<table>
<thead>
<tr>
<th>Title</th>
<th>Algebraic independence of certain power series associated with $d$-adic expansion of real numbers (Analytic Number Theory and Surrounding Areas)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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</tr>
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Kyoto University
Algebraic independence of certain power series associated with \(d\)-adic expansion of real numbers

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1 Introduction.

Let \(\omega > 0\) and let \(d\) be an integer greater than 1. The number \(\omega\) is expressed as a \(d\)-adic expansion

\[
\omega = \sum_{i=-l}^{\infty} \varepsilon_i d^{-i}, \quad l = \max\{\lfloor \log_d \omega \rfloor, 0\}, \quad \varepsilon_i \in \{0, 1, \ldots, d-1\},
\]

where \([x]\) denotes the largest integer not exceeding the real number \(x\). For those \(\omega\) having two ways of expression such as \(2 = 1.9999 \ldots\) (10-adic), we adopt only the left-hand side expression. Then this expansion is uniquely determined. Let

\[
a_k = [\omega d^k] \quad (k = 0, 1, 2, \ldots).
\]

It is clear that

\[
a_k = \sum_{i=-l}^{k} \varepsilon_i d^{k-i},
\]

namely the integer \(a_k\) is expressed as the \(d\)-adic number \(\varepsilon_{-l} \varepsilon_{-l+1} \ldots \varepsilon_{k-1} \varepsilon_k\). Hence we see that the sequence \(\{a_k\}_{k \geq 0}\) satisfies the recurrence formula

\[
a_0 = [\omega], \quad a_k = d a_{k-1} + \varepsilon_k \quad (k = 1, 2, 3, \ldots).
\]

The author [3] proved that the number \(\sum_{k=0}^{\infty} \alpha^{a_k}\) is transcendental for any algebraic number \(\alpha\) with \(0 < |\alpha| < 1\). In this paper we prove the following algebraic independence result. Let \(\omega_1, \ldots, \omega_m > 0\). Define

\[
f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m; \quad d = 2, 3, 4, \ldots). \quad (1)
\]

In what follows, \(\mathbb{Q}\) and \(\mathbb{R}\) denote the sets of rational and real numbers, respectively.
Theorem 1. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

Corollary 1. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the functions $f_{id}(z)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent over the field $\mathbb{C}(z)$ of rational functions.

**Example.** Let 

\[
\begin{align*}
    f_{1,d}(z) &= \sum_{k=0}^{\infty} z^{d^k}, \\
    f_{2,d}(z) &= \sum_{k=0}^{\infty} z^{\sqrt{2}d^k}, \\
    f_{3,d}(z) &= \sum_{k=0}^{\infty} z^{\sqrt{3}d^k}, \\
    f_{4,d}(z) &= \sum_{k=0}^{\infty} z^{\pi d^k} \\
    & \quad \text{for } (d = 2, 3, 4, \ldots).
\end{align*}
\]

For example we have

\[
\begin{align*}
    f_{2,10}(z) &= z + z^{14} + z^{141} + z^{1414} + z^{14142} + z^{141421} + \cdots, \\
    f_{3,10}(z) &= z + z^{17} + z^{173} + z^{1732} + z^{17320} + z^{173205} + \cdots,
\end{align*}
\]

and

\[
    f_{4,10}(z) = z^3 + z^{31} + z^{314} + z^{3141} + z^{31415} + z^{314159} + \cdots.
\]

Then by Theorem 1 the numbers $f_{i,d}(\alpha)$ ($i = 1, \ldots, 4; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$ since the numbers $1, \sqrt{2}, \sqrt{3}, \pi$ are linearly independent over $\mathbb{Q}$.

