A Note on Fowler's Differential Equation, II

By

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1. For the unique existence and vanishing in finiteness of the solution satisfying the initial condition

$$\theta(+0)=C, \quad (0<C<\infty) \quad \text{.......................(1)}$$

(Emden's solution) of Fowler's differential equation

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\xi^\lambda\theta^n=0, \quad (\lambda\geqq 0,\ n\geqq 0) \quad \text{..............(2)}$$

it is after Sansone necessary and sufficient that

$$2\lambda-n+1>0^{1} \quad \text{..................................(3)}$$

holds. To prove this, Mr. Sansone has introduced the most important identity,

$$\frac{2\lambda-n+1}{2}\int_0^\xi \xi^2\left(\frac{d\theta}{d\xi}\right)^2 d\xi=\xi^{\lambda+1}\theta^{n+1}(\xi)+(\lambda+1)\xi^2\theta(\xi)$$

$$+\frac{n+1}{2}\xi^3\theta'^2(\xi). \quad \text{......(4)}$$

In the previous note the present writer has given, probably the most natural method to arrive at this identity. In this note he intends chiefly to show briefly that quite the same method may be applied to the general Fowler's differential equation

$$\frac{d}{dx}\left(x^\rho\frac{dy}{dx}\right)+bx^\sigma y^n=0. \quad \text{....................(5)}$$

For simplicity we may suppose that the constants ρ, σ, n, b are positive. Then the required identity is (in the indefinite integral form)

$$\left[\sigma-\frac{n+1}{2}(\rho-1)+1\right]\int x^\rho\left(\frac{dy}{dx}\right)^2 dx=bx^{\sigma+1}y^{n+1}$$

$$+(\sigma+1)x^\rho y\frac{dy}{dx}+\frac{n+1}{2}x^{\rho+1}\left(\frac{dy}{dx}\right)^2+C. \quad \text{...(6)}$$

Here if we put $\rho=2$, $\sigma=\lambda$, we shall have (4).

2. The differential equation (5) is nothing but

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1. For the literature, see the previous note. These Memoirs vol. 24, No. 2.

$$x\frac{d^2y}{dx^2}+\rho\frac{dy}{dx}+bx^{\sigma-\rho+1}y^n=0. \quad \text{..............(7)}$$

To get rid of the factor of y'', we have to put

$$x=e^{\rho u} \quad \text{or} \quad u=\frac{1}{\rho}\log x. \quad \text{.....................(8)}$$

Then (7) becomes

$$\frac{d^2y}{du^2}+\rho(\rho-1)\frac{dy}{du}+b\rho^2e^{\rho(\sigma-\rho+2)u}y^n=0. \quad \text{...........(9)}$$

To chancell the term of y', we have to put

$$y=e^{-\left(\frac{\rho}{2}\right)u}v \quad \text{or} \quad v=ye^{\left(\frac{\rho}{2}\right)u}. \quad \text{.....................(10)}$$

Then (9) becomes

$$\left.\begin{aligned} &\frac{d^2v}{du^2}-\left(\frac{\rho}{2}\right)^2v+b\rho^2e^{pu}v^n=0, \\ \text{where} \quad &p\equiv\rho(\sigma-\rho+2)-(n-1)\left(\frac{\rho}{2}\right) \\ &=\rho\left[\sigma-\frac{n+1}{2}(\rho-1)+1\right]. \end{aligned}\right\} \quad \text{.....................(11)}$$

Remark that for $\rho=2$, $\sigma=\lambda$, we have $p=2\lambda-n+1$. As usual we write this equation as follows.

$$\frac{d}{du}\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]+\left(\frac{\rho}{2}\right)\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]=-b\rho^2e^{pu}v^n.$$

Multiplying by $2\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]$ and integrating, we shall have at last

$$\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]^2+2\left(\frac{\rho}{2}\right)\int\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]^2du$$
$$=-\frac{2b\rho^2}{n+1}e^{pu}v^{n+1}+2b\rho^2\left\{\frac{p}{n+1}+\left(\frac{\rho}{2}\right)\right\}\int e^{pu}v^{n+1}du+C,$$

under the assumption that $n+1\neq 0$. By aid of (11), we have after easy calculations

$$2b\rho^2\int e^{pu}v^{n+1}du=-2\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]v+2\int\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]^2du.$$

So we have finally the identity,

$$p\int\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]^2du$$
$$=b\rho^2e^{pu}v^{n+1}+\left\{p+(n+1)\left(\frac{\rho}{2}\right)\right\}\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]v$$
$$+\frac{n+1}{2}\left[\frac{dv}{du}-\left(\frac{\rho}{2}\right)v\right]^2+C. \quad \text{..............(12)}$$

Returning to the original variables, we have the required identity (6).

