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Algebraic independence of certain power series associated with $d$-adic expansion of real numbers

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1 Introduction.

Let $\omega > 0$ and let $d$ be an integer greater than 1. The number $\omega$ is expressed as a $d$-adic expansion

$$\omega = \sum_{i=-l}^{\infty} \epsilon_i d^{-i}, \quad l = \max\{[\log_d \omega], 0\}, \quad \epsilon_i \in \{0, 1, \ldots, d-1\},$$

where $[x]$ denotes the largest integer not exceeding the real number $x$. For those $\omega$ having two ways of expression such as $2 = 1.9999\ldots$ (10-adic), we adopt only the left-hand side expression. Then this expansion is uniquely determined. Let

$$a_k = [\omega d^k] \quad (k = 0, 1, 2, \ldots).$$

It is clear that

$$a_k = \sum_{i=-l}^{k} \epsilon_i d^{k-i},$$

namely the integer $a_k$ is expressed as the $d$-adic number $\epsilon_{-l} \epsilon_{-l+1} \ldots \epsilon_k$. Hence we see that the sequence $\{a_k\}_{k \geq 0}$ satisfies the recurrence formula

$$a_0 = [\omega], \quad a_k = da_{k-1} + \epsilon_k \quad (k = 1, 2, 3, \ldots).$$

The author [3] proved that the number $\sum_{k=0}^{\infty} \alpha^{a_k}$ is transcendental for any algebraic number $\alpha$ with $0 < |\alpha| < 1$. In this paper we prove the following algebraic independence result. Let $\omega_1, \ldots, \omega_m > 0$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m; \quad d = 2, 3, 4, \ldots).$$

In what follows, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational and real numbers, respectively.
Theorem 1. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

Corollary 1. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the functions $f_{id}(z)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1.

**Example.** Let

\[
\begin{align*}
f_{1,d}(z) &= \sum_{k=0}^{\infty} z^{d^k}, \\
f_{2,d}(z) &= \sum_{k=0}^{\infty} z^{[\sqrt{2}d^k]}, \\
f_{3,d}(z) &= \sum_{k=0}^{\infty} z^{[\sqrt{3}d^k]}, \\
f_{4,d}(z) &= \sum_{k=0}^{\infty} z^{[\pi d^k]} \quad (d = 2, 3, 4, \ldots).
\end{align*}
\]

For example we have

\[
\begin{align*}
f_{2,10}(z) &= z + z^{14} + z^{141} + z^{1414} + z^{14142} + z^{141421} + \cdots, \\
f_{3,10}(z) &= z + z^{17} + z^{173} + z^{1732} + z^{17320} + z^{173205} + \cdots, \\
and
f_{4,10}(z) &= z^3 + z^{31} + z^{314} + z^{3141} + z^{31415} + z^{314159} + \cdots.
\end{align*}
\]

Then by Theorem 1 the numbers $f_{i,d}(\alpha)$ ($i = 1, \ldots, 4; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$ since the numbers $1, \sqrt{2}, \sqrt{3}$, and $\pi$ are linearly independent over $\mathbb{Q}$.

Theorem 1 is proved by using the method developed from that of Nishioka used for proving the following:

**Theorem 2 (Nishioka [2, Theorem 1]).** Let

\[
f_d(z) = \sum_{k=0}^{\infty} \sigma_{dk} z^{d^k} \quad (d = 2, 3, 4, \ldots),
\]

where the $\sigma_{dk}$ ($k = 0, 1, 2, \ldots$) are in a finite set of nonzero algebraic numbers for every $d$. Then the numbers $f_d(\alpha)$ ($d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

We further obtain the following, which includes both Theorems 1 and 2.

**Theorem 3.** Let $\omega_1, \ldots, \omega_m > 0$. Define

\[
f_{id}(z) = \sum_{k=0}^{\infty} \sigma_{idk} z^{[\omega_d d^k]} \quad (i = 1, \ldots, m; d = 2, 3, 4, \ldots),
\]
where the $\sigma_{idk} (k = 0, 1, 2, \ldots)$ are in a finite set of nonzero algebraic numbers for every $i$ and for every $d$. If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

Theorem 3 implies the following result, which also includes Theorem 1.

