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On the Bending of a Rectangular Plate with Four Clamped Edges

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1. Introduction

As is well known, the problem of the bending of a rectangular plate under the action of surface pressure on one of its faces is one of the most important problems in the theory of elasticity. The problem was already investigated mathematically by Ritz (1), Hencky (2), Love (3), Timoshenko (4) and others, using various methods of solution. Love's analysis is very complicated, but his method of simplifying the boundary conditions by means of a conformal transformation seems to be effective for the treatment of the case of a plate with four clamped edges. In the present paper the writer gives an alternative analysis which is developed along a similar line to Love's, but is more concise.

Now, the interior of a rectangle in the z-plane can be mapped into the interior of a circle in the ζ-plane by the relation:

$$z = A\int_0^{\zeta} \frac{d\zeta}{\sqrt{1 - 2\gamma\zeta^2 + \zeta^4}}, \qquad (1)$$

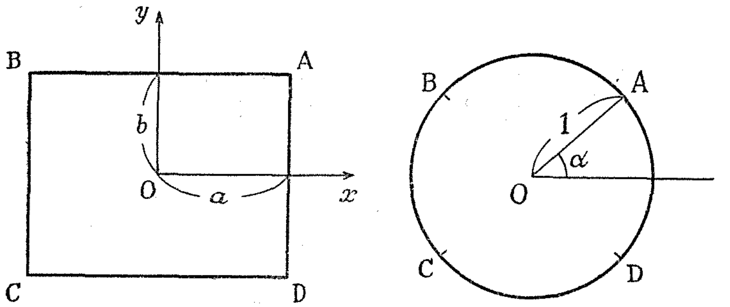
where

$$A = \frac{2a}{(1+k)K}, \qquad (2)$$

$$\gamma = \cos 2\alpha = 1 - 2\left(\frac{1-k}{1+k}\right)^2, \qquad (3)$$

and K is the complete elliptic integral of the first kind with modulus k. The correspondence of various points is shown in the annexed figure, and the ratio of the two sides of the rectangle is given by

$$b/a = K'/2K. \qquad (4)$$



z-plane. ζ-plane.

2. General solution

We consider a small deflection of a homogeneous isotropic rectangular plate with uniform thickness $2h$ which is subject to a surface pressure P on one of its faces. If w be the lateral displacement of a point in the middle plane, the equation of equilibrium takes the form :

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P}{D}, \tag{5}$$

where $D = \frac{2}{3}\frac{Eh^3}{1-\sigma^2}$ is the flexural rigidity, E Young's modulus and σ Poisson's ratio.

Let a particular solution of (5) be denoted by $w = w_1$. Then, the general solution takes the form :

$$w = w_1 + x\Phi_1 + y\Phi_2 + \Psi, \tag{6}$$

where Φ_1, Φ_2 and Ψ are any harmonic functions, and $x\Phi_1 + y\Phi_2 + \Psi$ is called a biharmonic function.

When the four edges of the plate are clamped, the boundary conditions are expressed by

$$\left.\begin{aligned} w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = \pm a; \\ w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = \pm b. \end{aligned}\right\} \tag{7}$$

If we introduce the complex conjugate variables $z = x + iy$ and $\bar{z} = x - iy$, equation (5) is transformed to

$$\frac{\partial^4 w}{\partial z^2 \partial \bar{z}^2} = \frac{P}{16D}, \tag{8}$$

and when the pressure distribution is uniform, this equation can be integrated immediately and we have

$$w = \frac{P}{64D}\left\{z^2\bar{z}^2 + \bar{z}F_1(z) + z\bar{F}_1(\bar{z}) + F_2(z) + \bar{F}_2(\bar{z})\right\}, \tag{9}$$

where $F_1(z)$ and $F_2(z)$ are arbitrary analytic functions of z.

