On Hayami's Turbulent Tensor

By

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1. On *Hydrological Studies on the Yangtze River, China*, Mr. Hayami[1] has introduced the following interesting turbulent tensor:

$$Y = C^2 |\operatorname{rot} \boldsymbol{v}| \frac{(h^2-z^2)^2}{2(h^2+z^2)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the origin of the rectangular coordinate system is taken on the free surface (of the river, assumed flat), the x-axis in the direction of the flow (as that of the stream along the surface), while the z-axis is taken vertically downward.

According to Gebelein,[2] the turbulent tensor is

$$(1) \qquad \begin{cases} Y = C^2 |\operatorname{rot} \boldsymbol{v}| \chi \begin{pmatrix} \frac{1}{2\varkappa} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\varkappa+1} \end{pmatrix}, \\ \chi = z^2 \end{cases}$$

where the coordinate origin is taken on the boundary of the flow (the bottom of the river). This formula is applicable when the fluid fills the semi-infinite space on one side of the boundary surface. In applying this formula to the flow of the river whose free surface should be kept in its state, Mr. Hayami has imagined the virtual eddy of the true one at the bottom of the river, just symmetric with respect to the surface of the river. Then the turbulent tensor will be obtained by putting

$$\left\{ \frac{1}{z^2} + \frac{1}{(2h-z)^2} \right\}^{-1} = \frac{z^2(2h-z)^2}{(2h-z)^2+z^2}$$

instead of z^2 in (1), where h is the depth of the river in consideration. Taking the coordinate origin on the free surface as stated above, we have

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1. Shôitirô Hayami, Hydrological Studies, IV. On the mechanics of flow in a wide river, Jour. Shanghai Sci. Inst., sec. I, vol. 1, No. 13 (1939).
2. H. Gebelein, Turbulenz (1935), p. 99.

$$(2)\qquad \begin{cases} Y = C^2|\text{rot}\,\boldsymbol{v}|\chi \begin{pmatrix} \frac{1}{2\varkappa} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\varkappa+1} \end{pmatrix}, \\ \chi = \dfrac{(h^2-z^2)^2}{2(h^2+z^2)}. \end{cases}$$

Supposing h is constant, and the inclination j of the stream to the horisontal line is very small, so that $\sin j = j$, the equation of the steady, uniform with respect to x and non kinetically viscous turbulent motion is given by

$$(3)\qquad \frac{\partial}{\partial z}\left\{\eta_{zz}\frac{\partial u}{\partial z}\right\} = -gj,$$

where

$$(4)\qquad \eta_{zz} = C^2\chi\left|\frac{\partial u}{\partial z}\right|\frac{1}{\varkappa+1} = C^2\frac{(h^2-z^2)^2}{2(h^2+z^2)}\left|\frac{\partial u}{\partial z}\right|\frac{1}{\varkappa+1}.$$

Here u is the mean velocity in the x-direction. According to Mr. Hayami, $\varkappa$ is experimentally very nearly zero, so that we have instead of (4)

$$(5)\qquad \eta_{zz} = C^2\chi\left|\frac{\partial u}{\partial z}\right| = C^2\frac{(h^2-z^2)^2}{2(h^2+z^2)}\left|\frac{\partial u}{\partial z}\right|.$$

This is Mr. Hayami's component of the turbulent tensor.[1]

2. Let us remark that to keep the free surface in its natural state, only a virtual eddy is not sufficient. So we shall imagine an infinite number of virtual eddies at the vertical distances $\pm 2h$, $\pm 4h$,......... Considering the coordinate origin on the bottom of the river, put

$$(6)\qquad \begin{aligned} \frac{1}{\chi} &= \frac{1}{z^2} + \frac{1}{(2h-z)^2} + \frac{1}{(4h-z)^2} + \cdots\cdots\cdots \\ &\quad + \frac{1}{(-2h-z)^2} + \frac{1}{(-4h-z)^2} + \cdots\cdots\cdots \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{(2nh-z)^2}. \end{aligned}$$

Then z^2 in Gebelein's turbulent tensor of (1) shall be replaced by the above χ.

