Two Theorems on the Connected Sets.

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Two point sets are said to be mutually separated if neither contains a point or limit point of the other. A set is said to be connected if it cannot be decomposed into two sets which are mutually separated.

[1]This note consists of two theorems on the connected sets:

I.[2] If M is a closed connected set, and if in M there is a largest division-set[3] A, then M is a simple arc.

II.[4] Let M be a closed connected set such that, if g is any connected subset of M, then $M-g$ consists of two components at most. Let the product of all division-sets of M not vanish, and let it be closed, then M consists of two simple closed curves which have a simple arc or a point in common.

Theorem I. If M is a closed connected set, and if in M there is a largest[5] division-set A, then M is a simple arc.

Proof.

Let $M-A=B+C$, where B is separated from C, then, $B+A$, as well as $C+A$, is connected. If $B=B_1+B_2$, where B_1 is separated from B_2, then $A+B_1$, as well as $A+B_2$, is connected. Then, $M-(A+B_1)=B_2+C$, where B_2 is separated from C, this contradicts to the fact that A is a largest division set. Hence B is connected. In the same way, C is also connected.

Now suppose B contains two points b_1 and b_2, then $B-b_1\neq 0$. Let $B-b_1$ be connected, then, if $B-b_1$ is not separated from A, $M-$

1. In this paper we consider exclusively the euclidean space of any dimensions.

2. Concerning this theorem, confer C. Zarankiewicz "Sur les points de division dans les ensembles connexes", Fund. Math. IX. p. 143. Nevertherless, the author cannot agree to this Zarankiewicz's theorem.

3. If A is a connected subset of M such that M-A is not connected, then A is called *a* division-set of M.

4. When M-g is always connected, M is a simple closed curve. J. R. Kline "Closed connected sets etc.", Fund. Math. V. p. 3-10.

5. Any division set is contained in A.

$\{A+(B-b_1)\}=C+b_1$, which is a contradiction; if $B-b_1$ is separated from A, then b_1 is not separated from A and $M-(A+b_1)=C+(B-b_1)$, which is a contradiction. Hence $B-b_1=B_1+B_2$, where B_1 is separated from B_2. As B is connected, b_1+B_1, as well as b_1+B_2, is connected. As $A+B$ is connected, A is not separated from one of b_1+B_1 and b_1+B_2, say from b_1+B_1, then $A+b_1+B_1$ is connected, and $M-(A+b_1+B_1)=C+B_2$, which is a contradiction. Hence B consists of only one point b, in the same way C cosists of only one point c.

We say that M is a irreducible[1] connected set between b and c. Suppose any point p contained in $M-(b+c)$, then $M-c-p$ is not connected clearly. Let $M-c-p=D+E$, where D is separated from E, then since $M-c$ is connected, $p+D$ and $p+E$ are also connected. Let b be contained in D, then $D-b$ and E are mutually separated, and since $M-(b+c)$ is connected, $(D-b)+p$, as well as $p+E$, is connected. If c is not separated from D, $c+(D-b)+p$ is connected, and $M-\{c+p+(D-b)\}=b+E$, this is a contradiction. Hence c is separated from D. If c is separated from E, $M-(p+D)=c+E$, which is a contradiction.

Therefore, $M-p=D+(E+c)$, where D is separated from $E+c$, and $b\in D$, $c\in E+c$.

Now, let P be any connected set contained in M, containing neither b nor c, and let p be any point contained in P. $M-p=H+K$, where H is separated from K, and where $b\in H$, $c\in K$. $M-P=(H-P)+(K-P)$, where $b\in H-P$ and $c\in K-P$, and where $H-P$ is separated from $K-P$.

Now, suppose N to be any connected set contained in M, containing b and c. If $M-N$ is not equal to zero, then,[2] let P be any component of $M-N$, $M-P$ is connected. On the other hand, as P contains neither b nor c, $M-P$ is not connected. Hence $M=N$, that is, M is a closed irreducible connected set between b and c. Therefore M is a simple arc.