3. We should observe whether our equation (5) may be transformed into an equation of the form (2). For this, by easy calculations, we have the result as follows.

If $\rho > 1$, then by the substitution

$$x = t^k, \quad k = \frac{1}{\rho - 1} > 0,$$

(5) may be transformed into

$$\frac{d}{dt}\left(t^2 \frac{dy}{dt}\right) + \frac{b}{(\rho-1)^2} t^{\frac{\sigma+1}{\rho-1}-1} y^n = 0. \quad \text{..............}(13)$$

Further if $n \neq 1$, $b > 0$, then by

$$Ay = \eta, \quad A = \sqrt[n-1]{\frac{b}{(\rho-1)^2}} > 0,$$

(13) becomes

$$\frac{d}{dt}\left(t^2 \frac{d\eta}{dt}\right) + t^{\frac{\sigma+1}{\rho-1}-1} \eta^2 = 0. \quad \text{..........................}(14)$$

Here we have $\lambda = \dfrac{\sigma + 1}{\rho - 1} - 1$. Hence the fundamental constant becomes

$$2\lambda - n + 1 = 2\frac{\sigma - \frac{n+1}{2}(\rho - 1) + 1}{\rho - 1} = \frac{p}{\binom{\rho}{2}}. \quad \text{......}(15)$$

Thus we see that not all equations (5) may be transformed into (2), (preserving the coordinate origin). Moreover we notice that for $\lambda > 0$, it is necessary that

$$\sigma - \rho + 2 > 0. \quad \text{...................................}(16)$$

4. Now we consider such solution

$$\left.\begin{aligned} &y(x) > 0, \quad \text{for} \quad 0 < x < x_0 \ (< \infty) \\ &y(+0) = C, \quad (0 < C < \infty) \end{aligned}\right\} \quad \text{..................}(17)$$

If our integral satisfies the condition that

$$x^\rho y' \Big|_{x \to +0} = 0 \quad \text{or} \quad x^{\frac{\rho+1}{2}} y' \Big|_{x \to +0} = 0, \quad \text{.....................}(18)$$

according as $\rho < 1$ or $\rho \geqq 1$, then by (6) we have

$$\left[\sigma - \frac{n+1}{2}(\rho - 1) + 1\right] \int_0^x x^\rho \left(\frac{dy}{dx}\right)^2 dx$$

$$= b x^{\sigma+1} y^{n+1} + (\sigma + 1) x^\rho y \frac{dy}{dx} + \frac{n+1}{2} x^{\rho+1} \left(\frac{dy}{dx}\right)^2. \quad \text{......}(19)$$

By (5), $x^\rho \dfrac{dy}{dx}$ decreases, a fortiori does so $\dfrac{dy}{dx}$. $(\rho \geqq 0)$ If $y(x_0) = 0$, then $\left(\frac{dy}{dx}\right)_{x=x_0}<0$. Hence by (18)

$$\left[\sigma-\frac{n+1}{2}(\rho-1)+1\right]\int_0^{x_0}x^\rho\left(\frac{dy}{dx}\right)^2dx=\frac{n+1}{2}x_0^{\rho+1}\left(\frac{dy}{dx}\right)_0^2>0.$$

So we have the necessary condition

$$\sigma-\frac{n+1}{2}(\rho-1)+1>0. \qquad \text{......(20)}$$

5. At first let us examine the first condition $\lim_{x\to+0}x^\rho y'(x)=0$. By (5) we have for $\varepsilon>0$

$$x^\rho y'(x)\Big|_\varepsilon^x=-b\int_\varepsilon^x x^\sigma y^n dx. \qquad \text{......(21)}$$