Next we shall summate the series (6). As we know

$$\cot\vartheta = \frac{1}{\vartheta} + \sum_{n=-\infty}^{\infty}{}'\left(\frac{1}{\vartheta - n\pi} + \frac{1}{n\pi}\right);$$

whence we have

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1. Here Gebelein, Loc. cit., p. 143 should be referred to.

$$(7)\qquad \operatorname{cosec}^2\vartheta=\sum_{n=-\infty}^{\infty}\frac{1}{(\vartheta-n\pi)^2}.$$

Therefore, comparing (6) and (7), we have

$$(8)\qquad \chi=\left(\frac{2h}{\pi}\right)^2\sin^2\frac{\pi z}{2h},$$

so that in the equation of the turbulent motion (3), we have to put instead of (5),

$$\eta_{zz}=C^2\chi\left|\frac{\partial u}{\partial z}\right|=C^2\left(\frac{2h}{\pi}\right)^2\sin^2\frac{\pi z}{2h}\left|\frac{\partial u}{\partial z}\right|.$$

Let us take the coordinate origin on the free surface and z-axis vertically downwards, then instead of the above formula, we have

$$(9)\qquad \eta_{zz}=C^2\chi\left|\frac{\partial u}{\partial z}\right|=C^2\left(\frac{2h}{\pi}\right)^2\cos^2\frac{\pi z}{2h}\left|\frac{\partial u}{\partial z}\right|.$$

Now integrating (3), we have (the tangential traction being zero),

$$\eta_{zz}\frac{\partial u}{\partial z}=-gjz.$$

Substituting the value of (9), we have

$$(10)\qquad \frac{\partial u}{\partial z}=-\frac{\sqrt{gjz}}{C\sqrt{\chi}}=-\frac{\sqrt{gjz}}{C\frac{2h}{\pi}\cos\frac{\pi z}{2h}}.$$

Changing the variable by

$$(11)\qquad z=h\zeta,$$

we have

$$(12)\qquad \frac{\partial u}{\partial \zeta}=-\frac{\pi\sqrt{gjh}}{2C}\frac{\sqrt{\zeta}}{\cos\frac{\pi}{2}\zeta},$$

where the ζ-axis directs vertically downwards and $\zeta=0$ on the free surface, while $\zeta=1$ at the bottom of the river. By (9) and (10), we have

$$(13)\qquad \eta_{zz}=C\sqrt{gjz}\sqrt{\chi}=\frac{2Ch\sqrt{gjh}}{\pi}\sqrt{\zeta}\cos\frac{\pi}{2}\zeta.$$

We shall put, as Mr. Hayami did,

$$(14)\qquad \Psi(\zeta)=\sqrt{\zeta}\cos\frac{\pi}{2}\zeta.$$

Then

$$(15)\qquad \eta_{zz}=\frac{2Ch\sqrt{gjh}}{\pi}\Psi(\zeta).$$

To find the maximum point ζ_0 of $\Psi(\zeta)$, we have to solve

$$\Psi'(\zeta)=\frac{1}{2\sqrt{\zeta}}\left(\cos\frac{\pi}{2}\zeta-\pi\zeta\sin\frac{\pi}{2}\zeta\right)=0.$$

Or putting $\frac{\pi}{2}\zeta_0=x_0$, we have

$$1-2x_0\tan x_0=0.$$

By the table and some simple calculations, we have (in comparison, Hayami's values[1] are written in the parenthesis)

$$x_0\doteqdot 0.6525,\ \zeta_0\doteqdot 0.415\ (0.40),\ \Psi(\zeta_0)\doteqdot 0.5121\ (0.4933)$$

Next $\int_0^1\Psi(\zeta)d\zeta\doteqdot 0.35739\quad(0.3633,\ 0.3666)$;

hence the mean is attained at

$$\zeta_1\doteqdot 0.133\ (0.14),\qquad \zeta_2\doteqdot 0.724\ (0.68)$$

Table of $\Psi(\zeta)=\sqrt{\zeta}\cos\frac{\pi}{2}\zeta$[2]

