Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXVI, Nos. 1 & 2, Article 2, 1950.

The Pressure Distributions on the Surface of an Obstacle in a Running Viscous Fluid at Small Reynolds Numbers

By

S. Tomotika and **T. Aoi**

(Received May 10, 1950)

1. Introduction

It is well known that the steady flow of an incompressible viscous fluid past a solid obstacle can be successfully dealt with on the basis of Oseen's linearized equations of motion. A number of investigations have been carried out so far by several writers, but they were mainly concerned with the discussion on the drag experienced by an obstacle.

It seems worth while, however, to discuss whether or not the actual flow pattern around a body as observed in experiments can be obtained theoretically from the solution of Oseen's linearized equations of motion. In a previous paper (1), we have therefore carried out detailed investigations on the steady flows of an incompressible viscous fluid past a sphere as well as past a circular cylinder, by making use of the exact analytical solutions of Oseen's equations. The flow patterns around these obstacles have been computed in detail, and the drags experienced by these bodies have also been discussed, making special reference to the pressure drag and the frictional drag separately. Thus, we have arrived at several interesting and important results which are in good agreement with the results of observation.

In this supplementary short note we intend to discuss the pressure distributions on the surface of a sphere and of a circular cylinder, each placed in the steady flow of a viscous fluid at small Reynolds numbers.

PART I. CASE OF A SPHERE

2. Pressure distributions on the surface of a sphere

In the first place, the essential parts of Goldstein's general solution (2) of Oseen's equations in the case of a sphere, which are necessary for our investigation, will be briefly reproduced for reference.

We assume that at a great distance from an obstacle the fluid flows with constant velocity U in the positive direction of the x-axis. If we denote by u, v and w the rectangular components of the velocity of perturbation due to the presence of the obstacle, Oseen's linearized equations of motion are given in the forms:

$$U\frac{\partial}{\partial x}(u, v, w) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p + \nu\nabla^2(u, v, w), \tag{1}$$

where p is the pressure, ρ the density of the fluid, ν its kinematic coefficient of viscosity and ∇^2 stands for $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The fluid being assumed to be incompressible, the equation of continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

The components of velocity of the perturbation which satisfy these equations are given by

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} + \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi, \\ v &= -\frac{\partial \phi}{\partial y} + \frac{1}{2k}\frac{\partial \chi}{\partial y}, \\ w &= -\frac{\partial \phi}{\partial z} + \frac{1}{2k}\frac{\partial \chi}{\partial z}; \end{aligned}\right\} \tag{3}$$

while the pressure relative to that at infinity is given by

$$p = \rho U\frac{\partial \phi}{\partial x}, \tag{4}$$

provided that

$$\nabla^2\phi = 0, \tag{5}$$

and

$$\nabla^2\chi - 2k\frac{\partial \chi}{\partial x} = 0, \tag{6}$$

where we have put $k = U/2\nu$.

Introducing spherical polar coordinates (r, θ, ω), the appropriate general solution of equation (5) is expressed in the form:

$$\phi = U\sum_{n=0}^{\infty} A_n \frac{P_n(\mu)}{r^{n+1}}, \tag{7}$$

where $P_n(\mu)$ denotes the Legendre function and A_n's are constants of integration.

On the other hand, the appropriate solution of equation (6) is given by

$$\chi = Ue^{\mu\xi}\sum_{m=0}^{\infty} B_m \chi_m(\xi) P_m(\mu), \tag{8}$$

where we have put $\xi = kr$, $\mu = \cos\theta$, and B_m's are constants of integration. The function $\chi_m(\xi)$ is expressed in terms of the modified Bessel function as:

$$\chi_m(\xi) = (2m + 1)\sqrt{\frac{\pi}{2\xi}} K_{m+\frac{1}{2}}(\xi). \tag{9}$$

