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Iteration of Elliptic Functions

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1. Let $F(z)$ be an elliptic function with periods ω, ω' and its fundamental parallelogram be D. In the following we shall consider the iteration of $F(z)$. Let $W=m\omega+n\omega'$ be any period, then we consider the point ζ such as

$$F(\zeta)=\zeta+W. \tag{1}$$

The existence of such a point may easily be known provided $|W|$ is great. For let ν be the order of $F(z)$, then in D there are just ν poles. Hence we may suppose $F(z)$ bounded on the contour of D, (or else a little modification of it is sufficient). We consider the integral along the contour

$$\mathfrak{J}=\frac{1}{2\pi i}\oint\frac{F'(z)-1}{F(z)-z-W}dz\,. \tag{2}$$

$|W|$ being sufficiently great, the denominator shall not be zero at any point of the contour. If we make $|W|$ great, $|\mathfrak{J}|$ becomes however small. Therefore in D there are ν points ζ such as (1).

The set of points ζ for all W is countable and isolated, the poles in D being the points of accumulation.

Now about ζ we have the expansion

$$F(z)=\zeta+W+s(z-\zeta)+\cdots,\qquad s=F'(\zeta).$$

For n-th iteration we have

$$F_n(z)=\zeta+W+s^n(z-\zeta)+\cdots,$$

where

$$F_n(\zeta)=\zeta+W,$$

$$F'_n(\zeta)=F'(F_{n-1}(\zeta))F'(F_{n-2}(\zeta))\cdots F'(F(\zeta))F'(\zeta).$$

If $|s|<1$, then since in a circle ($\mathfrak{S}$) sufficiently small, $\left|\frac{F(z)-\zeta-W}{z-\zeta}\right|$ becomes less than a number $\sigma<1$, we have

$$|F_n(z)-\zeta-W|<\sigma^n|z-\zeta|,\quad n=1, 2, \ldots.$$

ζ is (the equivalent point in D of) an attractive point and all points of the circle ($\mathfrak{S}$) by the iteration converge to ζ. All points of a certain domain (Δ) containing ($\mathfrak{S}$) in it shall have the same property. This is the immediate domain of attraction.

2. Next let us consider the attractive cycle. If

$$F(\zeta)=\zeta_1+W,\quad F(\zeta_1)=\zeta_2+W,\ \ldots\ldots,\ F(\zeta_{m-1})=\zeta+W, \qquad (3)$$

where W mean only certain periods, then $\zeta, \zeta_1, \zeta_2, \ldots, \zeta_{m-1}$ are the cycle corresponding to ζ of order m. ζ shall be found from the equation

$$F_m(\zeta)=\zeta+W. \qquad (4)$$

The existence of such point may be proved quite in the same way as in the case $m=1$. For that

$$F_m(z)=F_{m-1}(F(z))=\ldots=F(F_{m-1}(z))$$

shall be bounded, z must not pass through the poles of $F(z)$. There are ν poles in D. Let α be one of them and β be such that $F(\beta)=\alpha$. There are such ν points in D. Not only this, we must also avoid such points β in D: $F(\beta)=\alpha+W$. There are ν points β in D. Now varying W, we know that for a pole, there are always countably infinite number of β. Thus for ν poles in D, there are countably infinite number of β in D. Again for each β, we must take care of points γ such that $F(\gamma)=\beta+W$. Continuing this, we have a countable set E of avoidable points in D. These points are isolated but converge to the poles in D. This is clear, since if $|W|$ be bounded, the points $\beta, \gamma, \ldots$ are finite in number, therefore, for that they are infinite in number, it must be such that $W\to\infty$, so that $F\to\infty$. Therefore we may suppose that any point of E is not on the contour of D, so that $F_n(z)$ is bounded on the contour.

Now we seclude the poles of $F(z)$ by small circles, then there remain in D a finite number of points of E. We also seclude all the essential singularities of $F_n(z)$ by circles. Let D' be the remaining domain. There remain only a finite number of poles of $F_m(z)$ in D'. Now consider the integral as (2),

$$\mathfrak{J}=\frac{1}{2\pi i}\oint\frac{F_m'(z)-1}{F_m(z)-z-W}dz, \qquad (5)$$

we may conclude (4). As it is known, we have

$$F_m'(\zeta)=F'(\zeta_{m-1})F_{m-1}'(\zeta)=\dots=F'(\zeta_{m-1})F'(\zeta_{m-2})F'(\zeta),$$
$$F_m'(\zeta)=F_m'(\zeta_1)=\dots=F_m'(\zeta_{m-1}).$$

Hence in attractive case, for all z in a circle ($\mathfrak{S}$) about ζ, $F_{pm}(z)$ tend to ζ for $p\to\infty$; $F_{pm+1}(z)$ to ζ_1+W; ...; $F_{pm+m-1}(z)$ to $\zeta_{m-1}+W$, which we say $F_n(z)$ converges uniformly to the cycle.

3. Writing $F(\xi)=z+W$, where ζ is an attractive point such as $F(\zeta)=\zeta+W$, we consider the inverse function about $\xi=\zeta$. If ζ be not a branch point, then

$$\xi=F_{-1}(z+W)=\zeta+\bar{s}(z-\zeta)+\cdots,$$

where $s\bar{s}=1$, hence $|\bar{s}|>1$. If a circle $|z-\zeta|<r$ be in the domain of attraction, then its transformed domain by $\xi=F_{-1}(z+W)$ contains it. Therefore again we may consider the inverse function

$$\xi=F_{-1}(F_{-1}(z+W)+W),\ \dots.$$

If these functions are holomorphic in the circle then they must be bounded in it. Hence they form a normal family. This contradicts the fact that ζ is a repulsive point of these functions. Thus about the attractive point there must be a branch point of the inverse function of $F(\xi)=z+W$, that is $F'(\zeta)=0$. Since $F'(\xi)$ is elliptic, such points are finite in number in the fundamental parallelogram; hence the number of the attractive points is finite in the parallelogram.

Same consideration will be possible for the cycle.