ζ	$\sqrt{\zeta}$	$\cos\frac{\pi}{2}\zeta$	$\sqrt{\zeta}\cos\frac{\pi}{2}\zeta$	Diff.
0.00	0.00000000	1.0000000	0.0000	0.1000
0.01	0.10000000	0.9998766	0.1000	0.0413
0.02	0.14142136	0.9995066	0.1413	0.0317
0.03	0.17320508	0.9988899	0.1730	0.0266
0.04	0.20000000	0.9980267	0.1996	0.0233
0.05	0.22360680	0.9969173	0.2229	0.0210
0.06	0.24494897	0.9955620	0.2439	0.0191
0.07	0.26457513	0.9939610	0.2630	0.0176
0.08	0.28284271	0.9921147	0.2806	0.0164
0.09	0.30000000	0.9900237	0.2970	0.0153
0.10	0.31622777	0.9876883	0.3123	0.0145
0.11	0.33166248	0.9851093	0.3268	0.0135
0.12	0.34641016	0.9822873	0.3403	0.0128
0.13	0.36055513	0.9792228	0.3531	0.0121
0 14	0.37416574	0.9759168	0.3652	0.0114
0.15	0.38729833	0.9723699	0.3766	0.0108
0.16	0.40000000	0.9685832	0.3874	0.0103
0.17	0.41231056	0.9645574	0.3977	0.0098
0.18	0.42426407	0.9602937	0.4075	0.0091
0.19	0.43588989	0.9557930	0.4166	0.0087
0.20	0.44721360	0.9510565	0.4253	0.0083
0.21	0 45825757	0.9460854	0.4336	0.0077

1. Hayami, Loc. cit., p. 245.
2. These calculalions we owe to Mr. Hirosi Nakahata. Thanks are due to him.

ζ	$\sqrt{\zeta}$	$\cos\frac{\pi}{2}\zeta$	$\sqrt{\zeta}\ \cos\frac{\pi}{2}\zeta$	Diff.
0.22	0.46904158	0.9408808	0.4413	0.0073
0.23	0.47958315	0.9354440	0.4486	0.0069
0.24	0.48989795	0.9297765	0.4555	0.0065
0.25	0.50000000	0.9238795	0.4620	0.0060
0.26	0.50990195	0.9177546	0.4680	0.0056
0.27	0.51961524	0.9114033	0.4736	0.0052
0.28	0.52915026	0.9048271	0.4788	0.0048
0.29	0.53851648	0.8980276	0.4836	0.0044
0.30	0.54772256	0.8910065	0.4880	0.0041
0.31	0.55677644	0.8837656	0.4921	0.0036
0.32	0.56568542	0.8763067	0.4957	0.0033
0.33	0.57445626	0.8686315	0.4990	0.0029
0.34	0.58309519	0.8607420	0.5019	0.0025
0.35	0.59160798	0.8526402	0.5044	0.0022
0.36	0.60000000	0.8443279	0.5066	0.0018
0.37	0.60827625	0.8358074	0.5084	0.0014
0.38	0.61644140	0.8270806	0.5098	0.0012
0.39	0.62449980	0.8181497	0.5110	0.0007
0.40	0.63245553	0.8090170	0.5117	0.0004
0.41	0.64031242	0.7996847	0.5121	**0.5121**
0.42	0.64807407	0.7901550	0.5121	**0.5121**
0.43	0.65574385	0.7804304	0.5117	0.0004
0.44	0.66332496	0.7705132	0.5111	0.0006
0.45	0.67082039	0.7604060	0.5101	0.0010
0.46	0.67823300	0.7501111	0.5087	0.0014
0.47	0.68556546	0.7396311	0.5071	0.0016
0.48	0.69282032	0.7289686	0.5051	0.0020
0.49	0.70000000	0.7181263	0.5027	0.0024
0.50	0.70710678	0.7071068	0.5000	0.0027
0.51	0.71414284	0.6959128	0.4969	0.0031
0.52	0.72111026	0.6845471	0.4936	0.0033
0.53	0.72801099	0.6730125	0.4899	0.0037
0.54	0.73484692	0.6613119	0.4860	0.0039
0.55	0.74161985	0.6494480	0.4817	0.0043
0.56	0.74833148	0.6374240	0.4770	0.0047
0.57	0.75498344	0.6252427	0.4719	0.0051
0.58	0.76157731	0.6129071	0.4668	0.0051
0.59	0.76811457	0.6004202	0.4612	0.0056