**Theorem 4.** Let $\omega_1, \ldots, \omega_m > 0$ and $\eta_1, \ldots, \eta_m \in \mathbb{R}$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_{idk}+\eta_k]} \quad (i = 1, \ldots, m; d = 2, 3, 4, \ldots).$$

If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ ($i = 1, \ldots, m; d = 2, 3, 4, \ldots$) are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

## 2 Lemmas.

We prepare the notation for stating the lemmas. For any algebraic number $\alpha$, we denote by $|\alpha|$ the maximum of the absolute values of the conjugates of $\alpha$ and by $\text{den}(\alpha)$ the smallest positive integer such that $\text{den}(\alpha) \cdot \alpha$ is an algebraic integer and define

$$||\alpha|| = \max\{ \sqrt{\alpha}, \text{den}(\alpha) \}.$$

If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and if $z = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$ with $\mathbb{C}$ the set of complex numbers, we define the transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega z = \left( \prod_{j=1}^{n} z_{1}^{\omega_{1j}}, \prod_{j=1}^{n} z_{2}^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_{n}^{\omega_{nj}} \right).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of $n \times n$ matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega_{ij}^{(k)}) \quad \text{and} \quad \Omega^{(k)} z = (z_{1}^{(k)}, \ldots, z_{n}^{(k)}).$$

In what follows, $N$ and $N_0$ denote the sets of positive and nonnegative integers, respectively. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in (N_0)^n$, we define $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Let $K$ be an algebraic number field. Let $\{f_1^{(k)}(z)\}_{k \geq 0}, \ldots, \{f_m^{(k)}(z)\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \ldots, z_n]]$. Let $\chi = (z_1, \ldots, z_n)$ be the maximal ideal generated by $z_1, \ldots, z_n$ in the ring $K[[z_1, \ldots, z_n]]$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of $k$.

**Lemma 1** (cf. Nishioka [2, Theorem 2]). Assume that

$$f_i^{(k)}(z) \to f_i(z) \quad \text{as} \quad k \to \infty$$
with respect to the topology defined by $\chi$ for any $i$ ($1 \leq i \leq m$). Suppose that all the $f^{(k)}_{i}(z)$ ($k \geq 0$), $f_{i}(z)$ ($1 \leq i \leq m$) converge in the $n$-polydisc $\{z = (z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n} \mid |z_{j}| < r (1 \leq j \leq n)\}$. If $\alpha = (\alpha_{1}, \ldots, \alpha_{n})$ is a point of $K^{n}$ with $0 < |\alpha_{j}| < \min\{1, r\}$ ($1 \leq j \leq n$) and if the following three properties are satisfied, then the values $f^{(0)}_{1}(\alpha), \ldots, f^{(0)}_{m}(\alpha)$ are algebraically independent.

(I) There exists a sequence $\{\rho_{k}\}_{k \geq 0}$ of positive numbers such that

$$
\lim_{k \to \infty} \rho_{k} = \infty, \quad \omega^{(k)}_{ij} \leq c_{1}\rho_{k}, \quad \log |\alpha^{(k)}_{j}| \leq -c_{2}\rho_{k}.
$$

(II) If we put

$$
f^{(0)}_{i}(\alpha) = f_{i}^{(k)}(\Omega^{(k)}\alpha) + b^{(k)}_{i} \quad (1 \leq i \leq m),
$$

then $b^{(k)}_{i} \in K$ and

$$
\log ||b^{(k)}_{i}|| \leq c_{3}\rho_{k} \quad (1 \leq i \leq m).
$$

(III) For any power series $F(z)$ represented as a polynomial in $z_{1}, \ldots, z_{n}, f_{1}(z), \ldots, f_{m}(z)$ with complex coefficients of the form

$$
F(z) = \sum_{\lambda, \mu = (\mu_{1}, \ldots, \mu_{m})} a_{\lambda, \mu} z^{\lambda} f_{1}(z)^{\mu_{1}} \cdots f_{m}(z)^{\mu_{m}},
$$

where $a_{\lambda, \mu}$ are not all zero, there exists a $\lambda_{0} \in (\mathbb{N}_{0})^{n}$ such that if $k$ is sufficiently large, then

$$
|F(\Omega^{(k)}\alpha)| \geq c_{4}|(\Omega^{(k)}\alpha)^{\lambda_{0}}|.
$$

Although Theorem 2 of Nishioka [2] requires the assumption that the coefficients of $f^{(k)}_{i}(z)$ are in a finite set $S \subset K$ for all $i$ and $k$, it can be weakened as in Lemma 1, which is proved by the almost same way as in the proof of Theorem 2 of Nishioka [2].

