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Linear Conditions at a Point

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In algebraic geometry the "behavior", at a point O, of a curve or a surface is a fundamental notion. By this word is mostly meant the infinitely near base condition, and this is always a set of linear conditions for the coefficients of the equation of a curve or a surface. In this report we shall deal with the linear conditions in general; therefore the infinitely near base conditions are included in our "linear conditions."

Let O be a point of an n-dimensional affine space, $Ox_1x_2\cdots x_n$ a coordinate system in the space, and $R = K[x_1, x_2, \cdots, x_n]$ the ring of polynomials of the current coordinates $x_1, x_2, \cdots, x_n$ with coefficients from the field K.

A homogeneous condition for the coefficients $a_{i_1i_2\cdots i_n}$ of a polynomial $f(x) = \sum a_{i_1i_2\cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ can be regarded as a condition for the corresponding hypersurface f; $f(x)=0$. While, conditions for the coefficients of a polynomial, if its degree is not assigned, are rather attached to the coefficients of a formal power series than to those of a polynomial. Hence, we shall mainly consider the ring $\overline{R} = K[[x_1, x_2, \cdots, x_n]]$ of formal power series of $x_1, x_2, \cdots, x_n$ with coefficients from K.

A formal power series $f(x) = \sum \alpha_{i_1i_2\cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ can be represented by an infinite sequence $(\alpha) = (\alpha_{i_1i_2\cdots i_n})$ (i's running throughout all non-negative integers), where $\alpha_{i_1i_2\cdots i_n}$ precedes $\alpha_{j_1j_2\cdots j_n}$ when the corresponding n-tuples $(i_1, i_2, \cdots, i_n)$ and $(j_1, j_2, \cdots, j_n)$ are such that, either 1^0) $i_1 + i_2 + \cdots + i_n < j_1 + j_2 + \cdots + j_n$, or 2^0) $i_1 + i_2 + \cdots + i_n = j_1 + j_2 + \cdots + j_n$ and the first non-zero difference among $i_k - j_k$ $(k = 1, 2, \cdots, n)$ is positive; in either case we call $(i_1, i_2, \cdots, i_n) < (j_1, j_2, \cdots, j_n)$.

A linear homogeneous equation

$$\sum_{(i_1, i_2, \cdots, i_n) \leq (p_1, p_2, \cdots, p_n)} a_{i_1i_2\cdots i_n} \xi_{i_1i_2\cdots i_n} = 0 \quad (a_{i_1i_2\cdots i_n} \in K) \tag{1}$$

in the unknown $\xi_{i_1i_2\cdots i_n}$ (where the n-tuple $(p_1, p_2, \cdots, p_n)$ is an arbitrary

n-tuple, designating the upper limit of the range of the summation) is called a *linear condition* for the power series of $\overline{R}$.

Let $f(x) = \sum \alpha_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ be a series of $\overline{R}$. If $\xi_{i_1 i_2 \cdots i_n} = \alpha_{i_1 i_2 \cdots i_n}$ satisfies the equation (1), i. e. if

$$\sum_{(i_1, i_2, \cdots, i_n) \leqq (p_1, p_2, \cdots, p_n)} a_{i_1 i_2 \cdots i_n} \alpha_{i_1 i_2 \cdots i_n} = 0 , \qquad (2)$$

then we say that *the series* $f(x)$ *satisfies linear condition* (1).

Let (E):

$$\sum_{(i_1, i_2, \cdots, i_n) \leqq (p_1^{(\lambda)}, p_2^{(\lambda)}, \cdots, p_n^{(\lambda)})} a_{i_1 i_2 \cdots i_n} \xi_{i_1 i_2 \cdots i_n} = 0$$

(λ running throughout a set of indices) be a set of linear conditions. If the series $f(x)$ satisfies every condition of the set (E), we say that $f(x)$ *satisfies the linear condition* (E).

The linear condition (1) can be represented by an infinite sequence $(a) = (a_{i_1 i_2 \cdots i_n})$, where we mean $a_{i_1 i_2 \cdots i_n} = 0$ for every n-tuple $(i_1, i_2, \cdots \cdots, i_n) > (p_1, p_2, \cdots, p_n)$. That the series (α) satisfies the linear condition (a), we shall denote it by the notation $(a) \perp (\alpha)$. Moreover, that the series (α) satisfies the linear condition (E), we shall denote it by $(E) \perp (\alpha)$.

