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On a Method of Plunging of R_2 with Symmetric Projective Connexion into a Projective Space S_4.

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1. According to the result of a previous paper[1], a two-dimensional space $\boldsymbol{R}_2$ with a symmetric projective connexion $d\omega_\alpha^\beta = \Gamma_{\alpha i}^\beta dx^i$ $(\alpha, \beta = 0, 1, 2)$ can be so plunged into a four-dimensional projective space $\boldsymbol{S}_4$ that the space $\boldsymbol{R}_2$ becomes a ruled surface in $\boldsymbol{S}_4$. In this paper we investigate another way of plunging of $\boldsymbol{R}_2$ into $\boldsymbol{S}_4$.

By assumption we have

$$\Gamma_{ij}^\beta = \Gamma_{ji}^\beta \quad (\beta = 0, 1, 2, i, j = 1, 2),$$

and we may suppose always that

$$d\omega_0^0 = 0, \quad d\omega_0^1 = dx^1, \quad d\omega_0^2 = dx^2.$$

Consider a dominant projective connexion $d\Omega_\sigma^\tau = H_{\sigma i}^{\tau} dx^i (\sigma, \tau = 0, 1, 2, 3, 4)$ subjected to the condition[2]

$$\left.\begin{aligned} &H_{\alpha i}^\beta = \Gamma_{\alpha i}^\beta \ (\alpha, \beta, = 0, 1, 2\,;\ i = 1, 2), \\ &H_{0 i}^p = 0 \ (p = 3, 4\,;\ i = 1, 2) \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} H_{11}^3 &= 1, & H_{11}^4 &= 0, \\ H_{12}^3 &= 0, & H_{12}^4 &= 0, \\ H_{22}^3 &= 0, & H_{22}^4 &= 1. \end{aligned}\right\} \tag{2}$$

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1. J. Kanitani: Sur l'espace à connexion projective majorante, II, Jap. Jour. of Math. **19**, 395 (1948).

2. We denote by α, β, γ the suffixes which take the values 0, 1, 2; by $h, i, j, , k, l, m$ those which take the values 1, 2; by σ, τ, ρ those which take the values 0, 1, 2, 3, 4; by p, q, r those which take the values 3, 4.

We shall now prove that H_{pi}^{τ} ($\tau = 0, 1, 2, 3, 4$; $i = 1, 2$; $p = 3, 4$) can be so determined that the tensor of curvature $T_{\sigma ij}^{\tau}$ of the connexion $d\Omega_{\sigma}^{\tau}$ becomes zero.

2. Put

$$R_{\sigma ij}^{\tau} = \frac{\partial H_{\sigma i}^{\tau}}{\partial x^j} - \frac{\partial H_{\sigma j}^{\tau}}{\partial x^i} + H_{\sigma i}^{\gamma} H_{\gamma j}^{\tau} - H_{\sigma j}^{\gamma} H_{\gamma i}^{\tau} ,$$

$$(\sigma, \tau = 0, 2, 3, 4 \; ; \; i, j = 1, 2)$$

so that[3] for $\alpha, \beta = 0, 1, 2$ $R_{\alpha ij}^{\beta}$ becomes the tensor of curvature of the given connexion $d\omega_{\alpha}^{\beta}$ while we have, when $\sigma \leqq 2$, $\tau > 2$,

$$R_{012}^{3} = R_{012}^{4} = 0, \qquad R_{h12}^{3} = -\Gamma_{h2}^{1}, \qquad R_{h11}^{4} = \Gamma_{h1}^{2} \qquad (h = 1, 2) .$$

The equation $T_{\alpha 12}^{\tau} = 0$ can be written as[4]

$$R_{\alpha 12}^{\tau} + \Gamma_{\alpha 1}^{q} \Gamma_{q2}^{\tau} - \Gamma_{\alpha 2}^{q} \Gamma_{q1}^{\tau} = 0.$$

We get therefore

$$\Gamma_{32}^{\tau} = R_{121}^{\tau}, \qquad \Gamma_{41}^{\tau} = R_{212}^{\tau} \qquad (\tau = 0, 1, 2, 3\, 4) .$$