Theorem 1 is proved by using the method developed from that of Nishioka used for proving the following:

Theorem 2 (Nishioka [2, Theorem 1]). Let

\[
f_d(z) = \sum_{k=0}^{\infty} \sigma_{dk} z^{d^k} \quad (d = 2, 3, 4, \ldots),
\]

where the $\sigma_{dk}$ ($k = 0, 1, 2, \ldots$) are in a finite set of nonzero algebraic numbers for every $d$. Then the numbers $f_d(\alpha)$ ($d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

We further obtain the following, which includes both Theorems 1 and 2.

Theorem 3. Let $\omega_1, \ldots, \omega_m > 0$. Define

\[
f_{id}(z) = \sum_{k=0}^{\infty} \sigma_{idk} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m; d = 2, 3, 4, \ldots),
\]
where the $\sigma_{idk}$ $(k = 0, 1, 2, \ldots)$ are in a finite set of nonzero algebraic numbers for every $i$ and for every $d$. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ $(i = 1, \ldots, m; d = 2, 3, 4, \ldots)$ are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

Theorem 3 implies the following result, which also includes Theorem 1.

**Theorem 4.** Let $\omega_1, \ldots, \omega_m > 0$ and $\eta_1, \ldots, \eta_m \in \mathbb{R}$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega d^k + \eta_i]} \quad (i = 1, \ldots, m; d = 2, 3, 4, \ldots).$$

If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ $(i = 1, \ldots, m; d = 2, 3, 4, \ldots)$ are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

## 2 Lemmas.

We prepare the notation for stating the lemmas. For any algebraic number $\alpha$, we denote by $|\alpha|$ the maximum of the absolute values of the conjugates of $\alpha$ and by $\text{den}(\alpha)$ the smallest positive integer such that $\text{den}(\alpha) \cdot \alpha$ is an algebraic integer and define

$$||\alpha|| = \max\{|\alpha|, \text{den}(\alpha)\}.$$

If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and if $z = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$ with $\mathbb{C}$ the set of complex numbers, we define the transformation $\Omega : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\Omega z = \left( \prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of $n \times n$ matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega^{(k)}_{ij}) \quad \text{and} \quad \Omega^{(k)} z = (z_1^{(k)}, \ldots, z_n^{(k)}).$$

In what follows, $N$ and $N_0$ denote the sets of positive and nonnegative integers, respectively. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in (N_0)^n$, we define $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Let $K$ be an algebraic number field. Let $\{f_{i1}^{(k)}(z)\}_{k \geq 0}, \ldots, \{f_{im}^{(k)}(z)\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \ldots, z_n]]$. Let $\chi = (z_1, \ldots, z_n)$ be the maximal ideal generated by $z_1, \ldots, z_n$ in the ring $K[[z_1, \ldots, z_n]]$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of $k$.

**Lemma 1** (cf. Nishioka [2, Theorem 2]). Assume that

$$f_{i}^{(k)}(z) \rightarrow f_i(z) \quad \text{as} \quad k \rightarrow \infty.$$
with respect to the topology defined by $\chi$ for any $i$ ($1 \leq i \leq m$). Suppose that all the $f_i^{(k)}(z)$ ($k \geq 0$), $f_i(z)$ ($1 \leq i \leq m$) converge in the $n$-polydisc \( \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j| < 1 (1 \leq j \leq n)\}$ If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a point of $K^n$ with $0 < |\alpha_j| < \min\{1, r\}$ ($1 \leq j \leq n$) and if the following three properties are satisfied, then the values $f_1^{(0)}(\alpha), \ldots, f_m^{(0)}(\alpha)$ are algebraically independent.