Thus, the ring $\overline{R}$ can be regarded as a K-module of all infinite dimensional vectors (α): and the totality $\mathfrak{M}$ of all linear conditions as a K-module of all infinite dimensional vectors (a), where all but a finite number of components $a_{i_1 i_2 \cdots i_n}$ are zeros.

1. Module-theoretical preliminaries. Given a subset (E) of $\mathfrak{M}$, we denote by $\overline{\mathfrak{A}}_{(E)}$ the totality of all series (α) such that $(E) \perp (\alpha)$; $\overline{\mathfrak{A}}_{(E)}$ is the set of all series satisfying the linear condition (E), and it is a submodule of $\overline{R}$. If M is the submodule of $\mathfrak{M}$ which is generated by (E), then we see that $\overline{\mathfrak{A}}_M = \overline{\mathfrak{A}}_{(E)}$. Therefore, in what follows, we shall deal only with submodules of $\mathfrak{M}$.

Now, let $\overline{\mathfrak{A}}$ be a submodule of $\overline{R}$. If any enumerable set $(\alpha^{(\lambda)}_{i_1 i_2 \cdots i_n})$ $(\lambda = 1, 2, \cdots)$ of series of $\overline{R}$ such that there exists, for every n-tuple $(i_1, i_2, \cdots, i_n)$, a positive integer $r(i_1, i_2, \cdots, i_n)$ which has the property that $\alpha^{(\lambda)}_{i_1 i_2 \cdots i_n} = 0$ for all $\lambda > r(i_1, i_2, \cdots, i_n)$, has the sum-series

$$\sum_{\lambda=1}^{\infty} (\alpha^{(\lambda)}_{i_1 i_2 \cdots i_n}) = \Big(\sum_{\lambda=1}^{r(i_1, i_2, \cdots, i_n)} \alpha^{(\lambda)}_{i_1 i_2 \cdots i_n}\Big)$$

in $\overline{\mathfrak{A}}$, then we call $\overline{\mathfrak{A}}$ a *permissible submodule* of $\overline{R}$. We can prove that, if M is any submodule of $\mathfrak{M}$, $\overline{\mathfrak{A}}_M$ is a permissible submodule of $\overline{R}$.

Given a submodule $\overline{\mathfrak{A}}$ of $\overline{R}$, we denote by $M_{\overline{\mathfrak{A}}}$ the totality of all linear conditions (a) of $\mathfrak{M}$ such that $(a) \perp \mathfrak{A}$, i. e., (a) is satisfied by every series of $\mathfrak{A}$. We see that $M_{\mathfrak{A}}$ is a submodule of $\mathfrak{M}$.

It can be proved, by somewhat long but easy discussions which we have no space to describe here, that the correspondences $M \rightarrow \mathfrak{A}_M$ and $\mathfrak{A} \rightarrow M_{\mathfrak{A}}$, between the set of all submodules M of $\mathfrak{M}$ and the set of all *permissible* submodules $\mathfrak{A}$ of $\overline{R}$, are mutually inverses, and moreover that the correspondence is dual isomorphism between the lattice of all submodules of $\mathfrak{M}$ and the lattice of all *permissible* submodules of $\overline{R}$. The orderings in both of these lattices are the usual set-theoretical inclusion-orders.

2. Linear condition at a point. We are to be concerned with the linear condition which can be named "linear condition at the point O." For such condition it is natural to hope that, if a hypersurface f satisfies the condition, then the product hypersurface $f \cdot g$ also satisfies it for any hypersurface g. If a submodule M of $\mathfrak{M}$ is such that, if a series $f(x)$ of $\overline{R}$ satisfies the condition M, so does the product $f(x) \cdot g(x)$ for any series $g(x)$ of $\overline{R}$, then M is called a *linear condition at* O or a *D-submodule* of $\mathfrak{M}$.

We now enter into the stage where properties of $\overline{R}$, as a ring, are to be investigated. Hereafter, it is convenient to use Macauley's notation concerning the *inverse system*.[1] Namely, we represent a linear condition $(a_{i_1 i_2 \cdots i_n})$ by a polynomial:

$$\varphi(x^{-1}) = \varphi(x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}) = \sum a_{i_1 i_2 \cdots i_n} x_1^{-i_1} x_2^{-i_2} \cdots x_n^{-i_n}.$$

Thus, $\mathfrak{M}$ can be regarded as a K-module of all such polynomials.

