_2(x) \cdots M_l(x) = \begin{pmatrix} P_{\alpha, n-1} & P_{\alpha, n} \\ q_{\alpha, n-1} & q_{\alpha, n} \end{pmatrix} \quad (2.2)
\]

(ii) If \( l = k_n(x) + t, \ 1 \leq t < c_{\alpha, n+1}, \ n \geq 0 \),

\[
M_1(x)M_2(x) \cdots M_l(x) = \begin{pmatrix} u_{\alpha, n, t} & P_{\alpha, n} \\ v_{\alpha, n, t} & q_{\alpha, n} \end{pmatrix} \quad (2.3)
\]

The following is a direct consequence of Theorem 1.

Corollary 1. We have

\[
(M_1(x)M_2(x) \cdots M_l(x)) (\infty) = \begin{cases} e_{\frac{a_{1}n-1}{a.n-1}}q & \text{if } l = k_n(x), \ n \geq 1 \\ \frac{u_{\alpha_{1}n_1}}{v_{\alpha_1n_1}} & \text{if } 1 \leq t < c_{\alpha, n+1}, \ n \geq 0 \end{cases}
\]

Remark. In [3], the regular mediant convergents are obtained as

\[
(M_1(x)M_2(x) \cdots M_{l-1}(x))(1).
\]

3 The relation of \( \alpha \)-convergents and regular convergents

In this section, we describe a relation between the \( \alpha \)-convergents and the regular convergents. Here we divide into two cases for \( \alpha \), \( 0 < \alpha < \frac{1}{2} \) and \( \frac{1}{2} \leq \alpha \leq 1 \). First we have the following theorem in the case of \( \frac{1}{2} \leq \alpha \leq 1 \).

Theorem 2 (in the case of \( \frac{1}{2} \leq \alpha \leq 1 \)). For \( x \in \mathbb{I}_\alpha \) we suppose

\[
x = \frac{\varepsilon_{\alpha, 1}}{c_{\alpha, 1}} + \frac{\varepsilon_{\alpha, 2}}{c_{\alpha, 2}} + \frac{\varepsilon_{\alpha, 3}}{c_{\alpha, 3}} + \cdots
\]

is the \( \alpha \)-continued fraction expansion of \( x \). Then we have the following for any \( \frac{1}{2} \leq \alpha < 1 \):

(I) \( \left\{ \frac{P_{\alpha, n}}{q_{\alpha, n}}, \ n \geq 1 \right\} \subset \left\{ \frac{p_m}{q_m}, \ m \geq 1 \right\} \)

(II) If \( \frac{p_m}{q_m} \neq \frac{P_{\alpha, n}}{q_{\alpha, n}} \) for any \( n \geq 1 \), then \( m = n + l_n(x) \) for some \( n \geq 1, \varepsilon_{\alpha, n+1}(x) = -1 \), and

\[
\frac{u_{\alpha, n-1, c_{\alpha, n-1}}}{v_{\alpha, n-1, c_{\alpha, n-1}}} = \frac{p_m}{q_m} = \frac{u_{\alpha, n+1}}{v_{\alpha, n+1}},
\]

where

\[
l_n(x) := \#\{1 \leq k \leq n : \varepsilon_{\alpha, k}(x) = -1\}
\]

(III) \( \left\{ \frac{P_{\alpha, n}}{q_{\alpha, n}}, \ n \geq 1 \right\} \cup \left\{ \frac{u_{\alpha, n, t}}{v_{\alpha, n, t}} : 1 \leq t < c_{\alpha, n+1}, \ n \geq 0 \right\}
\]

\[
= \left\{ \frac{p_n}{q_n}, \ n \geq 1 \right\} \cup \left\{ \frac{u_{n, t}}{v_{n, t}} : 1 \leq t < c_{n+1}, \ n \geq 0 \right\}
\]
We can expand the above theorem to $S$-algorithm. We give the definition of $S$-algorithm by C. Kraaikamp [4]. At first, the following is called singularization:

\[
\ldots + \frac{1}{a_{n-1} + \frac{1}{1 + \frac{1}{a_{n+1} + \ldots}}},
\]

\[\downarrow \text{singularization}\]

\[
\ldots + \frac{1}{(a_{n-1} + 1) + \ldots + \frac{1}{(a_{n+1} + 1) + \ldots}}
\]

which follows from

\[
\begin{pmatrix}
0 & 1 \\
1 & a_{n-1}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & a_{n+1}
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 \\
1 & a_{n-1} + 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & a_{n+1} + 1
\end{pmatrix}.
\]

Next we define a map $T$ on $[0,1] \times [-\infty, -1]$ by

\[
T(x, y) = \left( \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{y} - \left\lfloor \frac{1}{x} \right\rfloor \right)
\]

\[= (Tx, \frac{1}{y} - a_{1})\]

Then we see

\[
T^n(x, -\infty)
\]

\[= \left( T^n x, -\frac{q_n}{q_{n-1}} \right)
\]

\[= \left( \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \ldots}} - \left( a_n + \frac{1}{a_{n-1} + \ldots} \right) \right).
\]

Let $S$ is subset of $[\frac{1}{2}, 1) \times [0,1]$. Then $S$ is called a singularization area if $m(\partial S) = 0$ and $S \cap TS = \emptyset$, where $m$ is 2-dimensional Lebesgue measure.