Therefore we have

$$\lim_{x\to+0}x^\rho y'(x)=a, \qquad \text{......(22)}$$

where a is finite determinate. If $a\neq0$, then supposing $a>0$, we may, for any number a', $(0<a'<a)$ find $\delta>0$, such that

$$0<a'<x^\rho y'(x)<a \quad \text{for} \quad 0<x<\delta.$$

Hence for $0<\varepsilon<x<\delta$, we have

$$\left.\begin{aligned}&\frac{a'}{1-\rho}\left(\frac{1}{x^{\rho-1}}-\frac{1}{\varepsilon^{\rho-1}}\right)<y(x)-y(\varepsilon)\\&\qquad<\frac{a}{1-\rho}\left(\frac{1}{x^{\rho-1}}-\frac{1}{\varepsilon^{\rho-1}}\right),\quad(\rho\neq1)\\&\text{or}\\&a'\log\frac{x}{\varepsilon}<y(x)-y(\varepsilon)<a\log\frac{x}{\varepsilon},\quad(\rho=1)\end{aligned}\right\} \qquad \text{......(23)}$$

If $\rho\geqq1$, we should have $y(+0)=-\infty$ which is contrary to the initial condition $y(+0)=C$. In case of $a<0$, the same contradiction occurs. Therefore it ought to be that

$$\lim_{x\to+0}x^\rho y'(x)=0, \quad (\rho>1) \qquad \text{......(24)}$$

Consequently by (21) we have

$$x^\rho y'(x)=-b\int_0^x x^\sigma y^n dx.$$

Now
$$\lim_{x\to+0}x^{\frac{\rho+1}{2}}y'(x)=-\lim_{x\to+0}\frac{b}{x^{\frac{\rho-1}{2}}}\int_0^x x^\sigma y^n dx$$
$$=-\frac{b}{\frac{\rho-1}{2}}\lim_{x\to+0}\frac{x^\sigma y^n}{x^{\frac{\rho-1}{2}-1}},$$

provided this last limit exists. It exists certainly. The limiting value is either zero or negative. If

$$x^{\frac{\rho+1}{2}} y'(x) < -A, \quad \text{for} \quad 0 < x < \delta,$$

then $$y(x) - y(\varepsilon) < -A \int_{\varepsilon}^{x} \frac{dx}{x^{\frac{\rho+1}{2}}}.$$

For $\varepsilon \to +0$, this inequality becomes absurd. Therefore we have

$$\lim_{x\to+0} x^{\frac{\rho+1}{2}} y'(x) = 0, \quad (\rho > 1) \qquad \text{(25)}$$

Thus one part of (18) has been verified and hence we may conclude that (19) is valid for $\rho \geqq 1$, without assuming $x^{\frac{\rho+1}{2}} y' |_{x\to+0} = 0$.

In case of $\rho < 1$, putting

$$\frac{x^{1-\rho}}{1-\rho} = u, \quad \text{or} \quad x = (1-\rho)^{\frac{1}{1-\rho}} u^{\frac{1}{1-\rho}}, \qquad \text{(26)}$$

(5) will be transformed into

$$\left.\begin{aligned} & x^{\rho} \frac{dy}{dx} = \frac{dy}{du}, \\ & \frac{d^2y}{du^2} + Bu^{\frac{\sigma+\rho}{1-\rho}} y^n = 0, \\ \text{where} \quad & B \equiv b(1-\rho)^{\frac{\sigma+\rho}{1-\rho}} > 0. \end{aligned}\right\} \qquad \text{(27)}$$

For our solution $y(u)$, we have by (22)

$$\lim_{u\to+0} \frac{dy}{du} = a. \quad \text{(finite determinate)}$$

Judging from (23), a may not be zero. For if $n \geqq 1$, then by the general existence theorem, (27) admits the unique integral with the initial conditions

$$y(0) = C, \quad y'(0) = a,$$

for any C and a. Therefore in the case of $\rho < 1$, the identity (19) is only true on condition that $x^{\rho} y' |_{x\to+0} = 0$.