By the definition, M is a D-submodule of $\mathfrak{M}$ if and only if $\mathfrak{A}_M$ is an ideal in $\overline{R}$. Since $\mathfrak{A}_M$ is a permissible submodule, this is the same thing as that $x_r \cdot f(x)$ $(r = 1, 2, \cdots, n)$ are all contained in $\mathfrak{A}_M$, provided $f(x)$ is contained in $\mathfrak{A}_M$. From this we can deduce that M is a D-submodule of $\mathfrak{M}$ if and only if the *x_r-derivative*

$$x_r \cdot \varphi(x^{-1}) = \sum a_{i_1 i_2 \cdots i_n} x_1^{-i_1} x_2^{-i_2} \cdots x_{r-1}^{-i_{r-1}} x_r^{-(i_r - 1)} x_{r+1}^{-i_{r+1}} \cdots x_n^{-i_n}$$

(where every term of at least one positive exponent is omitted out) is contained in M, provided $\varphi(x^{-1})$ is a polynomial in M.

Let us call any ideal of $\overline{R}$ which is a permissible submodule of $\overline{R}$ a *permissible ideal*. It can be proved that the lattice of all D-submodules M of $\mathfrak{M}$ and the lattice of all permissible ideals of $\overline{R}$ are sublattices

1. Macauley, The Algebraic Theory of Modular Systems, 1916.

of the lattice of submodules of $\mathfrak{M}$ and the lattice of permissible submodules of $\overline{R}$ respectively, and that they are dual isomorphic by the correspondence $M \rightarrow \overline{\mathfrak{A}}_M$, $\overline{\mathfrak{A}} \rightarrow M_{\overline{\mathfrak{A}}}$.

We see that every 0-dimensional and every $(n-1)$-dimensional ideal of $\overline{R}$ is permissible ideal. Furthermore, it can be proved that $\overline{\mathfrak{A}}_M$ is 0-dimensional ideal if and only if the corresponding D-submodule M of $\mathfrak{M}$ has a finite module-basis.

3. 0-Dimensional ideals. If we desire to use properly the word "linear condition at O," it is natural to limit our consideration only on those D-submodules M of $\mathfrak{M}$ such that $\overline{\mathfrak{A}}_M$ are 0-dimensional ideals, i. e. on D-submodules M which have finite module-bases.

Let M be a D-submodule of $\mathfrak{M}$ which has a finite module-basis and $\overline{\mathfrak{A}}$ the corresponding 0-dimensional ideal. It can be proved, on account of the dual isomorphism, that since $\overline{\mathfrak{A}}$ decomposes into an intersection of a finite number s of irreducible ideals $\overline{\mathfrak{A}}_1, \overline{\mathfrak{A}}_2, \cdots, \overline{\mathfrak{A}}_s$, M decomposes into a module-sum of s additively irreducible D-submodules $M_1, M_2, \cdots M_s$ which correspond to $\overline{\mathfrak{A}}_1, \overline{\mathfrak{A}}_2, \cdots, \overline{\mathfrak{A}}_s$ respectively, although the decomposition is not uniquely determined.

We see that every monomial $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ can be used as an operator of successive derivation of any polynomial $\varphi(x^{-1})$ of $\mathfrak{M}$, and moreover, that every element of $\overline{R}$ can be regarded as such operator. We call the result $g(x) \cdot \varphi(x^{-1})$ of operating $g(x)$ upon $\varphi(x^{-1})$ the $g(x)$-derivative of $\varphi(x^{-1})$.

Let $\varphi(x^{-1})$ be a polynomial of $\mathfrak{M}$. We denote by $(\varphi(x^{-1}))$ the totality of all $g(x)$-derivatives of $\varphi(x^{-1})$, $g(x)$ running throughout all series of $\overline{R}$. It is a D-submodule of $\mathfrak{M}$, and we call it a *principal D-submodule* of $\mathfrak{M}$. It can be proved that a D-submodule of $\mathfrak{M}$ is (additively) irreducible if and only if it is principal.

By these descriptions we see that the principal D-submodules of $\mathfrak{M}$ are the prototypes of the linear conditions at O and that they correspond to the irreducible 0-dimensional ideals of $\overline{R}$ belonging to the unique maximal prime ideal $\overline{\mathfrak{P}} = (x_1, x_2, \cdots, x_n)$.

If we take intersections with R of the ideals of $\overline{R}$, we can conclude that there is a reversible one-to-one correspondence between D-submodules, which has finite module-basis, and primary ideals of R belonging to the prime ideal $\mathfrak{P} = (x_1, x_2, \cdots, x_n)$ (which corresponds to the point O in question); and that, in this correspondence, to a principal D-submodule corresponds an irreducible ideal belong to $\mathfrak{P}$.