3. A transformed problem

Henceforth we consider the matter in the ζ-plane. Since z is an analytic function of ζ, the functions $F_1\{z(\zeta)\}$ and $F_2\{z(\zeta)\}$ are also analytic with respect to ζ. Let these be denoted by $f_1(\zeta)$ and $f_2(\zeta)$ respectively. Then, the solution (9) becomes

$$w = \frac{P}{64D}\left\{z^2\bar{z}^2 + \bar{z}f_1(\zeta) + z\bar{f}_1(\bar{\zeta}) + f_2(\zeta) + \bar{f}_2(\bar{\zeta})\right\}. \tag{10}$$

Taking account of the conformal character of the transformation it can readily be seen that the boundary conditions in the ζ-plane become (with $\zeta = re^{i\theta}$)

$$w = 0, \quad \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = 1, \tag{11}$$

or in complex forms

$$w = 0, \quad \zeta\frac{\partial w}{\partial \zeta} + \bar{\zeta}\frac{\partial w}{\partial \bar{\zeta}} = 0 \quad \text{at} \quad \zeta\bar{\zeta} = 1. \tag{12}$$

Substituting the expression for w as given by (10) in the right-hand sides of these equations, we have

$$[z^2\bar{z}^2 + \bar{z}f_1 + z\bar{f}_1 + f_2 + \bar{f}_2]_{\zeta\bar{\zeta}=1} = 0; \tag{13}$$

$$\left[2z\bar{z}\left(\bar{z}\zeta\frac{dz}{d\zeta} + z\bar{\zeta}\frac{d\bar{z}}{d\bar{\zeta}}\right) + \bar{z}\zeta\frac{df_1}{d\zeta} + z\bar{\zeta}\frac{d\bar{f}_1}{d\bar{\zeta}}\right.$$

$$\left. + \frac{dz}{d\zeta}\zeta\bar{f}_1 + \frac{d\bar{z}}{d\bar{\zeta}}\bar{\zeta}f_1 + \zeta\frac{df_2}{d\zeta} + \bar{\zeta}\frac{d\bar{f}_2}{d\bar{\zeta}}\right]_{\zeta\bar{\zeta}=1} = 0. \tag{14}$$

Now, if use is made of Legendre's polynominal $P_n(\gamma)$, the transformation equation (1) can also be expressed in the form of a series as:

$$z = A\sum_{n=0}^{\infty}\frac{P_n(\gamma)}{2n+1}\zeta^{2n+1}. \tag{15}$$

Let the Taylor series of $f_1(\zeta)$ and $f_2(\zeta)$ be respectively

$$f_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n ,$$

$$f_2(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n .$$

Since, in the present problem, the deflection of the plate is necessarily symmetrical with respect to both the x- and y-axes, it can easily be seen that all of the a_{2n}'s and b_{2n+1}'s vanish and the remaining coefficients are all real. Thus, we have

$$\left.\begin{aligned} f_1(\zeta) &= \sum_{n=0}^{\infty} a_{2n+1}\,\zeta^{2n+1} , \\ f_2(\zeta) &= \sum_{n=0}^{\infty} b_{2n}\,\zeta^{2n} . \end{aligned}\right\} \tag{16}$$

4. Fulfilment of the boundary conditions

Multiplying (15) and the corresponding conjugate complex expression side by side and putting $\zeta\bar{\zeta} = 1$, we obtain

$$\begin{aligned} \left[\frac{z\bar{z}}{A^2}\right]_{\zeta\bar{\zeta}=1} &= \Big[\left(P_0\zeta + \frac{1}{3}P_1\zeta^3 + \frac{1}{5}P_2\zeta^5 + \cdots\cdots\right) \\ &\qquad \times \left(P_0\bar{\zeta} + \frac{1}{3}P_1\bar{\zeta}^3 + \frac{1}{5}P_2\bar{\zeta}^5 + \cdots\cdots\right)\Big]_{\zeta\bar{\zeta}=1} \\ &= P_0^2 + \frac{1}{3\cdot 3}P_1^2 + \frac{1}{5\cdot 5}P_2^2 + \cdots\cdots \\ &\qquad + \left(\frac{1}{3}P_0P_1 + \frac{1}{3\cdot 5}P_1P_2 + \frac{1}{5\cdot 7}P_2P_3 + \cdots\right)(\zeta^2 + \bar{\zeta}^2) \\ &\qquad + \left(\frac{1}{5}P_0P_2 + \frac{1}{3\cdot 7}P_1P_3 + \frac{1}{5\cdot 9}P_2P_4 + \cdots\right)(\zeta^4 + \bar{\zeta}^4) \\ &\qquad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned}$$