(I) There exists a sequence $\{\rho_k\}_{k \geq 0}$ of positive numbers such that

$$
\lim_{k \to \infty} \rho_k = \infty, \quad \omega_{ij}^{(k)} \leq c_1 \rho_k, \quad \log |a_j^{(k)}| \leq -c_2 \rho_k.
$$

(II) If we put

$$
f_i^{(0)}(\alpha) = f_i^{(k)}(\Omega^{(k)}\alpha) + b_i^{(k)} \quad (1 \leq i \leq m),
$$

then $b_i^{(k)} \in K$ and

$$
\log \|b_i^{(k)}\| \leq c_3 \rho_k \quad (1 \leq i \leq m).
$$

(III) For any power series $F(z)$ represented as a polynomial in $z_1, \ldots, z_n, f_1(z), \ldots, f_m(z)$ with complex coefficients of the form

$$
F(z) = \sum_{\lambda, \mu = (\mu_1, \ldots, \mu_m)} a_{\lambda, \mu} z_1^{\lambda_1} f_1(z)^{\mu_1} \cdots f_m(z)^{\mu_m},
$$

where $a_{\lambda, \mu}$ are not all zero, there exists a $\lambda_0 \in (\mathbb{N}_0)^n$ such that if $k$ is sufficiently large, then

$$
|F(\Omega^{(k)}\alpha)| \geq c_4 |(\Omega^{(k)}\alpha)^{\lambda_0}|.
$$

Although Theorem 2 of Nishioka [2] requires the assumption that the coefficients of $f_i^{(k)}(z)$ are in a finite set $S \subset K$ for all $i$ and $k$, it can be weakened as in Lemma 1, which is proved by the almost same way as in the proof of Theorem 2 of Nishioka [2].

**Lemma 2** (Nishioka [2]). Let $f(z) = \sum_{\lambda_1, \ldots, \lambda_n} c_{\lambda_1, \ldots, \lambda_n} z_1^{\lambda_1} \cdots z_n^{\lambda_n} \in \mathbb{C}[[z_1, \ldots, z_n]]$ converge around the origin. If $z$ is sufficiently close to the origin, then

$$
\sum_{\lambda \geq H} |c_{\lambda_1, \ldots, \lambda_n}| \cdot |z_1|^{|\lambda_1|} \cdots |z_n|^{|\lambda_n|} \leq \gamma^{H+1} \max_{1 \leq i \leq n} |z_i|^H,
$$

where $\gamma$ is a positive constant depending on $f(z)$.

The following lemma is originally due to Masser [1] and improved by Nishioka [2].

**Lemma 3** (Masser [1], Nishioka [2]). Let $b_1 > \cdots > b_n \geq 2$ be pairwise multiplicatively independent integers. Let $\theta = \log b_1$ and $\theta_j = \theta / \log b_j$ ($1 \leq j \leq n$). Suppose that for each $\alpha$ in a finite set $A$ we are given real numbers $\lambda_{1\alpha}, \ldots, \lambda_{n\alpha}$, not all zero, and define the sequence

$$
S_{\alpha}(k) = \sum_{j=1}^n \lambda_{j\alpha} b_j^{\theta_j k} \quad (k = 0, 1, 2, \ldots).
$$
If \( \{k_l\}_{l \geq 1} \) is an increasing sequence of positive integers with \( k_{l+1} - k_l \) bounded, then there exists a positive number \( \delta \) such that

\[
R(\delta) = \{ k_l \mid \min_{a \in A} |S_a(k_l)| \geq \delta b_l \} = \{ m_l \}_{l \geq 1}, \quad m_l < m_{l+1},
\]
is an infinite set and \( \{m_{l+1} - m_l\}_{l \geq 1} \) is bounded.

Using Lemma 3, we have the following:

**Lemma 4.** Let \( b_1, \ldots, b_n \) be integers as in Lemma 3 and let \( \theta_1, \ldots, \theta_n \) be defined in Lemma 3. Let \( \omega_1, \ldots, \omega_m > 0 \) be linearly independent over \( \mathbb{Q} \). Then there exist an infinite set \( \Lambda \) of positive integers, a sequence \( \{\delta(l)\}_{l \geq 1} \) of positive numbers, and a total order \( \succ \) in \((\mathbb{N}_0)^{mn}\) such that if \( \lambda = (\lambda_{ij}) \succ \mu = (\mu_{ij}) \) with \( |\lambda| = \lambda_{11} + \cdots + \lambda_{mn}, |\mu| = \mu_{11} + \cdots + \mu_{mn} \leq l \), then

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \omega_i b_j^{[\theta_jq]} - \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{ij} \omega_i b_j^{[\theta_jq]} \geq \delta(l) \delta^q
\]

for all sufficiently large \( q \in \Lambda \). Moreover, any subset \( S \) of \((\mathbb{N}_0)^{mn}\) has the minimal element with respect to the total order \( \succ \).