**Definition 1 (S-algorithm).**

Let $S$ is a singularization area. Then an algorithm that induces continued fraction expansions is said to be $S$-algorithm if $T^n(x, -\infty) \in S$ induces the singularization at $n$th coefficients:

\[
\begin{pmatrix}
0 & 1 \\
1 & a_{1}
\end{pmatrix} \ldots 
\begin{pmatrix}
0 & 1 \\
1 & a_{n-1}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & a_{n}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & a_{n+1}
\end{pmatrix} \ldots
\]

\[\downarrow \text{singularization}\]

\[
\begin{pmatrix}
0 & 1 \\
1 & a_{1}
\end{pmatrix} \ldots 
\begin{pmatrix}
0 & 1 \\
1 & a_{n-1}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & a_{n+1}
\end{pmatrix} \ldots
\]
Remark 1 (C. Kraaikamp). For $\frac{1}{2} \leq \alpha \leq 1$, $T_\alpha$ is $S$-algorithm.

We have a generalization of Theorem 2 to $S$-algorithms.

Theorem 3. Suppose

$$x = \frac{\varepsilon_{s,1}}{c_{s,1}} + \frac{\varepsilon_{s,2}}{c_{s,2}} + \frac{\varepsilon_{s,3}}{c_{s,3}} + \ldots$$

is the $S$-expansion. Then we have the following:

(I) $\{\frac{p_{\alpha,n}}{q_{\alpha,n}}, n \geq 1\} \subset \{\frac{p_m}{q_m}, m \geq 1\}$

(II) If $\frac{p_m}{q_m} \neq \frac{p_{\alpha,n}}{q_{\alpha,n}}$ for any $n \geq 1$, then $m = n + l_n(x)$ for some $n \geq 1$, $\varepsilon_{s,n+1}(x) = -1$, and

$$\frac{u_{s,n-1,c_{s,n}+1}}{v_{s,n-1,c_{s,n}+1}} = \frac{p_m}{q_m} = \frac{u_{s,n}}{v_{s,n}},$$

where

$$l_n(x) := \#\{1 \leq k \leq n : \varepsilon_{s,k}(x) = -1\}$$

(III) $\left\{\frac{p_{\alpha,n}}{q_{\alpha,n}}, n \geq 1\right\} \cup \left\{\frac{u_{s,t,n}}{v_{s,t,n}}, 1 \leq t < c_{\alpha,n+1}, n \geq 0\right\} = \left\{\frac{p_m}{q_m}, m \geq 1\right\}$

For $0 < \alpha < \frac{1}{2}$, we see that $T_\alpha$ is not $S$-algorithm. However, we have the following theorem:

Theorem 4 (in the case of $0 < \alpha < \frac{1}{2}$). For any $0 < \alpha < \frac{1}{2}$, we have the following:

(I) There exists $x \in [\alpha - 1, \alpha]$ for which

$$\exists n \geq 1 \quad \text{s.t.} \quad \frac{p_{\alpha,n}}{q_{\alpha,n}} \neq \frac{p_m}{q_m}, m \geq 1,$$

that is,

$$\left\{\frac{p_{\alpha,n}}{q_{\alpha,n}}, n \geq 1\right\} \not\subset \left\{\frac{p_m}{q_m}, m \geq 1\right\}$$

(II) There exists $x \in [\alpha - 1, \alpha]$ such that $\frac{p_m}{q_m}$ appears 3-times in the sequence of $\alpha$-mediant convergents. (If $\alpha$ is small, $\frac{p_m}{q_m}$ appears 4-times, 5-times, ...)

(III) $\left\{\frac{p_{\alpha,n}}{q_{\alpha,n}}, n \geq 1\right\} \cup \left\{\frac{u_{s,t,n}}{v_{s,t,n}}, 1 \leq t < c_{\alpha,n+1}, n \geq 0\right\} = \left\{\frac{p_m}{q_m}, m \geq 1\right\}$

To prove Theorem 4, we use the following.

Lemma 1 (semi-singularization).

$$\begin{pmatrix} 0 & \pm 1 \\ 1 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0 \pm 1)(0 - 1)(0 - 1) \begin{pmatrix} 1 & 2 \\ 1 & l \end{pmatrix}$$

$$\begin{pmatrix} 0 & \pm 1 \\ 1 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix} = \begin{pmatrix} 0 \pm 1 \\ 1 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$
4 Some remarks

We recall the notion of semi-regular continued fractions by [4].

Definition 2 (semi-regular continued fractions). For a real number $x$,

$$x = b_0 + \frac{\varepsilon_1}{b_1 + \frac{\varepsilon_2}{b_2 + \ldots}},$$

where $b_0$ is an integer, $b_i (i \geq 1)$ is a positive integer and $\varepsilon_i = \pm 1 (i \geq 1)$. Above the continued fraction is called semi-regular

\[
\begin{align*}
\text{if } \varepsilon_{n+1} + b_n & \geq 1 \text{ and } \varepsilon_{n+1} + b_n \geq 2 \text{ infinitely often} \\
\text{ (in the case of the infinite continued fraction)} \\
\text{if } \varepsilon_{n+1} + b_n & \geq 1 \\
\text{ (in the case of the finite continued fraction).}
\end{align*}
\]

Definition 3 (semi-regular). An algorithm that induces continued fraction expansions is said to be semi-regular if induced continued fractions are always semi-regular.

We note the following, see [4].

Remark 2. Every $S$-algorithm is semi-regular.

Remark 3. If $0 < \alpha < \frac{1}{2}$, then $T_{\alpha}$ is not $S$-algorithm, but it is semi-regular.

Remark 4. $T_0$ is not semi-regular.

We have seen that the set of the $\alpha$-principal and the $\alpha$-mediant convergents coincides with the set of the regular’s. K. Dajani and C. Kraaikamp [1] showed that Lehner fractions induce the set of the regular principal and the regular mediant convergents. They also showed that this set includes all principal convergents arising from $S$-expansions. In this sense, they called this set “the mother of all semi-regular continued fractions”. Our claim is that we can construct the “mother” from any $\alpha$-continued fractions, $0 < \alpha \leq 1$, by producing the $\alpha$-mediant convergents.

References