We shall denote the components of the velocity of perturbation along r increasing, θ increasing and ω increasing by v_r, v_θ and v_ω respectively. Then, it is evident that $v_\omega = 0$, and after some reductions, we have

$$\left.\begin{aligned} v_r = {} & U\sum_{n=0}^{\infty} \frac{(n+1)A_n}{r^{n+2}} P_n(\mu) \\ & - \frac{U}{2} e^{\mu\xi} \sum_{m=0}^{\infty} B_m \left\{ \frac{m}{2m-1} \chi_{m-1}(\xi)\, P_m(\mu) + \mu\chi_m(\xi)\, P_m(\mu) \right. \\ & \qquad\qquad \left. + \frac{m+1}{2m+3} \chi_{m+1}(\xi)\, P_m(\mu) \right\}, \\ v_\theta = {} & U \sin\theta \sum_{n=1}^{\infty} \frac{A_n}{r^{n+2}} P_n'(\mu) \\ & + \frac{U}{2} \sin\theta\, e^{\mu\xi} \sum_{m=0}^{\infty} B_m \left\{ \frac{1}{2m-1} \chi_{m-1}(\xi)\, P_m'(\mu) + \chi_m(\xi)\, P_m(\mu) \right. \\ & \qquad\qquad \left. - \frac{1}{2m+3} \chi_{m+1}(\xi)\, P_m'(\mu) \right\}, \end{aligned}\right\} \tag{10}$$

where accents denote differentiation with respect to μ.

If the radius of the sphere be denoted by a, the boundary conditions at the surface of the sphere are given by

$$v_r = -U\cos\theta\,, \quad v_\theta = U\sin\theta \tag{11}$$

at $r = a$, and these conditions yield the relations between the constants of integration A_n and B_m. Further, after some calculations, it is found that the constants B_m can be determined by solving the following simultaneous equations:

$$\sum_{m=0}^{\infty} B_m \lambda_{m,n}(\xi_0) = \begin{cases} -6 & (n=1)\,, \\ 0 & (n=2,3,\cdots)\,, \end{cases} \tag{12}$$

where ξ_0 stands for ka. For the expressions of the functions $\lambda_{m,n}$, reference should be made to our previous paper (**1**) or to Goldstein's paper (**2**).*

We shall next proceed to the computation of the pressure distributions along a meridian line on the surface of the sphere. Now, the pressure p at any point in the fluid is given by (4), together with the expression for ϕ as given by (7). Since, however, $\partial\phi/\partial x \to 0$ as $r \to \infty$, this p represents, as mentioned already, the pressure relative to that at infinity.

Thus, if we denote the absolute pressure at any point in the fluid by p anew and the corresponding pressure at infinity by p_∞, we have

$$p - p_\infty = \rho U \frac{\partial\phi}{\partial x}\,. \tag{13}$$

In particular, if we denote the pressure on the surface of the sphere by p_s, we have

$$\frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = \frac{2}{U}\left(\frac{\partial\phi}{\partial x}\right)_{r=a}\,. \tag{14}$$

Equation (7) gives us immediately the following result:

$$\left(\frac{\partial\phi}{\partial x}\right)_{r=a} = -U\sum_{n=0}^{\infty}\frac{(n+1)A_n}{a^{n+2}}\,\mu\,P_n(\mu) + U\sum_{n=1}^{\infty}\frac{A_n}{a^{n+2}}(1-\mu^2)\,P_n'(\mu)\,.$$

But, this can be transformed into a more elegant form, if we make

* It is to be remarked here that the expressions for our functions $\lambda_{m,n}$ can be obtained from the corresponding Goldstein's by interchanging the suffixes m and n.

use of some recurrence formulae for the Legendre function as well as of the following relations:

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(n+1)A_n}{a^{n+2}}P_n(\mu) \\
&= -\mu + \frac{1}{2}e^{\mu\xi_0}\sum_{m=0}^{\infty}B_m\left\{\frac{m}{2m-1}\chi_{m-1}(\xi_0)\,P_m(\mu)\right. \\
&\qquad\qquad \left. + \mu\,\chi_m(\xi_0)\,P_m(\mu) + \frac{m+1}{2m+3}\chi_{m+1}(\xi_0)\,P_m(\mu)\right\}, \\
&\sum_{n=1}^{\infty}\frac{A_n}{a^{n+2}}P_n'(\mu) \\
&= 1 - \frac{1}{2}e^{\mu\xi_0}\sum_{m=0}^{\infty}B_m\left\{\frac{1}{2m-1}\chi_{m-1}(\xi_0)\,P_m'(\mu)\right. \\
&\qquad\qquad \left. + \chi_m(\xi_0)\,P_m(\mu) - \frac{1}{2m+3}\chi_{m+1}(\xi_0)\,P_m'(\mu)\right\},
\end{aligned}\right\} \tag{15}$$

which follow immediately from the boundary conditions (11) together with (10).