which may be written in the form:

$$\left[\frac{z\bar{z}}{A^2}\right]_{\zeta\bar{\zeta}=1} = Q_0 + Q_1(\zeta^2 + \bar{\zeta}^2) + Q_2(\zeta^4 + \bar{\zeta}^4) + \cdots\cdots , \tag{17}$$

where

$$\left.\begin{aligned} Q_0 &= P_0^2 + \frac{1}{3\cdot3}P_1^2 + \frac{1}{5\cdot5}P_2^2 + \cdots\cdots , \\ Q_1 &= \frac{1}{3}P_0P_1 + \frac{1}{3\cdot5}P_1P_2 + \frac{1}{5\cdot7}P_2P_3 + \cdots\cdots , \\ Q_2 &= \frac{1}{5}P_0P_2 + \frac{1}{3\cdot7}P_1P_3 + \frac{1}{5\cdot9}P_2P_4 + \cdots\cdots , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}\right\} \tag{18}$$

By squaring both sides of (17) we get

$$\left[\frac{z^2\bar{z}^2}{A^4}\right]_{\zeta\bar{\zeta}=1} = R_0 + R_1(\zeta^2 + \bar{\zeta}^2) + R_2(\zeta^4 + \bar{\zeta}^4) + \cdots , \tag{19}$$

where

$$\left.\begin{aligned} R_0 &= Q_0^2 + 2Q_1^2 + 2Q_2^2 + \cdots\cdots , \\ R_1 &= 2Q_0Q_1 + 2Q_1Q_2 + 2Q_2Q_3 + \cdots\cdots , \\ R_2 &= Q_1^2 + 2Q_0Q_2 + 2Q_1Q_3 + \cdots\cdots , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}\right\} \tag{20}$$

Finally, by combining equations (15) and the first of (16) we have

$$\begin{aligned} &\left[\frac{1}{A}(\bar{z}f_1 + z\bar{f}_1)\right]_{\zeta\bar{\zeta}=1} \\ &= 2\left(P_0a_1 + \frac{1}{3}P_1a_3 + \frac{1}{5}P_2a_5 + \cdots\cdots\right) \\ &\quad + \left(\frac{1}{3}P_1a_1 + \frac{1}{5}P_2a_3 + \frac{1}{7}P_3a_5 + \cdots\cdots \right. \\ &\qquad\qquad \left. + P_0a_3 + \frac{1}{3}P_1a_5 + \cdots\cdots\right)(\zeta^2 + \bar{\zeta}^2) \\ &\quad + \left(\frac{1}{5}P_2a_1 + \frac{1}{7}P_3a_3 + \frac{1}{9}P_4a_5 + \cdots\cdots \right. \\ &\qquad\qquad \left. + P_0a_5 + \cdots\cdots\right)(\zeta^4 + \bar{\zeta}^4) \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned} \tag{21}$$

Substituting (16), (19) and (21) into the first boundary condition (13) and equating all the terms with equal powers of ζ to zero, we obtain the following set of equations:

$$\left.\begin{aligned}
&2A\left(P_0a_1+\frac{1}{3}P_1a_3+\frac{1}{5}P_2a_5+\cdots\cdots\right)+2b_0+A^4R_0=0\,,\\
&A\left(\frac{1}{3}P_1a_1+\frac{1}{5}P_2a_3+\frac{1}{7}P_3a_5+\cdots\cdots\right.\\
&\qquad\left.+P_0a_3+\frac{1}{3}P_1a_5+\cdots\cdots\right)+b_2+A^4R_1=0\,,\\
&A\left(\frac{1}{5}P_2a_1+\frac{1}{7}P_3a_3+\frac{1}{9}P_4a_5+\cdots\cdots\right.\\
&\qquad\left.+P_0a_5+\cdots\cdots\right)+b_4+A^4R_2=0\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}\right\}\quad(22)$$

