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A Wave-mechanical Treatment of the Electron Oscillation

By

Isao Takahashi

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1. Introduction

The wave-mechanical treatment of an electron oscillation was made by Schuster[1] for the first time. His case where plane parallel electrodes were assumed was that the cathode and the anode were situated symmetrically relative to the grid and no anode d. c. voltage was applied. He adopted cylindrical functions for the exact wave functions, and obtained graphically the differences of the energies (proper values), in order to derive the frequency formulas in accordance with classical ones for his case, but he did not consider any transition probability.

The author[2] tried to solve the same problem for the case where the distances of the cathode and the anode from the grid were not necessarily equal to each other, the anode d. c. voltage being applied.

The author solved the wave equation with the aid of Jeffrey's approximation method (W. K. B. method) and derived the equation to determine the proper values. The differences of the energies (proper values) were shown to give the frequency formulas for B. K. wave and dwarf waves. In his calculation, the author limited the wave function within the space between the cathode and the plane where potential energy vanishes between the grid and the anode. But, an alternative is that the space is extended beyond the cathode and the anode with the potential gradient preserved, as is shown with the dotted lines in Fig. 2 and we make use of the quantum condition expressed by the integration between two classical reflecting points in W. K. B. method. The results

1. Ann. d. Phy. **76**, 1930.
2. The first part of "On the Electron Oscillation of Barkhausen and Kurz" lectured by W. Nakayama and its abstract in Nippon Sugaku-Butsurigakkaishi, **5** (1931), 214.

of the proper values in both assumptions, however, agree approximately, as is shown below, when the energy of electron is sufficiently small against eV_g, where V_g is the grid d. c. voltage. In this paper, the transition probabilities are calculated subject to the former assumption.

2. Unperturbed functions

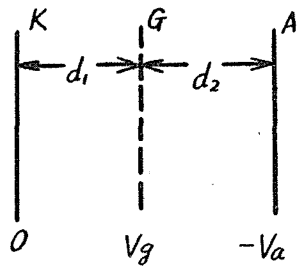

Fig. 1

K : cathode
G : grid
A : anode
d : distances between electrodes
V : d. c. voltages

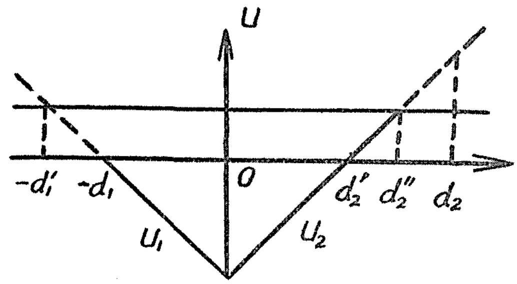

Fig. 2

U : potential energy of an electron
W : proper value
$-d_1, d_2$: points of zero potential
$-d_1', d_2''$: classical reflecting points

Since we assume that the potential is linear between electrodes as shown in Fig. 2, the wave functions satisfy the following equations:

$$\frac{d^2\psi_1}{dx^2} + \kappa^2(W - U_1)\psi_1 = 0 \quad \text{between K and G};$$

$$\frac{d^2\psi_2}{dx^2} + \kappa^2(W - U_2)\psi_2 = 0 \quad \text{between G and A},$$

where $\kappa^2 = \dfrac{8\pi^2 m}{h^2} \sim 1.67 \times 10^{28}$ and

$$W - U_1 = W + eV_g + a_1 x = W' + a_1 x\,,$$
$$W - U_2 = W + eV_g - a_2 x = W' - a_2 x$$

with $W' = W + eV_g\,,\quad a_1 = eV_g/d_1\,,\quad a_2 = e(V_g + V_a)/d_2\,.$

The approximate solutions between $-d_1$ and d_2' are

$$\psi_{\substack{1\\2}} = (W' \pm a_{\substack{1\\2}} x)^{-\frac{1}{4}} \left\{ A_{\substack{1\\2}} e^{i\kappa\int_0^x (W' \pm a_{\substack{1\\2}} x)^{1/2} dx} + B_{\substack{1\\2}} e^{-i\kappa\int_0^x (W' \pm a_{\substack{1\\2}} x)^{1/2} dx} \right\}, \quad (1)$$

where suffixes 1 and 2 correspond respectively to + and − of the double signs.

At $x=0$, for the derivatives of ψ,

$$\psi'_{\substack{1\\2}} \doteqdot i\kappa\,(W' \pm a_{\substack{1\\2}}x)^{-\frac{1}{4}}\left\{A_{\substack{1\\2}}e^{i\kappa\int_0^x} - B_{\substack{1\\2}}e^{-i\kappa\int_0^x}\right\} \qquad (2)$$

holds good, because of the extraordinarily large value of κ.