Q. E. D.

Theorem II. Let M be a closed connected set such that, if g is any connected subset of M, then $M-g$ consists of two components at most. Let the product of all division sets of M not vanish, and

1. There is no connected subset of M which is different from M and contains b and c. cf. B. Knaster and C. Kuratowski "Sur les ensembles connexes" Fund. Math. II. p. 214.
2. cf. "Sur les ensembles connexes" Fund, Math. II. p. 214.

let it be closed, then M consists of two simple closed curves which have a simple arc or a point in common.

Proof. Let A be the product of all division sets, and let B be its complement with respect to M.

In the first place we say that B should not consist of more than three components. If B has more than three components, then A should not be connected clearly. Suppose now a connected subset C containing A such that $M-C$ has two components.

$M-C$ contains points of two components of B at most, which may be written B_1 and B_2, where it is possible that $B_1=B_2$. Let us denote $(M-C)B_1=B_1'$ and $(M-C)B_2=B_2'$. Let B_3 be a component of B, where $B_3 \neq B_1$ and $B_3 \neq B_2$, then $C-B_3$ is not connected.

Consequently $C-B_3$ can be decomposed in the following form. $C-B_3=T+S$, where T is separated from S. Now, B_1', as well as B_2', must be separated from neither T nor S. If B_1' were separted from T or S, say from T, and B_2' were separated from T or S, say from T, then $M-B_3=B_1'+B_2'+T+S=(B_1'+B_2'+S)+T$, which is a contradiction. Hence one of B_1' and B_2', say B_1', is separated from neither T nor S. If B_2' is separated from T, then $M-(B_1'+B_3+S)$ $=B_2'+T$. If $T \cdot A=0$, then, as $M-(B_1'+S+B_3+B_2')=T$, T contains the points of two components of B at most, from this T is separated from B_3, which is a contradicton. Hence $T \cdot A \neq 0$ and so $M-(B_1'+$ $B_3+S)$ is connected, this is contradiction. Therefore, B_1', as well as B_2', is separated from neither T nor S. From this, after a short consideration, $A \cdot T \neq 0$ $A \cdot S \neq 0$, and so T and S are both connected. Next, we remove B_4 from C, where $B_4 \neq B_1$, $B_4 \neq B_2$, $B_4 \neq B_3$, and where B_4 is a component of B, then $C-B_4$ is not connected, Let us denote $C-B_4=T'+S'$, where T' is separated from S'. B_4 is contained in T or S entirely, say in T, then B_3+S is contained in $C-$ B_4. B_3+S is connected, B_3+S is contained in one of T' and S', say in S', entirely. Let us denote $S'-(B_3+S)=T_1$, $T'=T_2$. $S'+B_4$ is connected, and B_4 is separated from $S+B_3$, so $T_1 \neq 0$. We could say that $A \cdot T \neq 0$ $A \cdot S \neq 0$, and T and S are both connected. In the same way, $A \cdot T' \neq 0$ $A \cdot S' \neq 0$, and T' and S' are both connected. $A \cdot$ $T_1 \neq 0$. Because, if $T_1 \cdot A=0$, then $M-(B_4+T_2+B_1'+S+B_3+B_2')=T_1$ contains the points of two components of B at most, so T_1 is separated from B_3+S, which is a contradiction to the fact that T_1+B_3+S is connected. Now suppose that T_1 is separated from B_2', then $M-$ $(T_2+B_4+B_1'+S+B_3)=T_1+B_2'$, whereas T_1 contains points of A, this is impossible. $T_1+B_4+T_2=T$, which is a connected set. T_1 and T_2 are mutually separated, so T_1+B_4 is connected. T_1 is not separated from B_2', and $B_4+T_1+B_2'+S+B_1'$ is connected, so $M-(B_4+T_1+B_2'+S+B_1')=T_2+B_3$, whereas T_2 contains points of A, which is a contradiction. After all, B should consist of three components at most. Suppose now $B=B_1+B_2+B_3$, where B_1, B_2, B_3 are three components of B, then A is not connected, so $A=A_1+A_2$, where A_1 is separated from A_2. At least two components among B_1, B_2, B_3 are separated from neither A_1 nor A_2, say, B_1 and B_2 are separatod from neither A_1 nor A_2. Suppose that B_3 is separated from A_2 and not separated from A_1, then $A_2=M-(B_1+A_1+B_2+B_3)$ is connected. For $M-B_1=B_2+A_1+B_3+A_2$ is connected, and $M-(B_1+B_2)=(A_1+B_3)+A_2$, where A_1+B_3 is separated from A_2, and so $A_1+B_3+B_2$ is connected. A_1 cannot be connected, so $A_1=A_1'+A_2'$, where A_1' is separared from A_2'. As A_1+B_3 is connected, $A_1'+B_3$, as well as $A_2'+B_3$, is connected. A_1' and A_2' cannot be separated from one of B_1 and B_2. Let A_1' be not separated from B_1, then $A_2'=M-(A_1'+B_1+B_2+B_3+A_2)$ is connected, and in the same way A_1' is connected. Clearly A_1' is separated from B_2, and A_2' is separated from B_1. In the next place, suppose that B_1, B_2 and B_3 are separated from neither A_1 nor A_2. Let $M-(B_1+B_2)$ be connected, then A_1 and A_2 are both connected. Let $M-(B_1+B_2)$ be not connected, then we can write $A_1+A_2+B_3=(B_3+A_1')+A_2'$, where $A_1+A_2=A_1'+A_2'$, and where B_3+A_1' and A_2' are mutually separated. In this case, B_1 and B_2 are separated from neither A_1' nor A_2', and B_3 is separated from A_2', so $A_1'+A_2'=A_1+A_2=A$ consists of three components.