Lemma 2 (Nishioka [2]). Let $f(z) = \sum_{\lambda_{1}, \ldots, \lambda_{n}} c_{\lambda_{1}, \ldots, \lambda_{n}} z_{1}^{\lambda_{1}} \cdots z_{n}^{\lambda_{n}} \in \mathbb{C}[[z_{1}, \ldots, z_{n}]]$ converge around the origin. If $z$ is sufficiently close to the origin, then

$$
\sum_{\lambda \geq H} |c_{\lambda_{1}, \ldots, \lambda_{n}}| \cdot |z_{1}|^{\lambda_{1}} \cdots |z_{n}|^{\lambda_{n}} \leq \gamma^{H+1} \max_{1 \leq i \leq n} |z_{i}|^{H},
$$

where $\gamma$ is a positive constant depending on $f(z)$.

The following lemma is originally due to Masser [1] and improved by Nishioka [2].

Lemma 3 (Masser [1], Nishioka [2]). Let $b_{1} > \cdots > b_{n} \geq 2$ be pairwise multiplicatively independent integers. Let $\theta = \log b_{1}$ and $\theta_{j} = \theta / \log b_{j} (1 \leq j \leq n)$. Suppose that for each $\alpha$ in a finite set $A$ we are given real numbers $\lambda_{1\alpha}, \ldots, \lambda_{n\alpha}$, not all zero, and define the sequence

$$
S_{\alpha}(k) = \sum_{j=1}^{n} \lambda_{j\alpha} b_{j}^{\theta_{j}k} \quad (k = 0, 1, 2, \ldots).
$$
If \( \{k_l\}_{l \geq 1} \) is an increasing sequence of positive integers with \( \{k_{l+1} - k_l\}_{l \geq 1} \) bounded, then there exists a positive number \( \delta \) such that

\[
R(\delta) = \{k_l \mid \min_{a \in A} |S_a(k_l)| \geq \delta b^k_l\} = \{m_l\}_{l \geq 1}, \quad m_l < m_{l+1},
\]

is an infinite set and \( \{m_{l+1} - m_l\}_{l \geq 1} \) is bounded.

Using Lemma 3, we have the following:

**Lemma 4.** Let \( b_1, \ldots, b_n \) be integers as in Lemma 3 and let \( \theta_1, \ldots, \theta_n \) be defined in Lemma 3. Let \( \omega_1, \ldots, \omega_m > 0 \) be linearly independent over \( \mathbb{Q} \). Then there exist a positive number \( \delta \) such that if \( \lambda = (\lambda_{ij}) > \mu = (\mu_{ij}) \) with \( |\lambda| = \lambda_{11} + \cdots + \lambda_{mn}, |\mu| = \mu_{11} + \cdots + \mu_{mn} \leq l \), then

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} [\omega_i b_j^{[\theta_j q]}] - \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{ij} [\omega_i b_j^{[\theta_j q]}] \geq \delta(l) b^q_l
\]

for all sufficiently large \( q \in \Lambda \). Moreover, any subset \( S \) of \((\mathbb{N}_0)^{mn}\) has the minimal element with respect to the total order \( \succ \).

**Lemma 5** (Nishioka [2]). Let \( d \) be an integer greater than 1 and let

\[
f_l(z) = \sum_{k=0}^{\infty} s_h^{(l)} z^{d^k} \quad (l = 1, 2, \ldots),
\]

where the coefficients \( s_h^{(l)} \) are nonzero complex numbers. Then \( f_l(z) \) \( (l = 1, 2, \ldots) \) are algebraically independent over \( \mathbb{C}(z) \).

### 3 Proof of Theorems 1 and 4.

**Proof of Theorem 1.** Let

\[
D = \{d \in \mathbb{N} \mid d \neq a^n \ (a, n \in \mathbb{N}, n \geq 2)\}.
\]

Then

\[
\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\},
\]

which is a disjoint union since any two distinct elements of \( D \) are multiplicatively independent by the definition of \( D \). Let \( d_1 > \cdots > d_n \) be elements of \( D \) and let \( z = (z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn}) \), where \( z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn} \) are distinct variables. For any \( i \) \( (1 \leq i \leq m) \) and for any \( d_j \in D \ (1 \leq j \leq n) \), we define the sequence \( \{r_k^{(i,j)}\}_{k \geq 0} \) by

\[
r_0^{(i,j)} = 1, \quad r_k^{(i,j)} = [\omega_i d_j^k] \quad (k \geq 1)
\]
and define
\[ f_{ijl0}(z) = \sum_{h=0}^{\infty} \alpha^{r^{(i,j)}_h - d^h_{j}^{lhd_{j}^{lh}}} z_{ij}^h \quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t). \]

Letting $\alpha = (\alpha, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha)$, we have
\[ f_{ijl0}(\alpha) = \sum_{h=0}^{\infty} \alpha^{r^{(i,j)}_h} = \alpha + \sum_{h=1}^{\infty} \alpha^{[\omega_i d^h_{j}]} = f_{id}^{(i,j)}(\alpha) - \alpha^{|\omega_i|} + \alpha, \]

where $f_{id}$ is defined by (1). Hence it suffices to prove the algebraic independency of the values $f_{ijl0}(\alpha)$ ($1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t$). For the purpose we apply Lemma 1.