We thus have

$$\begin{aligned}\left(\frac{\partial\phi}{\partial x}\right)_{r=a} = U - \frac{U}{2}e^{\mu\xi_0}\sum_{m=0}^{\infty}B_m&\left\{\frac{m}{2m-1}\chi_{m-1}(\xi_0)\,P_{m-1}(\mu)\right. \\
&\left. + \chi_m(\xi_0)\,P_m(\mu) + \frac{m+1}{2m+3}\chi_{m+1}(\xi_0)\,P_{m+1}(\mu)\right\}. \end{aligned} \tag{16}$$

Hence, bearing in mind that the function $\chi_m(\xi)$ is given by (9), we have ultimately

$$\begin{aligned}\frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = 2 - e^{\mu\xi_0}\sqrt{\frac{\pi}{2\xi_0}}\sum_{m=0}^{\infty}B_m&\left\{mK_{m-\frac{1}{2}}(\xi_0)\,P_{m-1}(\mu)\right. \\
&+ (2m+1)K_{m+\frac{1}{2}}(\xi_0)\,P_m(\mu) \\
&\left. + (m+1)K_{m+\frac{3}{2}}(\xi_0)\,P_{m+1}(\mu)\right\}, \end{aligned} \tag{17}$$

and this gives the exact general expression for the pressure coefficient on the surface of the sphere.

Making use of this exact expression, detailed numerical calculations of the values of the pressure coefficient on the surface of the sphere have been carried out in two cases in which the Reynolds number of the flow $R = Ud/\nu$, where d is the diameter of the sphere, is equal to 1 and 2 respectively. The results are shown in Table I and Fig. 1. It is found that there is a good qualitative agreement with experiment.

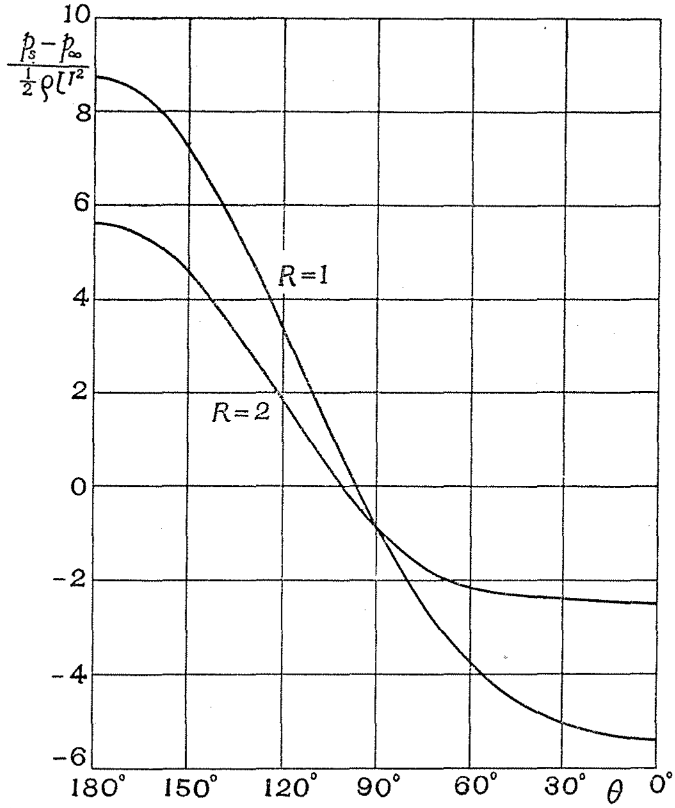

Fig. 1. Pressure distributions on a sphere.

TABLE I.