If B consists of two components, then, after a short consideration A consists of two components at most.

Consequently the following five cases can arise:

1) A is a connected set, and B is also a connected set.
2) A is a connected set, and B consists of two components.
3) A consists of two components, and B consists of two components.
4) A consists of two components, and B consists of three components.
5) A consists of three components, and B consists of three components.

In the first place let us consider the case 2).

Let $M-A=C+D$, where C and D are components of B, then

$A+C$ and $A+D$ are connected. If E be any connected set contained in $A+C$, the following five cases may arise:

i. E is contained in C.

ii. E is contained in A.

iii. E contains C, and yet $E\cdot A$ is not zero.

iv. E contains A, and yet $E\cdot C$ is not zero.

v. $E\cdot C$, $E\cdot A$, $A-E$, and $C-E$ are not zero.

Let us first consider the case i), Suppose $(A+C)-E=F+G$, where F is separated from G, then, as A is connected, A is contained in one of F and G entirely, say in F, and so G is contained in C. Hence G is separated from D. From this, $F+D$ is separated from G. $M-E=(A+C+D)-E=(D+F)+G$, whereas E contains no point of A, which is a contradiction. Hence $(A+C)-E$ is connected.

ii) If $E=A$, then $(A+C)-E=C$, which is connected. Suppose $A-E$ is not zero. If E were separated from D, then, as $M-\{(A+C)-E\}=D+E$, $(A+C)-E$ is not connected. Hence $(A+C)-E=F+G$, where F is separated from G. As C is connected, C is contained in one of F and G, say in F, so G is contained in $A-E$. $E+F$ and $E+G$ are both connected. As $M-E=D+F+G$, $D+F$ and $D+G$ are both connected. If $C=F$, then $G=A-E$ and $M-E=(A-E)+D+C$, where C is separated from $(A-E)+D$; this is a contradiction. Therefore $F-C$ is not zero. $A-E=(F-C)+G$, where $F-C$ is separated from G. $M-\{E+(F-C)\}=C+(G+D)$, where C is separated from $G+D$. On the other hand, G is contained in A, which is a contradiction. Therefore E should not be separated from D, and $(A+C)-E=M-(D+E)$, which is connected.