**Lemma 5** (Nishioka [2]). Let \( d \) be an integer greater than 1 and let

\[
f_l(z) = \sum_{h=0}^{\infty} s_h^{(l)} z^{dh} \quad (l = 1, 2, \ldots),
\]

where the coefficients \( s_h^{(l)} \) are nonzero complex numbers. Then \( f_l(z) \) \((l = 1, 2, \ldots)\) are algebraically independent over \( \mathbb{C}(z) \).

### 3 Proof of Theorems 1 and 4.

**Proof of Theorem 1.** Let

\[
D = \{ d \in \mathbb{N} \mid d \neq a^n \ (a, n \in \mathbb{N}, \ n \geq 2) \}.
\]

Then

\[
\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\},
\]

which is a disjoint union since any two distinct elements of \( D \) are multiplicatively independent by the definition of \( D \). Let \( d_1 > \cdots > d_n \) be elements of \( D \) and let \( z = (z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn}) \), where \( z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn} \) are distinct variables. For any \( i \) \((1 \leq i \leq m)\) and for any \( d_j \in D \) \((1 \leq j \leq n)\), we define the sequence \( \{r_k^{(i,j)}\}_{k \geq 0} \) by

\[
r_0^{(i,j)} = 1, \quad r_k^{(i,j)} = [\omega_i d_j^k] \quad (k \geq 1)
\]

(2)
and define

$$f_{ijl0}(z) = \sum_{h=0}^{\infty} \alpha^{r_{ijl0}^{(i,j)} - d_{j}^{lh}} z_{ij}^{d_{j}^{lh}} \quad (1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t).$$

Letting $\alpha = (\alpha, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha)$, we have

$$f_{ijl0}(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_{ijl0}^{(i,j)} - d_{j}^{lh}} + \sum_{h=1}^{\infty} \alpha^{[\omega_{i}d_{j}^{lh}]} = f_{id}^{(i,j)}(\alpha) - \alpha^{[\omega_{i}]} + \alpha,$$

where $f_{id}$ is defined by (1). Hence it suffices to prove the algebraic independency of the values $f_{ijl0}(\alpha)$ $(1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t)$. For the purpose we apply Lemma 1.

Put $b_{j} = d_{j}^{lh}$, $\theta = \log b_{1}$, and $\theta_{j} = \theta / \log b_{j}$ $(1 \leq j \leq n)$. Noting that

$$0 \leq r_{ijl0}^{(i,j)} - r_{ijl0}^{(i,j)} - d_{j}^{lh} \leq d_{j}^{lh} - 1 \quad (1 \leq i \leq m),$$

we put

$$\Sigma_{q} = \left( \alpha^{r_{ijl0}^{(i,j)} - r_{ijl0}^{(i,j)} - d_{j}^{lh}} \right)_{1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t, \ h \geq 0}$$

$$\in \prod_{h=0}^{\infty} \prod_{j=1}^{n} \prod_{l=1}^{t} \{ 1 + \ldots, \alpha^{d_{j}^{lh} - 1} \}^{m}$$

for any $q \in \Lambda$ with the $\Lambda$ defined in Lemma 4. Since the right-hand side is a compact set, there exists a converging subsequence $\{ \Sigma_{q_{k}} \}_{k \geq 1}$ of $\{ \Sigma_{q} \}_{q \in \Lambda}$, where $q_{1}$ will be chosen sufficiently large. Let