If we substitute these values of Γ_{32}^{τ}, Γ_{41}^{τ}, the equation $\Gamma_{p12}^{\sigma} = 0$ becomes

$$\left.\begin{aligned} \frac{\partial \Gamma_{31}^{\sigma}}{\partial x^2} &= \frac{\partial R_{121}^{\sigma}}{\partial x^1} - \Gamma_{31}^{\alpha} \Gamma_{\alpha 2}^{\sigma} - \Gamma_{31}^{3} R_{121}^{\sigma} - \Gamma_{31}^{4} \Gamma_{42}^{\sigma} \\ &\qquad + R_{121}^{\alpha} \Gamma_{\alpha 1}^{\sigma} + R_{121}^{3} \Gamma_{31}^{\sigma} + R_{121}^{4} R_{212}^{\sigma} , \\ \frac{\partial \Gamma_{42}^{\sigma}}{\partial x^1} &= \frac{\partial R_{212}^{\sigma}}{\partial x^2} + R_{212}^{\alpha} \Gamma_{\alpha 2}^{\sigma} + R_{212}^{3} R_{121}^{\sigma} + R_{212}^{4} \Gamma_{42}^{\sigma} \\ &\qquad - \Gamma_{42}^{\alpha} \Gamma_{\alpha 1}^{\sigma} - \Gamma_{42}^{3} \Gamma_{31}^{\sigma} - \Gamma_{42}^{4} R_{212}^{\sigma} \; . \end{aligned}\right\} \qquad (3)$$

Thus the functions Γ_{31}^{σ}, Γ_{42}^{σ} are determined as solutions of this system of simultaneous partial differential equations. Now suppose that the functions Γ_{ij}^{α} ($\alpha = 0, 1, 2$; $i, j = 1, 2$) are analytic in the vicinity of x_0^i. Let $\Phi_{31}^{\sigma}(x^1)$ and $\Phi_{42}^{\sigma}(x^2)$ be arbitrary functions which are analytic in the vicinity of x_0^1 and x_0^2 respectively. Then the system (3) admits one and only one system of solutions Γ_{31}^{σ}, Γ_{42}^{σ} ($\sigma = 0, 1, 2, 3, 4$) such that Γ_{31}^{σ} are reduced to $\Phi_{31}^{\sigma}(x^1)$ when $x^2 = x_0^2$, while Γ_{42}^{σ} are reduced to $\Phi_{42}^{\sigma}(x^2)$ when $x^1 = x_0^1$.

3. By the said convention, the summation is made from 0 to 2 with respect to γ.
4. The summation is made from 3 to 4 with respect to q.

This can be proved by the method of dominant function. Put

$$x = x^1 - x_0^1, \qquad y = y^1 - y_0^1,$$

$$Z^\sigma(x, y) = \Gamma_{31}^{\sigma}(x + x_0^1, y + x_0^2) - \Phi_{31}^{\sigma}(x + x_0^1),$$

$$W^\sigma(x, y) = \Gamma_{42}^{\sigma}(x + x_0^1, y + x_0^2) - \Phi_{42}^{\sigma}(y + x_0^2).$$

Then the system of equations (3) can be written in the form:

$$\frac{\partial Z^\sigma}{\partial y} = F_1(x, y, Z^\sigma, W^\tau), \qquad \frac{\partial W^\sigma}{\partial x} = F_2(x, y, Z^\sigma, W^\tau). \tag{4}$$

$$(\sigma, \tau = 0, 1, 2, 3, 4)$$

Making use of these equations and

$$Z^\sigma(x, 0) = W^\sigma(0, y) = 0,$$

we can evaluate the successive partial derivatives of Z^σ, W^σ for $x = y = 0$ in only one way. To verify the convergency of Taylor's series with the coefficients thus determined, we consider the system of equations

$$\frac{\partial Z^\sigma}{\partial y} = \frac{M}{1 - \frac{1}{\rho}(x + y + Z^0 + W^0 + \cdots + Z^5 + W^5)},$$

$$\frac{\partial W^\sigma}{\partial x} = \frac{M}{1 - \frac{1}{\rho}(x + y + Z^0 + W^0 + \cdots + Z^5 + W^5)},$$

where ρ, M are the constants such that

$$|F_1| < M, \ |F_2| < M \text{ for } |x| + |y| + |Z^0| + \cdots\cdots + |W^5| < \rho.$$