iii) As $E\cdot A$ is not zero, E is not separated from D, and $(A+C)-E=M-(E+D)$, which is connected.

iv) E contains A, $(A+C)-E$ is connected clearly.

v) As $E\cdot A$ is not zero, E is not separated from D. $(A+C)-E=M-(E+D)$. As $A-E$ is not zero, $(A+C)-E$ is connected.

Therefore $A+C$ is a closed connected set such that, if E is any connected subset of $A+C$, then $(A+C)-E$ is also connected. As $A+C$ contains more than one point, by Kline's theorem, $A+C$ is a simple closed curve. In the same way $A+D$ is a simple closed curve. A is a simple arc or one point. In the case 2), A should not be a simple arc. Therefore M consists of two simple closed curves which have one point in common. Conversely such M satisfies the condition of theorem II clearly.

Now let us consider the case 3) and let $M=A+B$, $A=A_1+B_1$, $B=C_1+D_1$, where A_1 and B_1 are components of A, and C_1 and D_1 are components of B. Clearly A_1 is separated from neither C_1 nor D_1; B_1 is also unseparated in the same way. Let D_2 be a set of limit points of D_1, contained in A_1. If D_2 contains two points a and b, then $A_1-(a+b)=M-\{D_1+a+b+B_1+C_1\}$ is connected, and $A_1-(a+b)$ is not separated from C_1. So $M-\{A-(a+b)+C_1+B_1+D_1\}=a+b$, which is impossible. Hence D_2 consists of only one point d_2, and by the same reason, a set of limit points of C_1 contained in A consists of one point c_2. When c_2 is coincident with d_2, A_1 is coincident with d_2 clearly. When c_2 is not coincident with d_2, A_1 is a irreducible cennected set between d_2 and c_2, that is, a simple arc. Let c_3 be a limit point of C_1 contained in B_1, and let d_3 be a limit point of D_1 contained in B_1. B_1 is a point c_3 or a simple arc of which extremties are c_3 and d_3. $C_1+c_2+c_3$ is clearly closed connected set, and let E be any connected subset of $C_1+c_2+c_3$, which contains only one point of c_2 and c_3, say c_2, then $C_1+c_2+c_3-E$ is a connected set, because $M-\{E+A_1+D_1+(B_1-c_3)\}=C_1+c_2+c_3-E$. If[1] E contains both c_2 and c_3, then $C_1+c_2+c_3-E$ is connected, because $M-E=(C_1+c_2+c_3-E)+\{D_1+(A_1-c_2)+(B_1-c_3)\}$, and yet $C_1+c_2+c_3-E$ is separated from $D_1+(A_1-c_2)+(B_1-c_3)$. Therefore $C_1+c_2+c_3$ is a closed connected set containing a largest division set C_1, so by theorem I, $C_1+c_2+c_3$ is a simple arc, of which extremities are c_2 and c_3. In the same way, $D_1+d_2+d_3$ is also a simple arc. Hence M should be a simple closed curve, whereas the simple closed curve does not satisfy the condition of theorem II. Therefore the case 3) can not arise.