The boundary conditions of ψ_1 and ψ_2 are

$$\psi_1 = 0 \qquad \text{at} \quad x = -d_1\,,$$

$$\psi_1 = \psi_2\,, \quad \psi'_1 = \psi'_2 \quad \text{at} \quad x = 0\,,$$

$$\psi_2 = 0 \qquad \text{at} \quad x = d'_2\,,$$

which give

$$A_1 = A_2 = A\,, \quad B_1 = B_2 = B\,,$$

and

$$\frac{2\kappa}{3}\,(W'^{3/2} - W^{3/2})\left(\frac{1}{a_1} + \frac{1}{a_2}\right) = n\pi\,, \qquad (3)$$

n being any positive integer. Corresponding to the large value of the left-hand side of (3), n must be very large. The quantum condition in W. K. B. method:

$$\int_{-d_1'}^{d_2''} \kappa\sqrt{W-U}\,dx = \left(n + \frac{1}{2}\right)n$$

gives, instead of (3),

$$\frac{2\kappa}{3}\,W'^{3/2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) = \left(n + \frac{1}{2}\right)\pi\,. \qquad (4)$$

For large values of n and $eV_g \gg W$, the above two expressions (3) and (4) agree with each other. The difference of energy can be given as differential ΔW in the equation:

$$\kappa\left(\frac{1}{a_1} + \frac{1}{a_2}\right)(W'^{1/2} - W^{1/2})\,\Delta W = \Delta n\pi\,, \qquad (5)$$

for $\Delta n \ll n$ which is very large.

When we apply the frequency condition:

$$\nu = \frac{\Delta W}{h}\,,$$

the classical wave length formulas for B. K. wave and dwarf waves can be derived from (5). Thus,

$$\lambda = \frac{c}{\nu} = c\sqrt{\frac{8m}{cV_g}}\,\frac{(d_1 + d_2)V_g + d_1 V_g}{V_g + V_a}\,\frac{1}{\Delta n}\,.$$

Equation (1) is transformed so as to adapt the boundary conditions and the subsequent calculation, as follows:

$$\psi_{n\frac{1}{2}} = C_{n\frac{1}{2}}\,(W_n' \pm a_1 x)^{-\frac{1}{4}}\left\{e^{i\kappa I n\frac{1}{2}} - e^{-i\kappa I n\frac{1}{2}}\right\}\,, \tag{6}$$

with $C_{n1} = C_{n2}\cos n\pi$, where

$$I_{n\frac{1}{2}} = \pm\frac{2\kappa}{3a_{\frac{1}{2}}}\left\{(W_n' \pm a_{\frac{1}{2}}\,x)^{3/2} - W_n'^{3/2}\right\}\,. \tag{7}$$

In order to show the orthogonality of ψ_n's and to normalize ψ_n, we put

$$(W_n' \pm a_{\frac{1}{2}}\,x)^{3/2} = y\,. \tag{8}$$

Then, we get

$$x = \pm\frac{1}{a_{\frac{1}{2}}}(y^{2/3} - y_0^{2/3})\,,\qquad dx = \pm\frac{2}{3a_{\frac{1}{2}}}\,y^{-1/3}dy\,,$$

with $y_1 = W_n'^{3/2}$, $y_0 = W_n'$ and

$$I_{n\frac{1}{2}} = \pm\,\alpha_{\frac{1}{2}}(y - y_1)\,,\quad I_{m\frac{1}{2}} = I_{n\frac{1}{2}} - \Delta I_{\frac{1}{2}}\,,\quad \Delta I_{\frac{1}{2}} = \pm\,\beta_{\frac{1}{2}}(y^{1/3} - y_1^{1/3})\,,$$

where $\alpha_{\frac{1}{2}} = \dfrac{2\kappa}{3a_{\frac{1}{2}}}$, $\beta_{\frac{1}{2}} = \dfrac{\kappa}{a_{\frac{1}{2}}}\Delta W$.