$$\lim_{k \rightarrow \infty} \Sigma_{q_{k}} = \left( \alpha^{s_{h}^{(i,j,l)}} \right)_{1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t, \ h \geq 0}$$

and define

$$f_{ijk}(z) = \sum_{h=0}^{\infty} \alpha^{r_{ijk}^{(i,j)} - r_{ijk}^{(i,j)} - d_{j}^{lh}} z_{ij}^{d_{j}^{lh}} \quad (1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t, \ k \geq 1)$$

and

$$f_{ijl}(z) = \sum_{h=0}^{\infty} \alpha^{s_{h}^{(i,j,l)}} z_{ij}^{d_{j}^{lh}} \quad (1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq l \leq t).$$

Then

$$\lim_{k \rightarrow \infty} f_{ijk}(z) = f_{ijl}(z).$$
Define the $mn \times mn$ matrix 

$$\Omega^{(k)} = \text{diag} \left( [\omega_{1} b_{1}^{[\theta_{1} q_{k}]}, \ldots, [\omega_{m} b_{1}^{[\theta_{1} q_{k}]}], \ldots, [\omega_{1} b_{n}^{[\theta_{1} q_{k}]}], \ldots, [\omega_{m} b_{n}^{[\theta_{1} q_{k}]}]] \right).$$

We assert first that $\{\Omega^{(k)}\}_{k \geq 1}$, $\alpha = (\alpha, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha)$, and $\rho_k = b_j^{q_k}$ $(k \geq 1)$ satisfy the assumptions (I) and (II) of Lemma 1. Since $b_1 > \cdots > b_n$, we have

$$b_1^{q_k-1} \leq b_1^{-1} b_1^{q_k} < b_j^{[\theta_{j} q_{k}]} \leq b_1^{q_k}$$

and so

$$\frac{1}{2} \left( \min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k-1} \leq b_1^{q_k-1} - \left( \min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k-1} - 1 < \left[ \omega_i b_j^{[\theta_{j} q_{k}]} \right] \leq b_1^{q_k} \max_{1 \leq i \leq m} \omega_i$$

for any $i$ $(1 \leq i \leq m)$, $j$ $(1 \leq j \leq n)$, and for all $k \geq 1$, if $q_1$ is sufficiently large. Hence the assumption (I) is satisfied.

Let $K = \mathbb{Q}(\alpha)$. Then $f_{ijlk}(z) \in K[[z]]$ $(1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$, $k \geq 0)$ and

$$f_{ijlk}(\Omega^{(k)}\alpha) = \sum_{n=0}^{\infty} \alpha^{r_{ijlk}} = f_{ijlk}(\alpha) - \sum_{n=0}^{(t/|\theta_j q_k|-1)} \alpha^{r_{ijlk}}$$

$$(1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$, $k \geq 1).$$

Since $r_{ijlk}^{(k+1)} > r_{ijlk}^{(k)}$ $(1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all sufficiently large $k$ by the definition, there is a positive constant $C$ such that $\max_{0 \leq h \leq k-1} r_{ijlk}^{(h)} \leq C r_{ijlk}^{(k)}$ $(1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all $k \geq 1$. Hence

$$\log \left| \sum_{h=0}^{(t/|\theta_j q_k|-1)} \alpha^{r_{ijlk}} \right| \leq \log(t/|\theta_j q_k|) + \left( \max_{0 \leq h \leq (t/|\theta_j q_k|-1)} r_{ijlk}^{(h)} \right) \log ||\alpha||$$

$$\leq \left( 1 + C(\max_{1 \leq i \leq m} \omega_i) \log ||\alpha|| \right) \rho_k,$$

and the assumption (II) is satisfied.