Let us now consider the case 4), and let $M=A+B$, $A=A_1+B_1$, $B=C_1+D_1+E_1$, where A_1 and B_1 are components of A, and C_1, D_1 and E_1 are components of B. B_1, as well as A_1, is not separated from C_1, D_1 and E_1 clearly. Suppose any connected set G contained in B_1. If G is not separated from one of C_1, D_1 and E_1, say from C_1, then $B_1-G=M-(G+C_1+D_1+E_1+A_1)$ is connected. Next, let G be separated from C_1, D_1 and E_1. In this case, if B_1-G is not connected, then $B_1-G=T+S$, where T is separated from S. Suppose that T is separated from C_1 and yet S is not separated from one of D_1 and E_1, then $M-(S+G+D_1+A_1+E_1)=C_1+T$; on the other

1. When $A_1=c_2$ and $B_1=c_3$, we cannot directly say that $C_1+c_2+c_3-E$ is zero.
When A_1 is not equal to c_2 or B_1 is not equal to c_3, it goes without saying that $C_1+c_2+c_3-E$ is zero.

hand, $S+G+D_1+A_1+E_1$ does not entirely contain A, which is a contradiction. Hence, if T is separated from C_1, then S is separated from D_1 and E_1. As S is separated from D_1 and E_1, T is separated from C_1, D_1, and E_1. Hence $B_1=G+S+T$ is separated from D_1; this is impossible. T, as well as S, is not separated from C_1, D_1 and E_1. $M-(T+G+C_1+D_1+E_1)=S+A_1$, which is impossible. Hence B_1 is a closed connected set such that, if E is any connected subset of B_1, B_1-E is also connected. Hence B_1 is one point or a simple closed curve. Suppose that B_1 is a simple closed curve. In this case, let c_1, d_1, e_1 be a point of a set of limit points of C_1, D_1, E_1 respectively, contained in B_1, then $M-(C_1+D_1+E_1+\widehat{c_1d_1e_1})=A_1+(B_1-\widehat{c_1d_1e_1})$, where $\widehat{c_1d_1e_1}$ is a simple arc contained in B_1, which contains c_1, d_1, e_1 and of which extremities are c_1 and e_1. $\widehat{c_1d_1e_1}+C_1+D_1+E_1$ does not entirely contain A, and yet its complement with respect to M consists of two components; which is impossible. Therefore B_1 consists of only one point b_1, and, in the same way, A_1 consists of only one point a_1. $b_1+a_1+C_1$ is a closed and irreducible connected set between b_1 and a_1, that is, $b_1+a_1+C_1$ is a simple arc. $b_1+a_1+D_1$ and $b_1+a_1+E_1$ are also simple arcs. Hence M consists of two simple closed curves which have a simple arc in common. Conversely such M suffices the condition of theorem II.

Let us now consider the case 5) and let $A=A_1+B_1+C_1$, $B=D_1+E_1+F_1$, where A_1, B_1 and C_1 are components of A, and D_1, E_1 and F_1 are components of B. After a short consideration, D_1 is not separated from two of A_1, B_1 and C_1, and is separated from the other one. It is the same for E_1 and F_1. We may suppose that D_1 is not separated from both A_1 and B_1, and is separated from C_1, and that E_1 is not separated from both A_1 and C_1, and is separated from B_1, and that F_1 is not separated from both B_1 and C_1, and is separated from A_1. A set of limit points of D_1, contained in A_1, consists of only one point d_1, and a set of limit points of E_1, contained in A_1, consists of only one point e_1. If d_1 is coincident with e_1, then A_1 consists of only one point d_1. If d_1 is not coincident with e_1, then A_1 is an irreducible connected set between d_1 and e_1. Hence A_1 is one point or a simple arc, as B_1 and C_1 also are. By the same reason as with the case 3), M is a simple closed curve, whereas a simple closed curve does not suffice the condition of theorem II. Therefore case 5) cannot arise.