Hence, putting $(W_n'^{3/2} \pm a_{\frac{1}{2}}\,x)^{-1/4} = (y^{2/3} - \Delta W)^{-1/4} \doteqdot y^{-1/6}$, we have

$$\int_{-d_1}^{d_2'}\psi_n^*\,\psi_m\,dx$$

$$= C_{n1}^*\,C_{m1}\left(+\frac{2}{3a_1}\right)\int_{y_1}^{y_0} y^{-\frac{2}{3}}(e^{-i\Delta I_1} - e^{-2iIn_1}\,e^{i\Delta I_1} - e^{2iIn_1}\,e^{-i\Delta I_1} + e^{i\Delta I_1})\,dy$$

$$+ C_{n2}^*\,C_{m2}\left(-\frac{2}{3a_2}\right)\int_{y_0}^{y_1} y^{-\frac{2}{3}}(e^{-i\Delta I_2} - e^{-2iIn_2}\,e^{i\Delta I_2} - e^{2iIn_2}\,e^{-i\Delta I_2} + e^{i\Delta I_2})\,dy\,.$$

Since, in this integral, $e^{\pm 2iI_n}$ oscillate very rapidly, we may take their mean value which is equal to 0. Therefore

$$\int_{-d_1}^{d_2'} \psi_n^* \psi_m \, dx = C_{n1}^* C_{m1} \left(+ \frac{4}{3a_1}\right) \int_{y_1}^{y_0} y^{-2/3} \cos(\Delta I_1) \, dy$$
$$+ C_{n2}^* C_{m2} \left(- \frac{4}{3a_2}\right) \int_{y_0}^{y_1} y^{-2/3} \cos(\Delta I_2) \, dy \,. \qquad (9)$$

If we make the transformation:

$$y^{\frac{1}{3}} = z \,, \qquad (10)$$

then we have

$$\int_{-d_1}^{d_2'} \psi_n^* \psi_m \, dx = C_{n1}^* C_{m1} \left(+ \frac{4}{a_1}\right) \int_{z_1}^{z_0} \cos\left\{+ \beta_1 (z - z_1)\right\} dz$$
$$+ C_{n2}^* C_{n2} \left(- \frac{4}{a_2}\right) \int_{z_0}^{z_1} \cos\left\{- \beta_2 (z - z_1)\right\} dz \,,$$

with $z_1 = y_1^{1/3} = W_n^{1/2}$, $z_0 = y_0^{1/3} = W_n'^{1/2}$.

According as $\Delta n = n - m$ is even or odd,

$$\left.\begin{aligned} \sin\{\beta_1 (z_0 - z_1)\} &= \mp \sin\{\beta_2 (z_0 - z_1)\} \,, \\ \cos\{\beta_1 (z_0 - z_1)\} &= \pm \cos\{\beta_2 (z_0 - z_1)\} \,, \end{aligned}\right\} \qquad (11)$$

because of (5) expressed in the form $\beta_1 (z_0 - z_1) + \beta_2 (z_0 - z_1) = \Delta n \pi$, and moreover

$$C_{n1}^* C_{m1} = \pm \, C_{n2}^* C_{m2} \,, \qquad (12)$$

corresponding to Δn being even or odd.

Thus, we have finally

$$\int_{-d_1}^{d_2} \psi_n^* \psi_m \, dx = 0 \,, \qquad \text{for } n \neq m \qquad (13)$$

and the condition of normalization $\int_{-d_1}^{d_2} \psi_n^* \psi_n \, dx = 1$ gives

$$|C_{n2}|^2 = \frac{1}{4 (W_n'^{1/2} - W_n^{1/2}) \left(\dfrac{1}{a_1} + \dfrac{1}{a_2}\right)} \,. \qquad (14)$$

3. Transition

We can calculate the transition probabilities between two energy states as in the case of an atom. We assume that the oscillating field is an oscillating statical field, but we have to take into account that the oscillating field between K and G is generally not equal to that between G and A, both in amplitude and in sense. Therefore, we cannot take formally $\left| \int_{-d_1}^{d_2} \psi_n^* x \, \psi_m dx \right|^2$ to give the measure of transition probability. We thus proceed to treat a perturbation problem with the perturbing potentials u_1 and u_2 such that

$$u_1 = (-f_1 x - g) \sin \omega t \,, \quad u_2 = (f_2 x - g) \sin \omega t \,, \tag{15}$$

with $f_1 = \dfrac{ev_1}{d_1}$, $f_2 = \dfrac{e(v_1 + v_2)}{d_2}$, $ev_1 = g$, where $v_1 \sin \omega t$ and $- v_2 \sin \omega t$ are oscillating voltages on the grid and on the anode respectively as shown in Fig. 3.

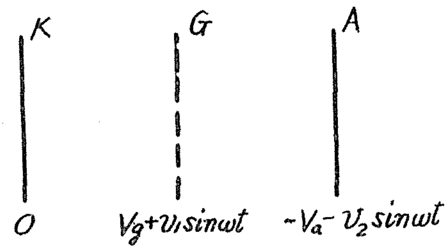

Fig. 3.

