A Note on Fowler's Differential Equation

By

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(Received March 30, 1942)

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1. After G. Sansone,[1] Fowler's differential equation is

$$\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right]+\xi^\lambda\theta^n=0, \quad (\lambda>0,\ n\geqq 0) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

and its Emden's solution is such one that it satisfies the initial condition

$$\theta(+0)=C, \quad (0<C<\infty) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Emden[2] has studied the case $\lambda=2$ and Fowler[3] has continued the study for several cases. Sansone[1] has recently given a complete proof of the theorem in its general case: For that such a solution vanish at certain point of $0<\xi<\infty$, it is necessary and sufficient that $2\lambda-n+1>0$.

This number comes as a coefficient of an identity fundamental to his proof. But it would be impossible to perceive it at the first glance of the differential equation (1). The author intends to bring forward this number $2\lambda-n+1$ by the ordinary transformation of the differential equation.

After the differentiation, the equation (1) is

$$\xi\frac{d^2\theta}{d\xi^2}+2\frac{d\theta}{d\xi}+\xi^{\lambda-1}\theta^n=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1')$$

By the substitution

$$\xi=e^{2u}, \quad u=\frac{1}{2}\log\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

the equation becomes

$$\frac{d^2\theta}{du^2}+2\frac{d\theta}{du}+4e^{2\lambda u}\theta^n=0 \quad \ldots\ldots\ldots\ldots\ldots(4)$$

To have the differential equation without the term of the first order, we put as usual

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1. *Sulle soluzioni di Emden dell' equazione di Fowler;* Rendiconti di Matematica e delle sue applicazioni, **1** (1940), p. 163.
2. Gaskugeln, (1907).
3. Quartery Journ., **45** (1914); Ibid (Oxford series), **2** (1931).

$$\theta = ve^{-u}, \quad v = \theta e^{u}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

then the equation becomes

$$\frac{d^2v}{du^2} - v + 4e^{(2\lambda-n+1)u}v^n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Here we see the number $2\lambda - n + 1$ which would play a special rôle.

2. To prove the necessity of the condition, Mr. Sansone has used the identity:

$$\frac{2\lambda - n + 1}{2}\int_0^{\xi} \xi^2 \left(\frac{d\theta}{d\xi}\right)^2 d\xi = \xi^{\lambda+1}\theta^{n+1}(\xi) + (\lambda+1)\xi^2\theta(\xi)\theta'(\xi)$$
$$+ \frac{n+1}{2}\xi^3\theta'^2(\xi) \ldots\ldots\ldots (7)$$

It is not explicitly remarked from what consideration he has derived it. For our equation (6), we derive it as follows:

From (6) we have

$$\left(\frac{d^2v}{du^2} - \frac{dv}{du}\right) + \left(\frac{dv}{du} - v\right) = -4e^{(2\lambda-n+1)u}v^n \ldots\ldots\ldots (6')$$

Multiplying both sides by $2\left(\frac{dv}{du} - v\right)$ and integrating, we have

$$\left(\frac{dv}{du} - v\right)^2 + 2\int\left(\frac{dv}{du} - v\right)^2 du = -8\int e^{(2\lambda-n+1)u}v^n \frac{dv}{du} du$$
$$+ 8\int e^{(2\lambda-n+1)u}v^{n+1} du + C.$$

If $n+1 \neq 0$ (for our case $n+1 \geqq 0$), we have

$$\int e^{(2\lambda-n+1)u}v^n \frac{dv}{du} du = \frac{1}{n+1}\Big\{e^{(2\lambda-n+1)u}v^{n+1}$$
$$-(2\lambda - n + 1)\int e^{(2\lambda-n+1)u}v^{n+1} du\Big\}.$$

Hence

$$\left(\frac{dv}{du} - v\right)^2 + 2\int\left(\frac{dv}{du} - v\right)^2 du = -\frac{8}{n+1}e^{(2\lambda-n+1)u}v^{n+1}$$
$$+ \frac{16(\lambda+1)}{n+1}\int e^{(2\lambda-n+1)u}v^{n+1} du + C.$$

Now by (6)

$$4\int e^{(2\lambda-n+1)u}v^{n+1} du = \int\left(v - \frac{d^2v}{du^2}\right)v\, du$$
$$= \int\left(v - \frac{dv}{du}\right)v\, du + \int\left(\frac{dv}{du} - \frac{d^2v}{du^2}\right)v\, du.$$

The second integral in the right is equal to

$$\left(v-\frac{dv}{du}\right)v-\int\left(v-\frac{dv}{du}\right)\frac{dv}{du}du$$
$$=\left(v-\frac{dv}{du}\right)v+\int\left(\frac{dv}{du}-v\right)^2du+\int\left(\frac{dv}{du}-v\right)v\,du.$$

Hence after a short calculation, we have

$$(2\lambda-n+1)\int\left(\frac{dv}{du}-v\right)^2du=4e^{(2\lambda-n+1)u}v^{n+1}+2(\lambda+1)\left(\frac{dv}{du}-v\right)v$$
$$+\frac{n+1}{2}\left(\frac{dv}{du}-v\right)^2+C'.$$

For $u\to-\infty$, v and $\frac{dv}{du}$ vanish and

$$e^{(2\lambda-n+1)u}v^{n+1}=e^{(2\lambda+1)u}v(ve^{-u})^n\to 0.$$

Hence $C'=0$ and the required identity for the equation (6) becomes

$$(2\lambda-n+1)\int_{-\infty}^{u}\left(\frac{dv}{du}-v\right)^2du=4e^{(2\lambda-n+1)u}v^{n+1}+2(\lambda+1)\left(\frac{dv}{du}-v\right)v$$
$$+\frac{n+1}{2}\left(\frac{dv}{du}-v\right)^2. \quad \ldots\ldots(8)$$

We remark that the identity may more easily be obtained provided $2\lambda-n+1\neq 0$. Multiplying both sides of the equation (6′) by v and integrating we have

$$\left(\frac{dv}{du}-v\right)v-\int\left(\frac{dv}{du}-v\right)\frac{dv}{du}du+\int\left(\frac{dv}{du}-v\right)v\,du$$
$$=-4\int e^{(2\lambda-n+1)u}v^{n+1}du.$$

Hence

$$\left(\frac{dv}{du}-v\right)v-\int\left(\frac{dv}{du}-v\right)^2du=-\frac{4}{2\lambda-n+1}e^{(2\lambda-n+1)u}v^{n+1}$$
$$+\frac{4(n+1)}{2\lambda-n+1}\int e^{(2\lambda-n+1)u}v^n\frac{dv}{du}du.$$

Observe that

$$4\int e^{(2\lambda-n+1)u}v^n\frac{dv}{du}du=\int\left(v-\frac{d^2v}{du^2}\right)\frac{dv}{du}du$$
$$=\frac{1}{2}v^2-\frac{1}{2}\left(\frac{dv}{du}\right)^2.$$

After a short calculation we shall have the identity.