If we put

$$Z^\sigma = W^\sigma = f(x + y), \qquad x + y = u,$$

this system are reduced to the ordinary differential equation

$$\frac{df}{du} = \frac{M}{1 - \frac{1}{\rho}(u + 10f)},$$

which admits an analytical solution satisfying the condition $f(0) = 0$.

3. Conversely, suppose that, in a four-dimensional projective space S_4, there exists a surface V_2 whose generic point A_0 are defined by

$$dA_\sigma = H^\tau_{\sigma i}\, dx^i\, A_\tau ,$$

where $H^\beta_{\alpha i}$, H^p_{0i} $(\alpha, \beta = 0, 1, 2\,;\; i = 1, 2\,;\; p = 3, 4)$ are subjected to the condition (1) and

$$|\, A_0\ A_1\ A_2\ A_3\ A_4 \,| \neq 0 .$$

Let B^σ_τ be algebraic complement of A^τ_σ, divided by this determinant. We can take B^σ_0, B^τ_1, B^σ_2, B^σ_3, B^σ_4 as coordinates of the hyper-plane

$$B^\sigma = A_0 \cdots A_{\sigma-1}\ A_{\sigma+1} \cdots A_4 ,$$

and we have

$$dB^\sigma = -\, H^\sigma_{\tau i}\, dx^i\, B^\tau .$$

A hyper-plane passing through the tangent plane $A_0 A_1 A_2$ of V_2 at A is expressed by $\nu_p B^p$. When we associate such hyper-plane to every point A_0 of V_2, the characteristic of the envelope of the hyper-planes along a curve C on V_2 is the intersection of the hyper-planes $\nu_p B^p$, $d\,(\nu_p B^p)$. This intersection will coincide with the tangent plane $A_0 A_1 A_2$ if and only if

$$\nu_p H^p_{1i}\, dx^i = 0, \qquad \nu_p H^p_{2i}\, dx^i = 0.$$

Eliminating ν_3, ν_4 from these equations we get

$$\begin{vmatrix} H^3_{11}\, dx^1 + H^3_{12}\, dx^2 & H^3_{21}\, dx^1 + H^3_{22}\, dx^2 \\ H^4_{11}\, dx^1 + H^4_{12}\, dx^2 & H^4_{21}\, dx^1 + H^4_{22}\, dx^2 \end{vmatrix} = 0 .$$

If the osculating planes at a point A_0 of the curves lying on V_2 and passing through A_0 are not all contained in a hyper-plane, there exist two families of curves satisfying this equation. We shall call them characteristic curves of V_2. Taking these curves as parameter curves and performing a transformation of the form

$$A_3 = \mu^3_3\, A^1_3 + \mu^4_3\, A^1_4 , \qquad A_4 = \mu^3_4\, A^1_3 + \mu^4_4\, A^1_4 ,$$

we can deduce the equations (2).

Hence we have the following proposition.

A space $\boldsymbol{R}_2$ with a symmetric projective connexion can be so plunged into $\boldsymbol{S}_4$ that $\boldsymbol{R}_2$ becomes a surface V_2 and a net of curves arbitrarily given upon $\boldsymbol{R}_2$ becomes the net of characteristic curves of V_2.

Since $\Phi^{\sigma}_{31}(x^1)$ and $\Phi^{\sigma}_{42}(x^2)$ to which Γ^{σ}_{31}, Γ^{σ}_{42} are reduced for $x^2 = x^2_0$ and $x^1 = x^1_0$ respectively may be arbitrary functions, we can say as follows:

Let C, C' be any two geodesics on $\boldsymbol{R}_2$ passing through a point and having distinct tangents at this point. Then the space $\boldsymbol{R}_2$ can be so plunged into $\boldsymbol{S}_4$ that the curves C, C' become conics on V_2.