Now, let us consider the case 1). A set of limit points of B, contained in A, consists of only one point. If it consists of two points c and d, then A is clearly irreducible between c and d, and so is $B+c+d$ also. Hence M is a simple closed curve, which is impossible. If a set of limit points of B, contained in A, contains three points c, d and e, then $M-(B+c+d)=A-(c+d)$, which is connected. $M-[B+\{A-(c+d)\}]=c+d$: this is a contradiction. Hence a set of limit points of B, contained in A, consists of only one point a. $M-a=B+(A-a)$: hence A is coincident with the point a. Suppose a set C, where $C+a$ is connected, such that $M-(C+a)$ consists of two components. If we choose an adequate C, then C may be supposed connected. Suppose it were impossible. Let $M-(a+C)=E+F$, where E is separated from F, then $C=C_1+C_2$, where C_1 is separated from C_2. $E+F+C_1+C_2=B$ is connected, so at least one of E and F, say F, is separated from neither C_1 nor C_2. Suppose that E is separated from C_2, and that it is not separated from C_1. If $C_2=C_2'+C_2''$, where C_2' is separated from C_2'', then C_2' and C_2'' are connected. F is not separated from one of C_2' and C_2'', say C_2', then $M-(C_2'+F+a)=C_2''+(C_1+E)$. On the other hand, $C_2'+F$ is connected, and C_2'' is separated from C_1+E. This is a contradiction. Hence C_2 is connected. In the same way, C_1 is connected. Now as C_1+F is connected, $M-(C_1+F+a)=E+C_2$, which is a contradiction. Hence E, as well as F, is not separated from both C_1 and C_2. In this case, if one of C_1 and C_2 is not connected, then the other is also not connected. Suppose that $C_1=C_1'+C_1''$, where C_1' and C_1'' are two components of C_1, and that $C_2=C_2'+C_2''$, where C_2' and C_2'' are two components of C_2. One of C_1' and C_1'' is separated from E and not from F, and the other is separated from F and not from E. Let C_1' be separated from F, and let C_1'' be separated from E. In the same way, let C_2' be separated from F and let C_2'' be separated from E. $C_1'+C_2'$ is separated from $C_1''+C_2''$ and yet $C_1'+C_2'$ is separated from F, which is impossible. Hence C_1 and C_2 are both connected sets. E, as well as F, is clearly not separated from the point a. Let e_1 be a limit point of E, contained in C_1, then C_1-e_1 is a connected set. If e_1 is a limit point of F, then $E+F+e_1$ is connected, and $M-(E+F+e_1+a)=(C_1-e_1)+C_2$, which is impossible. If e_1 is not a limit point of F, then $(C_1-e_1)+F+C_2+E$ is connected, and $M-\{(C_1-e)+F+C_2+E\}=a+e_1$, which is impossible. Hence there is no limit points of E contained in C_1. In the same way, there is no limit points of C_1 contained in E. This contradicts to the fact that C_1 is not separated from E.

Therefore we may choose an adequate connected set C such that $M-(C+a)=E+F$, where E is separated from F. As $M-C=E+a+F$, which is connected, it is clear that $E+a$ and $F+a$ are both connected. A set of limit points of C contained in F consists of only one point at most. If it contains two points f_1 and f_2, then $F-f_1$, as well as $F-f_2$, is connected. $(F-f_1)+C+E$ is connected, and $M-\{(F-f_1)+C+E\}=f_1+a$, which is a contradiction. In the same way, a set of limit points of C contained in E consists of only one point at most. We put these points in C, then we may consider $M-(C_1+a)=E_1+F_1$, where C_1 is connected, and where E_1 is separated from F_1 and $\bar{C}_1$[1]$(E_1+F_1)=0$. A set of limit points of E_1, contained in C_1, consists of one point. If it contains two points e_1 and e_2, then $(E_1+e_1+e_2)+a+F_1$ is connected. $M-\{(E_1+e_1+e_2)+a+F_1\}=C_1-e_1-e_2$, and this consists of two components at most. Suppose that $C_1-e_1-e_2=C_2+C_3$, where C_2 is separated from C_3, then at least one of $C_2+e_1+C_3$ and $C_2+e_2+C_3$, say $C_2+e_2+C_3$, is connected. $M-(E_1+e_1+e_2+a)=C_2+C_3+F_1$, and so F_1 is not separated from one of C_2 and C_3, say from C_2, then $M-(E_1+e_2+C_2+C_3+F_1)=e_1+a$, which is a contradiction. Hence $C_1-e_1-e_2$ must be connected. If F_1 is separated from $C_1-e_1-e_2$, then F_1 is not separated from one of e_1 and e_2, say e_1, so $M-\{F_1+E_1+e_1+(C_1-e_1-e_2)\}=e_2+a$, which is a contradiction. F_1 is not separated from $C_1-e_1-e_2$, so $M-\{F_1+(C_1-e_1-e_2)+e_1+E_1\}=e_2+a$, which is impossible. Therefore a set of limit points of E_1, contained in C_1, consists of one point e_1. In the same way a set of limit points of F_1, contained in C_1, consists of one point f_1.