ζ	$\sqrt{\zeta}$	$\cos\frac{\pi}{2}\zeta$	$\sqrt{\zeta}\cos\frac{\pi}{2}\zeta$	Diff.
0.60	0.77459667	0.5877853	0.4553	0.0059
0.61	0.78102497	0.5750053	0.4491	0.0062
0.62	0.78740079	0.5620834	0.4426	0.0065
0.63	0.79372539	0.5490228	0.4357	0.0069
0.64	0.80000000	0.5358268	0.4286	0.0071
0.65	0.80622577	0.5224986	0.4212	0.0074
0.66	0.81240384	0.5090414	0.4135	0.0077
0.67	0.81853528	0.4954587	0.4056	0.0079
0.68	0.82462113	0.4817537	0.3973	0.0083
0.69	0.83066239	0.4679298	0.3888	0.0085
0.70	0.83666003	0.4539905	0.3799	0.0089
0.71	0.84261498	0.4399392	0.3707	0.0092
0.72	0.84852814	0.4257793	0.3613	0.0094
0.73	0.85440037	0.4115144	0.3516	0.0097
0.74	0.86023253	0.3971479	0.3417	0.0099
0.75	0.86602540	0.3826834	0.3314	0.0103
0.76	0.87177979	0.3681246	0.3209	0.0105
0.77	0.87749644	0.3534748	0.3102	0.0107
0.78	0.88317609	0.3387379	0.2991	0.0111
0.79	0.88881944	0.3239174	0.2879	0.0112
0.80	0.89442719	0.3090170	0.2764	0.0115
0.81	0.90000000	0.2940403	0.2646	0.0118
0.82	0.90553851	0.2789911	0.2526	0.0120
0.83	0.91104336	0.2638731	0.2404	0.0122
0.84	0.91651514	0.2486899	0.2279	0.0125
0.85	0.92195445	0.2334454	0.2153	0.0126
0.86	0.92736185	0.2181432	0.2023	0.0130
0.87	0.93273791	0.2027873	0.1892	0.0131
0.88	0.93808315	0.1873813	0.1758	0.0134
0.89	0.94339811	0.1719291	0.1622	0.0136
0.90	0.94868330	0.1564345	0.1484	0.0138
0.91	0.95393920	0.1409012	0.1344	0.0140
0.92	0.95916630	0.1253332	0.1202	0.0142
0.93	0.96436508	0.1097343	0.1058	0.0144
0.94	0.96953597	0.0941083	0.0912	0.0146
0.95	0.97467943	0.0784591	0.0765	0.0147
0.96	0.97979590	0.0627905	0.0615	0.0150

ζ	$\sqrt{\zeta}$	$\cos\frac{\pi}{2}\zeta$	$\sqrt{\zeta}\cos\frac{\pi}{2}\zeta$	Diff.
0.97	0.98488578	0.0471065	0.0464	0.0151
0.98	0.98994949	0.0314108	0.0311	0.0153
0 99	0.99498744	0.0157073	0.0156	0.0155
1.00	1.00000000	0.0000000	0.0000	0.0156
			35.7396	

The graph of the function is very similar to that of Mr. Hayami.[1]

3. Instead of (9), if we take Mr. Hayami's value (5), then we have

(16) $$\Psi(\zeta)=\frac{(1-\zeta^2)\sqrt{\zeta}}{\sqrt{1+\zeta^2}}.$$

He has considered that this function is approximately near to the well known empirical formula:

(17) $$\Psi(\zeta)=\zeta(1-\zeta).$$

In our case it will not be too rough if we approximate the sine curve of $\cos\frac{\pi}{2}\zeta$ by the parabolic arc of $1-\zeta^2$. Then we have from (14),

(18) $$\Psi(\zeta)=\sqrt{\zeta}(1-\zeta^2).$$

In this case the equation (12) becomes

$$\frac{\partial u}{\partial\zeta}=-\frac{\pi\sqrt{gjh}}{2C}\frac{\sqrt{\zeta}}{1-\zeta^2}.$$