Also, by a similar analysis, the second boundary condition (14) gives another set of equations as follows:

$$\left.\begin{aligned}
&4A(P_0a_1+P_1a_3+P_2a_5+\cdots\cdots)+2A^4T_0=0\,,\\
&A\left[\left(1+\frac{1}{3}\right)P_1a_1+\left\{(1+3)P_0+\left(1+\frac{3}{5}\right)P_2\right\}a_3\right.\\
&\qquad\left.+\left\{\left(1+\frac{5}{3}\right)P_1+\left(1+\frac{5}{7}\right)P_3\right\}a_5+\cdots\cdots\right]+2b_2+2A^4T_1=0\,,\\
&A\left[\left(1+\frac{1}{5}\right)P_2a_1+\left(1+\frac{3}{7}\right)P_3a_3\right.\\
&\qquad\left.+\left\{(1+5)P_0+\left(1+\frac{5}{9}\right)P_4\right\}a_5+\cdots\cdots\right]+4b_4+2A^4T_2=0\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}\right\}\quad(23)$$

where T_0, T_1, T_2, $\cdots\cdots$ are given by

$$\left.\begin{aligned}
&T_0=Q_0S_0+2Q_1S_1+2Q_2S_2+\cdots\cdots\,,\\
&T_1=Q_0S_1+Q_1S_0+Q_1S_2+Q_2S_1+\cdots\cdots\,,\\
&T_2=Q_1S_1+Q_0S_2+Q_2S_0+Q_1S_3+Q_3S_1+\cdots\cdots\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}\right\}\quad(24)$$

with

$$\left.\begin{aligned} S_0 &= 2\left(P_0^2 + \frac{1}{3}P_1^2 + \frac{1}{5}P_2^2 + \cdots\cdots\right), \\ S_1 &= \left(1 + \frac{1}{3}\right)P_0P_1 + \left(\frac{1}{3} + \frac{1}{5}\right)P_1P_2 + \left(\frac{1}{5} + \frac{1}{7}\right)P_2P_3 + \cdots, \\ S_2 &= \left(1 + \frac{1}{5}\right)P_0P_2 + \left(\frac{1}{3} + \frac{1}{7}\right)P_1P_3 + \left(\frac{1}{5} + \frac{1}{9}\right)P_2P_4 + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}\right\} \tag{25}$$

5. Results of numerical computations

Elimination of b_{2n}'s from (22) and (23) gives equations for determining a_{2n+1}'s which can be written in the form:

$$\begin{bmatrix} 2P_0, & 2P_1, & 2P_2, & \cdot & \cdot \\ \frac{1}{3}P_1, & P_0 + \frac{3}{5}P_2, & P_1 + \frac{5}{7}P_3, & \cdot & \cdot \\ \frac{1}{5}P_2, & \frac{3}{7}P_3, & P_0 + \frac{5}{9}P_4, & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ \cdot \\ \cdot \end{bmatrix} = A^3 \begin{bmatrix} -T_0 \\ R_1 - T_1 \\ 2R_2 - T_2 \\ \cdot \\ \cdot \end{bmatrix} . \tag{26}$$

The deflection of the centre of the plate is given by

$$\begin{aligned} w_{\max} &= \frac{P}{64D} \cdot 2b_0 \\ &= -\frac{P}{64D}\left\{2A\left(P_0a_1 + \frac{1}{3}P_1a_3 + \frac{1}{5}P_2a_5 + \cdots\cdots\right) + A^4R_0\right\} . \end{aligned} \tag{27}$$

With the purpose of examining the convergence of the present procedure, numerical calculations have been carried out for the case of a square plate, by taking two, three and four terms of a_{2n+1}'s into account respectively. The results obtained are

$$w_{\max}/(Pa^4/8D) = 0.1736,\ 0.1658 \text{ and } 0.1638 .$$

These results may be compared with Love's approximate value 0.16597 and also with more accurate values 0.1622 and 0.1617 which were obtained by Hencky and Timoshenko respectively.

In conclusion, the writer wishes to express his hearty thanks to Professor S. Tomotika for his continual guidance throughout this work.

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