Therefore, if the assumption (III) is also satisfied, the proof is completed. Noting that $z_{11}, \ldots, z_{m1}, \ldots, z_{in}, \ldots, z_{mn}$ are distinct variables, we see by Lemma 5 that the functions $f_{ijl}(z)$ $(1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t)$ are algebraically independent over $\mathbb{C}(z_{11}, \ldots, z_{m1}, \ldots, z_{in}, \ldots, z_{mn})$. Let

$$F(z) = \sum_{\mu=(\mu_{ij}), \nu=(\nu_{ij})} a_{\mu, \nu} z^\mu f_{111}^{\nu_{11}} \cdots f_{mnt}^{\nu_{mnt}} = \sum_{\lambda=(\lambda_{ij}) \in (\mathbb{N}_0)^{mn}} c_{\lambda} z^\lambda,$$

where the coefficients $a_{\mu, \nu}$ are not all zero, and let $\lambda_0 = (\lambda_{ij}^{(0)})$ be the minimal element in $(\mathbb{N}_0)^{mn}$ with respect to the total order $\succ$ defined in Lemma 4 among $\lambda$ with $c_\lambda \neq 0$. Let
\( l = 2(\lambda_0 + 1) \left( \frac{\max_{1 \leq i \leq m} \omega_i}{\min_{1 \leq i \leq m} \omega_i} \right) + 1 \). If \( k \) is sufficiently large, then by Lemma 2

\[
\sum_{|\lambda| \geq l} |c_{\lambda}| \cdot |\alpha|^{|\lambda_1| + \ldots + |\lambda_m|} \leq \gamma^{l+1} |\alpha|^{(\max_{1 \leq i \leq m} \omega_i) b^1} \max_{1 \leq i \leq m} \omega_i + 1 \]

\[
\leq \gamma^{l+1} \left( |\alpha|^{|\frac{1}{2} \max_{1 \leq i \leq m} \omega_i} b^k - 1 \right)^l
\]

\[
\leq \gamma^{l+1} |\alpha|^{\omega_{\lambda_0} b^1} (l_0 + 1).
\]

Since

\[
\lambda_{i_1}^{(0)} [\omega_1 b^1] + \ldots + \lambda_{i_m}^{(0)} [\omega_m b^1] + \ldots + \lambda_{i_m}^{(0)} [\omega_m b^1] + \ldots + \lambda_{i_m}^{(0)} [\omega_m b^1] 
\]

\[
\leq |\lambda_0| (\max_{1 \leq i \leq m} \omega_i) b^1,
\]

we have

\[
\frac{\left| \sum_{|\lambda| \geq l} c_{\lambda} (\Omega^{(k)} \alpha) \right|}{|(\Omega^{(k)} \alpha)|} \leq \gamma^{l+1} |\alpha|^{|\lambda_0| (\max_{1 \leq i \leq m} \omega_i) b^1} 
\]

if \( k \) is sufficiently large. If \( |\lambda| < l \) and \( \lambda \neq \lambda_0 \), then by Lemma 4

\[
\frac{|c_{\lambda} (\Omega^{(k)} \alpha)|}{|(\Omega^{(k)} \alpha)|} \leq |c_{\lambda}| \cdot |\alpha|^{|\delta(i) b^1} 
\]

for all sufficiently large \( k \). Therefore

\[
|F(\Omega^{(k)} \alpha)/(\Omega^{(k)} \alpha)_{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty),
\]

which implies (III), and the proof of the theorem is completed.

**Proof of Theorem 4.** Define

\[
g_{id}(\alpha) = \sum_{k=0}^{\infty} \alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k] z^{[\omega_i d^k]}} \quad (i = 1, \ldots, m; \ d = 2, 3, 4, \ldots). 
\]

Then

\[
\alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} \in \{\alpha^{[\eta_i]}, \alpha^{[\eta_i] + 1}\},
\]

since \( 0 \leq [\omega_i d^k + \eta_i] - [\omega_i d^k] - [\eta_i] \leq 1 \) for any \( i, d \), and for all \( k \). By Theorem 3 the numbers \( g_{id}(\alpha) \ (i = 1, \ldots, m; \ d = 2, 3, 4, \ldots) \) are algebraically independent, which implies the theorem.

**References**