Put $b_j = d^h_{j}^{lhd_{j}^{lh}}$, $\theta = \log b_1$, and $\theta_j = \theta / \log b_j$ ($1 \leq j \leq n$). Noting that
\[ 0 \leq r^{(i,j)}_{ijh+1\theta_j} - r^{(i,j)}_{ijh\theta_j} \leq d^h_{j}^{lh} - 1 \quad (1 \leq i \leq m), \]

we put
\[ \Sigma_q = \left( \alpha^{r^{(i,j)}_{ijh+1\theta_j} - r^{(i,j)}_{ijh\theta_j}} d^h_{j}^{lh} \right)_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, h \geq 0} \in \prod_{h=0}^{\infty} \prod_{j=1}^{n} \prod_{l=1}^{t} \{1, \alpha, \ldots, \alpha^{d^h_{j}^{lh}-1}\} \cdot m \]

for any $q \in \Lambda$ with the $\Lambda$ defined in Lemma 4. Since the right-hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k \geq 1}$ of $\{\Sigma_q\}_{q \in \Lambda}$, where $q_1$ will be chosen sufficiently large. Let
\[ \lim_{k \to \infty} \Sigma_{q_k} = \left( \alpha^{s^{(i,j,l)}_h} \right)_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, h \geq 0} \]

and define
\[ f_{ijlk}(z) = \sum_{h=0}^{\infty} \alpha^{r^{(i,j,l)}_{ijh+1\theta_{q_k}\theta_j} - r^{(i,j,l)}_{ijh\theta_{q_k}\theta_j}} d^h_{j}^{lh} z_{ij}^h \quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t, k \geq 1) \]

and
\[ f_{ijl}(z) = \sum_{h=0}^{\infty} \alpha^{s^{(i,j,l)}_h} d^h_{j}^{lh} z_{ij}^h \quad (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq t). \]

Then
\[ \lim_{k \to \infty} f_{ijlk}(z) = f_{ijl}(z). \]
Define the $mn \times mn$ matrix
\[ \Omega^{(k)} = \text{diag} \left( [\omega_1 b_1^{[\theta_1 q_k]}], \ldots, [\omega_m b_m^{[\theta_m q_k]}] \right). \]

We assert first that \( \{ \Omega^{(k)} \}_{k \geq 1} \) and $\alpha = (\alpha, \ldots, \alpha, \ldots, \alpha)$, and $\rho_k = b_1^{q_k}$ ($k \geq 1$) satisfy the assumptions (I) and (II) of Lemma 1. Since $b_1 > \cdots > b_n$, we have
\[ b_1^{q_k - 1} \leq b_j^{-1} b_1^{q_k} < b_j^{[\theta_j q_k]} \leq b_1^{q_k} \]
and so
\[ \frac{1}{2} \left( \min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k - 1} \leq \left( \min_{1 \leq i \leq m} \omega_i \right) b_1^{q_k - 1} - 1 < [\omega_i b_j^{[\theta_j q_k]}] \leq b_1^{q_k} \]
for any $i$ ($1 \leq i \leq m$), $j$ ($1 \leq j \leq n$), and for all $k \geq 1$, if $q_1$ is sufficiently large. Hence the assumption (I) is satisfied.

Let $K = \mathbb{Q}(\alpha)$. Then $f_{ijkl}(z) \in K[[z]]$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$, $k \geq 0$) and
\[ f_{ijkl}(\Omega^{(k)}\alpha) = \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha_{hl}^{(i,j)} = f_{ijkl}(\alpha) - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha_{hl}^{(i,j)} \]
(1 $\leq i \leq m$, 1 $\leq j \leq n$, 1 $\leq l \leq t$, $k \geq 1$).

Since $r_{(i,j)}^{(l+1)} > r_{(i,j)}^{(l)}$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all sufficiently large $k$ by the definition, there is a positive constant $C$ such that $\max_{0 \leq h \leq k-1} r_{(i,j)}^{(l,h)} \leq C r_{(i,j)}^{(l)}$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) for all $k \geq 1$. Hence
\[ \log \left| - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha_{hl}^{(i,j)} \right| \leq \log(t!/l)[\theta_j q_k] + \left( \max_{0 \leq h \leq (t!/l)[\theta_j q_k]-1} r_{(i,j)}^{(l,h)} \right) \log \| \alpha \| \]
\[ \leq \left( 1 + C(\max_{1 \leq i \leq m} \omega_i) \log \| \alpha \| \right) \rho_k, \]
and the assumption (II) is satisfied.

