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On a Property of the Conjugate Net in Connexion with Generalized Projective Deformation

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Consider a non-ruled surface S in a three-dimensional projective space. Associate to a generic point E of S a moving frame of reference $[E, E_1, E_2, E_3]$, of which the vertices E_1, E_2 lie on the tangent plane of S at E and the vertex E_3 is outside of this tangent plane. The movement of this frame is defined by the system of the equations of the form:

$$dE = dw_o^0 E + dw_o^i E_i,$$

$$dE_i = dw_i^0 E + dw_i^l E_l + dw_i^3 E_3, \quad (i = 1, 2)$$

$$dE_3 = dw_3^0 E + dw_3^l E_l + dw_3^3 E_3.$$

Choosing the vertices E_1, E_2 properly, we can make

$$dw_o^0 = 0, \quad dw_o^1 = dx^1, \quad dw_o^2 = dx^2,$$

where x^1, x^2 are the parameters on which the generic point E depends. Take the point E_3 so that the frame $[E, E_1, E_2, E_3]$ becomes a frame of Darboux, namely, the surface S referred to the frame $[E, E_1, E_2, E_3]$ is defined by the equation:

$$z^3 = \frac{1}{2} H_{ij} z^i z^j - \frac{1}{3} K_{ijl} z^i z^j z^l + \cdots\cdots\cdots\cdots,$$

where

$$H_{ij} z^i z^j = 0$$

gives the asymptotic tangents, while

$$K_{ijl} z^i z^j z^l = 0$$

gives the tangents of Darboux so that

$$H^{ij}K_{ijl}=0.$$

Then, by choosing properly the common factor of the coordinates of the point E_3, we can make

$$dw_i^3 = H_{ij}dx^j, \quad dw_3^3 = 0,$$
$$dw_i^l = (K_{ij}^l + \Gamma_{ij}^l)\,dw^j,$$

where Γ_{ijl} ($i, j, l = 1, 2$) are Christoffel's symbols with respect to the quadric from $H_{ij}dx^i dx^j$.

We now regard the surface S as the space $\boldsymbol{R}_2$ with the connexion dw_α^β ($\alpha, \beta = 0, 1, 2$). If we develop any curve drawn from a point A on S into the tangent plane at A, defining the moving frame $[A, A_1, A_2]$ by means of the equations

$$dA_\alpha = dw_\alpha^\beta A_\beta \quad (A_0 \equiv A\,;\, \alpha = 0, 1, 2),$$

and giving to A, A_1, A_2 the same initial values as those of E, E_1, E_2, then the infinitely small variations of E, E_1, E_2 are obtained by projecting the homologous variations of A, A_1, A_2 from the point E_3.

If we take as the point E_3 another point satisfying the above-mentioned condition, namely, a point on the line EE_3, *the space corresponding to the new center of projection is projectively deformable*[1] *to the original space.*

Now suppose that the parameter curves upon S form a conjugate net. Then, by choosing the common factor of the coordinates of the point E, we can make

$$K_{11}^2 + \Gamma_{11}^2 = 0,$$

so that the point E_3 becomes a point on the plane osculating at E to the curve $x^2 =$ const passing through this point.

Thus, we get

$$dE = dx^i E_i,$$
$$dE_1 = dw_1^0 E + dw_1^1 E_1 + (K_{12}^2 + \Gamma_{12}^2)\,dx^2 E_2.$$

The Laplace transform on the line $E\,E_1$ is expressed by

$$E_1 - (K_{12}^2 + \Gamma_{12}^2)\,E.$$

1. J. Kanitani: On a generalisation of the projective deformation, This Memoirs, Vol. **25**, p. 23.

We have, therefore, the following proposition.

Let S be a non-ruled surface in a three-dimensional projective space, and A be a generic point of this surface. Consider a family of non-asymptotic curves upon S. Denote by l the tangent at A to the curve L passing through A and belonging to this family. Associate to S a projective connexion obtained by the projection from a point on the osculating plane of L at A. Then, the curves under consideration become geodesic lines for the space $\boldsymbol{R}_2$ with the connexion thus determined and consequently, this space can be so plunged[2] into a four-dimensional projective space that it becomes a ruled surface whose generating lines correspond to these geodesic lines and that, at a particular position of A, the characteristic of the envelope of the tangent hyperplane along generating line intersects the axis of projection. *The stationary point corresponding to this position of A, coincides with the Laplace transform of A on l with respect to the conjugate net of which the family under consideration forms a part.*

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2. J. Kanitani: *loc. cit.*