Hence

(19) $$u=u_0+\frac{\pi\sqrt{gjh}}{2C}\left(\arctan\sqrt{\zeta}-\frac{1}{2}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right).$$

Or expanding into the series,

$$\arctan\sqrt{\zeta}=\sqrt{\zeta}-\frac{\sqrt{\zeta}^3}{3}+\frac{\sqrt{\zeta}^5}{5}-\ldots\ldots,$$

$$\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}=2\left(\sqrt{\zeta}+\frac{\sqrt{\zeta}^3}{3}+\frac{\sqrt{\zeta}^5}{5}+\ldots\ldots\right),$$

$$(0\leqslant\zeta<1)$$

we have

(20) $$u=u_0-\frac{\pi\sqrt{gjh}}{C}\sqrt{\zeta}^3\left(\frac{1}{3}+\frac{\zeta^2}{7}+\ldots\ldots\right).$$

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1. Hayami, Loc. cit., p. 246.

This gives the vertical distribution of the velocity.

4. To study the bed of the river, we shall now consider the suspension of the matter in the river.[1] Let θ be the concentration of the suspended matter (weight of the matter per unit volume). Then

$$\frac{\partial}{\partial z}\left(\eta_{zz}\frac{\partial\theta}{\partial z}\right)=w_0\frac{\partial\theta}{\partial z}, \tag{21}$$

where w_0 is the setting velocity of the suspended particle and the coordinate system is taken as the above. Then integrating we have

$$\eta_{zz}\frac{\partial\theta}{\partial z}=w_0\theta+A,$$

where A is the integrating constant. Writing θ instead of $\theta+\dfrac{A}{w_0}$, we have

$$\eta_{zz}\frac{\partial\theta}{\partial z}=w_0\theta \quad \text{or} \quad \eta_{zz}\frac{\partial\theta}{\partial\zeta}=w_0 h\theta.$$

Hence by aid of (15),

$$\log\frac{\theta}{\theta_0}=-w_0 h\int_\zeta^{\zeta_0}\frac{d\zeta}{\eta_{zz}}=-\frac{\pi w_0}{2C\sqrt{gjh}}\int_\zeta^{\zeta_0}\frac{d\zeta}{\Psi(\zeta)}.$$

We cannot take $\zeta_0=1$, since then the improper integral does not exist. According to Mr. Hayami[2]

$$\zeta_0=0.9999.$$

Put

$$\Omega(\zeta)=\int_\zeta^{\zeta_0}\frac{d\zeta}{\Psi(\zeta)}=\int_\zeta^{\zeta_0}\frac{d\zeta}{\sqrt{\zeta}(1-\zeta^2)}. \tag{22}$$

Then we have

$$\Omega(\zeta)=\arctan\sqrt{\zeta}+\frac{1}{2}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\Bigg|_\zeta^{\zeta_0}.$$

Therefore

$$\log\frac{\theta}{\theta_0}=-\frac{\pi w_0}{2C\sqrt{gjh}}\Omega(\zeta) \tag{23}$$

$$=-\frac{\pi w_0}{2C\sqrt{gjh}}\left[\arctan\sqrt{\zeta}+\frac{1}{2}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right]_\zeta^{\zeta_0}.$$

Using Mr. Hayami's data,[3]

$$\theta_0=10^5\text{g/cm}^3,\quad \frac{\theta}{\theta_0}=10^{-5},\quad C=0.2,\quad (\log_{10}e=0.4343)$$

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1. Hayami, Loc. cit., p. 253.
2. Hayami, Loc. cit., VII. On the stability of an erodible river bed with special reference to the lower Yangtze River. Jour. Shanghai Sci. Inst. New Series vol. 1, No. 1 (1941).
3. Hayami, Loc. cit., VII, p. 47.

we have by (23),

$$(24) \qquad \frac{0.4343\pi}{2\times 0.2}\ \frac{w_0}{\sqrt{gjh}}\Omega(\zeta)=5.$$