Therefore, if the assumption (III) is also satisfied, the proof is completed. Noting that $z_{11}, \ldots, z_{m1}, \ldots, z_{m1}, \ldots, z_{mn}$ are distinct variables, we see by Lemma 5 that the functions $f_{ijl}(z)$ ($1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq l \leq t$) are algebraically independent over $\mathbb{C}(z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn})$. Let
\[ F(z) = \sum_{\mu=(\mu_{ij}), \nu=(\nu_{ij})} a_{\mu,\nu} z^{\mu} f_{111}^{\nu_11} \cdots f_{mnt}^{\nu_{mnt}} = \sum_{\lambda=(\lambda_{ij}) \in (\mathbb{N}_0)^{mn}} c_{\lambda} z^{\lambda}, \]
where the coefficients $a_{\mu,\nu}$ are not all zero, and let $\lambda_0 = (\lambda_{ij}^{(0)})$ be the minimal element in $(\mathbb{N}_0)^{mn}$ with respect to the total order $\succ$ defined in Lemma 4 among $\lambda$ with $c_{\lambda} \neq 0$. Let
\[ l = 2(|\lambda_0| + 1) \left( \frac{\max_{1 \leq i \leq m} \omega_i}{\min_{1 \leq i \leq m} \omega_i} \right) + 1 \] 

If \( k \) is sufficiently large, then by Lemma 2
\[
\sum_{|\lambda| \geq l} |c_\lambda| \cdot |\alpha|^{\lambda_1[\omega_1 b_1^{[\theta_{1\|k}]}] \cdots |\alpha|^{\lambda_m[\omega_m b_m^{[\theta_{m\|k}]}] \cdots |\alpha|^{\lambda_m[\omega_m b_m^{[\theta_{m\|k}]}]}
\leq \gamma^{l+1} \left( |\alpha| \left( \min_{1 \leq i \leq m} \omega_i \right) b_1^{qk} \right)^l
\leq \gamma^{l+1} \left( |\alpha| \left( \max_{1 \leq i \leq m} \omega_i \right) b_1^{qk} \right)^{(|\lambda_0|+1)}.
\]

Since
\[
|\lambda_{11}[\omega_1 b_1^{[\theta_{1\|k}]}] + \cdots + |\lambda_{m1}[\omega_m b_1^{[\theta_{1\|k}]}] + \cdots + |\lambda_{1n}[\omega_1 b_n^{[\theta_{1\|k}]}] + \cdots + |\lambda_{mn}[\omega_m b_n^{[\theta_{1\|k}]}] \leq |\lambda_0| \left( \max_{1 \leq i \leq m} \omega_i \right) b_1^{qk},
\]
we have
\[
\frac{|\sum_{|\lambda| \geq l} c_\lambda (\Omega^{(k)}\alpha)^{\lambda}|}{|\Omega^{(k)}\alpha|^{\lambda_0}} \leq \gamma^{l+1} \left( |\alpha| \left( \max_{1 \leq i \leq m} \omega_i \right) b_1^{qk} \right)^{(|\lambda_0|+1)}
\]
if \( k \) is sufficiently large. If \( |\lambda| < l \) and \( \lambda \neq \lambda_0 \), then by Lemma 4
\[
\frac{|c_\lambda (\Omega^{(k)}\alpha)^{\lambda}|}{|\Omega^{(k)}\alpha|^{\lambda_0}} \leq |c_\lambda| \cdot |\alpha|^{\delta(k) b_1^{qk}}
\]
for all sufficiently large \( k \). Therefore
\[
|F(\Omega^{(k)}\alpha)/(\Omega^{(k)}\alpha)^{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty),
\]
which implies (III), and the proof of the theorem is completed.

**Proof of Theorem 4.** Define
\[
g_{id}(z) = \sum_{k=0}^{\infty} \alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k] + \eta_i} (i = 1, \ldots, m; \quad d = 2, 3, 4, \ldots).
\]
Then
\[
\alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} \in \{ \alpha^{[\eta_i]}, \alpha^{[\eta_i]+1} \},
\]
since \( 0 \leq [\omega_i d^k + \eta_i] - [\omega_i d^k] - [\eta_i] \leq 1 \) for any \( i, d \), and for all \( k \). By Theorem 3 the numbers \( g_{id}(\alpha) (i = 1, \ldots, m; \quad d = 2, 3, 4, \ldots) \) are algebraically independent, which implies the theorem.

**References**

