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On a Generalization of the Projective Deformation

By

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1. Let $\boldsymbol{R}_2$ be a two-dimensional space with a projective connexion $\Gamma^{\alpha}_{\beta i}\,dx^i$ ($\alpha, \beta=0, 1, 2$; $\Gamma^{\alpha}_{0i}=\delta^{\alpha}_{i}$). The triangle of reference $[C, C_1, C_2]$ in a projective plane $\boldsymbol{S}_2$ which moves along the development of a curve L drawn on $\boldsymbol{R}_2$ is defined by

$$(1\cdot1)\qquad \begin{cases} dC=dx^iC_i\,, \\ dC_i=\Gamma^{\alpha}_{ij}\,dx^j\,C_\alpha \qquad (i=1, 2;\ C_0\equiv C)\,. \end{cases}$$

Consider another space $\boldsymbol{R}'_2$ with a projective connexion. The spaces $\boldsymbol{R}_2$ and $\boldsymbol{R}'_2$ will be said to be projectively deformable to each orther, if we can realize a one to one correspondence between the points of these spaces in the following manner.

Let M and M' be any two corresponding points on $\boldsymbol{R}_2$ and $\boldsymbol{R}'_2$ respectively. Develop any homologous curves L and L' passing through M and M' respectively, taking a point C on $\boldsymbol{S}_2$ as the common image of M and M', and giving at this point a common initial position to the moving frames of reference which move along the developments of L and L' respectively. Take, in the vicinity of C, two homologous points P and Q on these developments. Then the *écart* $[PQ]$ is an infinitesimal of the third order at least with regard to the *écart* $[CP]$.

Choose the coordinates of the generating points of $\boldsymbol{R}_2$ and $\boldsymbol{R}'_2$ in such way that the corresponding points are determined by the same values of the coordinates x^i. Let $\Gamma'^{\alpha}_{\beta i}dx^i$ be the connexion for $\boldsymbol{R}'_2$. Then, the condition of the projective deformability of $\boldsymbol{R}'_2$ to $\boldsymbol{R}_2$ is written

$$\Gamma'^{l}_{ij}+\Gamma'^{l}_{ji}=\Gamma^{l}_{ij}+\Gamma^{l}_{ji} \qquad (l, i, j=1, 2).$$

We see therefore that if two spaces $\boldsymbol{R}_2$ and $\boldsymbol{R}'_2$ are projectively deformable to each other, *the geodesic lines are in correspondence*, and that if two spaces with normal projective connexion are projectively deformable to each other, they are equivalent.

We can make the given connexion $\Gamma^{\beta}_{\alpha i}\, dx^i$ symmetry without changing the geodesic lines. We shall suppose hereafter that this transformation is previously made, namely, we have

$$\Gamma^{\alpha}_{ij}=\Gamma^{\alpha}_{ji} \qquad (\alpha=0, 1, 2\,;\ i, j=1, 2).$$

2. The space R_2 can be plunged into a four-dimensional projective space S_4 in such way that the space R_2 becomes a ruled surface V_2 in S_4, and that the osculating planes at a point P of this surface to various curves on the surface passing through P are not all contained in a three-dimensional projective space. We can take any ∞^1 geodesic lines of R_2 as the lines becoming the generating lines of V_2. If we designate these geodesic lines, the surface V_2 depends on eight arbitrary functions of an argument[1].

Choose the coordinates x^1, x^2 so that the ∞^1 geodesic lines in question are given by x^2=const. Then we have $\Gamma^2_{11}=0$. Consider, in S_4, a moving frame of reference $[A, A_1, A_2, A_3, A_4]$ which depends on the parameters x^1, x^2.

Let

$$dA_\sigma=H^{\tau}_{\sigma i}A_\tau \qquad (\sigma=0, 1, 2, 3, 4)$$

be the system of equations defining the movement of this frame[2]. If the vertex A describes the surface V_2 as the frame moves, we can make

$$H^{\beta}_{\alpha i}=\Gamma^{\beta}_{\alpha i} \quad (\alpha, \beta=0, 1, 2), \qquad H^{p}_{0i}=H^{p}_{11}=0 \quad (p=3, 4),$$

$$H^{\beta}_{31}=R^{\beta}_{112} \quad (\beta=0, 1, 2), \qquad H^{3}_{31}=\Gamma^{1}_{11}-\Gamma^{2}_{12}, \quad H^{4}_{31}=0,$$

$$H^{\beta}_{41}=R^{\beta}_{212}+H^{\beta}_{32}, \quad H^{3}_{41}=\Gamma^{1}_{12}-\Gamma^{2}_{22}+H^{3}_{32}, \quad H^{4}_{41}=\Gamma^{2}_{12}+H^{4}_{32},$$

while H^{τ}_{pi} are determined by the normal system of partial differential equations:

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1. J. Kanitani: 'Sur l'espace à connexion projective majorante, II.', *Jap. Jour. of Math.*

2. We denote by α, β, γ the surffixes which take the values 0, 1, 2; by h, i, j, k, l, m those which take the values 1, 2; by σ, τ, ρ those which take the values 0, 1, 2, 3, 4; by p, q, r those which take the values 3, 4.

$$(2\cdot1)\qquad \frac{\partial H_{\nu 2}^{\tau}}{\partial x^1}=\frac{\partial H_{\nu 1}^{\tau}}{\partial x^2}+H_{\nu 1}^{\rho}H_{\rho 2}^{\tau}-H_{\nu 2}^{\rho}H_{\rho 1}^{\tau}.$$

The tangent planes of the ruled surface V_2 along the generating line AA_1 determine the hyper-plane $\Pi=AA_1A_2A_3$. We shall call it tangent hyper-plane along AA_1 of V_2. The position of this tangent hyper-plane depends only on the parameter x^2. When this parameter varies, the hyper-plane Π is enveloped by a developable hyper-surface of which the characteristic is the plane determined by the points

$$A,\ A_1,\ A_3-H_{32}^{4}A_2.$$

The surface V_2 is therfore contained in this developable hyper-surface, and the characteristic is the tangent plane of V_2 at the point

$$E=A_1-H_{41}^{4}A.$$

We shall name it *stationary point*, for it remains unchanged when the parameter x^1 varies. The three hyper-planes Π, $d\Pi$, $d^2\Pi$ intersect at a line l passing through E, while the four hyper-planes Π, $d\Pi$, $d^2\Pi$, $d^3\Pi$ intersect at a point E_1 on l, which will be called *cuspidal point*. When the point A describes the ruled surface V_2, the cuspidal point E_1 describes a curve which will be called edge of regression, and consequently the tangent l to this curve at E_1 describes a ruled surface U_2. The osculating plane to the edge of regression at E_1 is the characteristic of the envelope of the tangent hyper-plane Π of V_2 along the generating line AA_1. This plane touches U_2 along l and touches V_2 at the stationary point E.

3. Consider a particular point A on V_2. We may suppose that $x^1=0$, $x^2=0$ for this point. The vertices A_1, A_2 of the frame $[A, A_1, A_2, A_3, A_4]$ lie on the tangent plane of V_2 at the point A. If we develop any curve drawn from a point C on V_2 into the tangent plane S_2 at C, defining the moving frame $[C, C_1, C_2]$ by means of the equations $(1\cdot1)$, and giving initial values so that $C_\alpha=A_\alpha(\alpha=0, 1, 2)$ for $x^1=x^2=0$, the points $C_\alpha+\alpha C_\alpha$ are obtained by projecting the points $A_\alpha+dA_\alpha$ from the line A_3A_4. We shall name this line *axis of projection*.

Remarking that we can determine the integrals $H_{\nu 2}^{\tau}$ of the system of partial differential equations $(2\cdot1)$ by giving arbitrarily the functions $\varphi_{\nu 2}^{\tau}(x^2)$ to which $H_{\nu 2}^{\tau}$ are reduced for $x^1=0$, we can prove the following proposition.

The space $\boldsymbol{R}_2$ can be so plunged in S_4 that it becomes a ruled surface V_2 passing through a point A arbitrarily given in S_4 and, at this point, the characteristic of the envelope of the tangent hyper-plane along the generating line intersects the axis of projection in S_4, the point of intersection becoming the cuspidal point.

Let $\boldsymbol{R}'_2$ be a space which is projectively deformable to $\boldsymbol{R}_2$. Take the coordinates so that the corresponding points are determined by the same values of the parameters. Plunge also $\boldsymbol{R}'_2$ into S_4 in the above said way, giving to the moving frames a common initial position for $x^1=x^2=0$.

Then the homologous stationary points E and E′ are in coincidence for $x^1=x^2=0$, *but the tangents at these points to the loci of the stationary points are different. We may choose the space* $\boldsymbol{R}'_2$ *so that the tangent to the locus on* V_2' *becomes an arbitrarily given line through E on the plane* AA_1A_3.