$a+E_1+e_1$ is clearly a closed connected set, and let G be any connected subset of $a+E_1+e_1$, where G contains at least one of a and e_1, then $(a+E_1+e_1)-G$ is connected. Hence $a+F_1+e_1$ is a simple arc, of which extremities are a and e_1. In the same way, $a+F_1+f_1$ is a simple arc, of which extremities are a and f_1. If e_1 is coincident with f_1, then C_1+a is a simple arc, of which extremities are a and f_1. Hence M consists of two simple closed curves which have a simple arc in common. This is impossible in case 1). Therefore e_1 is not coincident with f_1. A proper connected subset of C_1+a clearly does not simultaneously contain a, e_1 and f_1. Let H be a connected subset ef C_1+a, which contains two of a, f_1 and e_1 simultaneously and does not contain the other one, then a complement of H with respect to C_1+

1. $\bar{C}_1$ denotes a closure of C_1.

a is connected. Let H be a connected subset of C_1+a, which contains e_1 and does not contain f_1 and a, then $(C_1+a)-H$ consists of two components at most. Let $(C_1+a)-H=G+K$, where G is separated from K, and where $a \in G$ and $f_1 \in K$. $G=M-(K+F_1+H+E_1)$, so G is connected. As any proper connected subset of C_1+a cannot simultaneously contain f_1, a, e_1, K is connected. Let H be a connected subset of C_1+a, which contains f_1 and does not contain e_1 and a, then $(C_1+a)-H$ consists of two components at most. Let H be a connected subset of C_1+a, which contains a and does not contain e_1 and f_1, then $(C_1+a)-H$ consists of two components at most. Let H be a connected subset of C_1+a, which does not contain a, e_1 and f_1, then $(C_1+a)-H$ consists of three components at most.

Therefore C_1+a does not contain a[1] continuum of condensation, so[2] C_1+a is a Jordan continuum. Hence[3] a can be joined with e_1 by a simple arc $\widehat{ae_1}$ contained in C_1+a, where the extremities of $\widehat{ae_1}$ are a and e_1. In the same way a can be joined with f_1 by a simple arc $\widehat{af_1}$ contained in C_1+a, where the extremities of $\widehat{af_1}$ are a and f_1. It is clear that $C_1+a=\widehat{ae_1}+\widehat{af_1}$. $M=\widehat{ae_1}+\widehat{af_1}+(a+E_1+e_1)+(a+F_1+f_1)$, which is impossible in case 1). Therefore 1) cannot arise.

Q. E. D.

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1. A set g is said to be a continuum of condensation of a set M, if g is a closed connected subset of M containing more than one point such that every point of g is a limit point of $M-g$. Suppose a connected set M such that, if N is any division set of M, $M-N$ consists of three components at most, then M does not contain a continnum of condensation. We can prove this easily.

2. Cf. S. Mazurkiewicz "Sur les lignes de Jordan" Fund. Math. 1. p. 176.

3. Cf. Moore "A theorem concerning continuous curves" Bull. Amer. Soc., Vol. 23 (1917). S. Mazurkiewicz "Sur les lignes de Jordan".