From this equation we may find ζ corresponding to several values of

$$s=\frac{w_0}{\sqrt{gjh}}.$$

We have to solve the equation with respect to ζ:

$$(25) \qquad \Omega(\zeta)=\frac{K}{s}, \quad \left(K=\frac{5\times 2\times 0.2}{0.4343\pi}\doteqdot 1.48125\right)$$

into the form

$$(26) \qquad \zeta=f(s).$$

The graph of this function has the shape of the integral sign. In Hayumi's case, the abrupt increase occurs between $s=0.2$ and 0.3.[1] In our case it occurs between $s=0.3$ and 0.4 as the following calculations[2] show.

$$\zeta_0=0.9999$$

$$\arctan\sqrt{\zeta_0}+\frac{1}{2}\log\frac{1+\sqrt{\zeta_0}}{1-\sqrt{\zeta_0}}\doteqdot 5.7363$$

i. When $s=\mathbf{0.3}$ we have

$$\frac{K}{s}\doteqdot 4.9375.$$

Hence we have to solve

$$\arctan\sqrt{\zeta}+\frac{1}{2}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}=0.7988$$

But for	$\zeta=0.16$, ,, ,,	$=0.8038$
	$=0.15$,	$=0.7775$
	0.01,	0.0263

Hence $\zeta\doteqdot\mathbf{0.1581}$

ii. When $s=\mathbf{0.4}$ we have

$$\frac{K}{s}\doteqdot 3.664$$

Hence we have to solve

$$\arctan\sqrt{\zeta}+\frac{1}{2}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}=2.0699$$

1. Hayami, Loc. cit., VII, p. 48.
2. We owe these to Mr. Keniti Koseki. Thanks are due to him.

But for $\zeta=0.77$, „ „ $=2.0840$
$=0.76$, $=2.0566$

0.01, 0.0274

Hence $\zeta \fallingdotseq \mathbf{0.7648}$

Thus we see the abrupt increase of $f(s)$ occurs between $s=0.3$ and 0.4. Moreover $f(0.26) \fallingdotseq 0$.

5. We remark that the graph of the function $\zeta=f(s)$ is of the shape of Γ, is common to all the improper integrals which are not convergent.

Let the integral be

$$\Omega(\zeta)=\int_{\zeta}^{\zeta_0}\frac{d\zeta}{\Psi(\zeta)} \quad \text{and} \quad \int_{\zeta}^{1}\frac{d\zeta}{\Psi(\zeta)}=\infty,$$

where

$$\Psi(\zeta)>0, \qquad \Psi(1)=0.$$

The function $\zeta=f(s)$ should change abruptly in an interval, say (a, b). It must be noted that the position of this interval in $(0, 1)$ depends upon the value of ζ_0. The interval (a, b) displaces in the direction of $s>0$, provided ζ_0 be taken nearer to $\zeta=1$.

For ζ_1 near to ζ_0, we have

$$\int_{\zeta}^{\zeta_0}\frac{d\zeta}{\Psi(\zeta)}=\frac{\zeta_0-\zeta}{\Psi(\bar{\zeta})}, \quad \zeta_1<\zeta<\bar{\zeta}<\zeta_0.$$

Hence by (25), we have

$$\frac{\zeta_0-\zeta}{\Psi(\zeta_1)}<\frac{K}{s}<\frac{\zeta_0-\zeta}{\Psi(\zeta_0)},$$

or

$$K\Psi(\zeta_0)<s(\zeta_0-\zeta)<K\Psi(\zeta_1),$$

wher $\Psi(\zeta)$ is supposed by hypothesis to decrease monotonously to zero. Therefore if we consider the hyperbolas

$$xy=K\Psi(\zeta_0)$$

and

$$x_1 y=K\Psi(\zeta_1),$$

then we have for $y=\zeta_0-\zeta$, the inequalities

$$x<s<x_1,$$

i. e., s increases more rapidly than x of the first hyperbola, but slower than x_1 of the second one in $\zeta_1<\zeta<\zeta_0$. If ζ_0, ζ_1 are very near to 1, both hyperbolas turn very rapidly about their vertices. Therefore the Γ-form of $\zeta=f(s)$ becomes very aucute. Thus the rapid increase of $\zeta=f(s)$ in the neighbourhood of $\zeta=1$ depends upon the choice of the initial point ζ_0.