We have to solve the following unstationary wave equation:

$$\left\{ -\frac{1}{\kappa^2} \frac{\partial^2}{\partial x^2} + U + u \right\} \Psi = i\hbar \frac{\partial}{\partial t} \Psi \,,$$

where

$$\Psi = \sum_m a_m(t) \Psi_m \,, \quad \Psi_m = \psi_m e^{-\frac{iW_m}{\hbar} t}$$

a_m's being determined by the equations:

$$i\hbar \dot{a}_n = \sum_m U_{nm} a_m \,, \qquad U_{nm} = \int_{-d_1}^{d_2'} \Psi_n^* U \Psi_m dx \,.$$

Assuming U in the form:

$$U_{\frac{1}{2}} = (\mp f_{\frac{1}{2}} x - g)(e^{i\omega t} - e^{-i\omega t})/2i \,,$$

we obtain

$$U_{nm} = \frac{1}{2i}\left\{e^{i(Wn - Wm + \hbar\omega)\frac{t}{\hbar}} - e^{i(Wn - Wm - \hbar\omega)\frac{t}{\hbar}}\right\}\int_{-d_1}^{d_2'}(\mp f_{\frac{1}{2}} x)\,\psi_n^* \psi_m\,dx\,.$$

With the aid of the transformations (8) and (10), we have

$$\begin{aligned}
U_{nm}^{(0)} &= \int_{-d_1}^{d_2'} \psi_n^* (\mp f_{\frac{1}{2}} x)\,\psi_m\,dx \\
&= 4C_{n2}^* C_{m2} \\
&\quad\times\left\{\mp \frac{f_1}{(\kappa\Delta W)^2}\left[2z_0\cos\{\beta_1(z_0 - z_1)\} - 2z_1 - \frac{2}{\beta_1}\sin\{\beta_1(z_0 - z_1)\}\right]\right. \\
&\quad\left. + \frac{f_2}{(\kappa\Delta W)^2}\left[2z_1 - 2z_0\cos\{\beta_2(z_0 - z_1)\} + \frac{2}{\beta_2}\sin\{\beta_2(z_0 - z_1)\}\right]\right\},
\end{aligned}$$

where $-$ or $+$ of the double signs in front of f_1 should be taken according as Δn is even or odd.

Since the ratio $z_1 : 1/\beta$ is nearly equal to the ratio $W_n^{1/2} : W_n'^{1/2}$, we can neglect z_1 in the above expression and therefore, by use of (11), we get

$$\begin{aligned}
U_{nm}^{(0)} = \frac{8C_{n2}^* C_{m2}}{(\kappa\Delta W)^2}&\left\{-(f_1 + f_2)\,z_0\cos\{\beta_2(z_0 - z_1)\}\right. \\
&\left. + \left(\frac{f_2}{\beta_2} - \frac{f_1}{\beta_1}\right)\sin\{\beta_2(z_0 - z_1)\}\right\}.
\end{aligned}$$

If we put

$$\frac{f_2}{\beta_2} - \frac{f_1}{\beta_1} = R\cos\varphi\,, \qquad (f_1 + f_2)\,z_0 = R\sin\varphi\,,$$

then we have

$$U_{nm}^{(0)} = \frac{8C_{n2}^* C_{m2}}{(\kappa\Delta W)^2} R\sin\{\beta_2(z_0 - z_1) - \varphi\}\,, \tag{16}$$

where

$$\tan\varphi = (f_1 + f_2)\,z_0 \Big/ \left(\frac{f_2}{\beta_2} - \frac{f_1}{\beta_1}\right),$$

and

$$R^2 \doteq \left(\frac{f_2}{\beta_2} - \frac{f_1}{\beta_1}\right)^2 + (f_1 + f_2)^2 z_0^2\,,$$

Since $\beta_2(z_0 - z_1) = \dfrac{\Delta n\pi}{1 + a_2/a_1}$, because of (5), the forbidden transitions are given by $\dfrac{\Delta n\pi}{1 + a_2/a_1} - \varphi = N\pi$, namely:

$$1 + \frac{a_2}{a_1} = \frac{\Delta n}{N + \varphi/\pi}, \tag{17}$$

where N is an integer. Thus, it is found that the forbidden transitions are characterized by the tube dimensions, applied d. c. voltages, and external circuit relations which are implied in β_1 and β_2.

4. Conclusion

In this paper, the author has calculated the measure of the transition probability and has shown the possibility of the occurrence of B. K. wave and the dwarf waves and that of the forbidden transition.