θ	$R = 1$	$R = 2$
0°	-5.419	-2.505
30°	-5.070	-2.432
60°	-3.738	-2.160
90°	-0.863	-0.876
120°	3.325	1.800
150°	7.192	4.521
180°	8.770	5.642

PART II. CASE OF A CIRCULAR CYLINDER

3. Pressure distributions on the surface of a circular cylinder

In the previous paper (1) we have obtained the exact analytical solution of Oseen's equations in two dimensions in the case of steady flow of an incompressible viscous fluid past a circular cylinder, by developing similar analysis to Goldstein's in the case of a sphere. Only the essentials necessary for our present discussion will be given here briefly for reference.

We assume that the fluid at infinity flows with constant velocity U in the positive direction of the x-axis, and let u and v be the rectangular components of the velocity of perturbation caused by the presence of the body, which become vanishingly small at a great distance from the body. Then, Oseen's linearized equations of motion are given by

$$U\frac{\partial}{\partial x}(u, v) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)p + \nu\nabla^2(u, v), \tag{18}$$

where p, ρ and ν have the same meanings as before, and ∇^2 stands for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

The fluid being assumed to be incompressible, the equation of continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{19}$$

A general solution of these equations can be expressed in the form:

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} + \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi, \\ v &= -\frac{\partial \phi}{\partial y} + \frac{1}{2k}\frac{\partial \chi}{\partial y}; \end{aligned}\right\} \tag{20}$$

and

$$p = \rho U\frac{\partial \phi}{\partial x}, \tag{21}$$

provided

$$\nabla^2\phi = 0, \tag{22}$$

and

$$\nabla^2\chi - 2k\frac{\partial \chi}{\partial x} = 0, \tag{23}$$

where, as before, we have put $k = U/2\nu$.

Introducing polar coodinates (r, θ), the appropriate general solution of equation (22) is given by

$$\phi = UA_0 \log r - U\sum_{n=1}^{\infty} \frac{A_n}{n} \frac{\cos n\theta}{r^n}, \tag{24}$$

where A_n's are constants of integration.

Also, it is found that a general solution of equation (23) which is appropriate to our problem is given by

$$\chi = Ue^{kr\cos\theta} \sum_{m=0}^{\infty} B_m K_m(kr) \cos m\theta, \tag{25}$$

where K_m is the modified Bessel function and B_m's are constants of integration.

Instead of using the rectangular components of velocity u, v, the radial and circumferential components may conveniently be used. If we denote them by v_r and v_θ, we have, after some calculations,

$$\left.\begin{aligned} v_r = &- U\sum_{n=0}^{\infty} A_n \frac{\cos n\theta}{r^{n+1}} - \frac{U}{4} e^{kr\cos\theta} \sum_{m=0}^{\infty} B_m \Big\{K_{m-1}(kr)\cos m\theta \\ &+ 2K_m(kr)\cos\theta\cos m\theta + K_{m+1}(kr)\cos m\theta\Big\}, \\ v_\theta = &- U\sum_{n=1}^{\infty} A_n \frac{\sin n\theta}{r^{n+1}} + \frac{U}{4} e^{kr\cos\theta} \sum_{m=0}^{\infty} B_m \Big\{K_{m-1}(kr)\sin m\theta \\ &+ 2K_m(kr)\sin\theta\cos m\theta - K_{m+1}(kr)\sin m\theta\Big\}. \end{aligned}\right\} \tag{26}$$

Denoting the radius of the circular cylinder by a, the boundary conditions to be satisfied at its surface are

$$v_r = -U\cos\theta, \quad v_\theta = U\sin\theta \tag{27}$$

at $r = a$, and these conditions give us immediately the relations between the constants A_n and B_m. Further, it is found that B_m's can be determined by the following simultaneous equations:

$$\sum_{m=0}^{\infty} B_m \lambda_{m,n}(\xi_0) = \begin{cases} 4 & (n=1), \\ 0 & (n=2, 3, \cdots), \end{cases} \tag{28}$$

where $\xi_0 = ka$ and the functions $\lambda_{m,n}$ are expressed in terms of the modified Bessel functions I_n and K_n as follows:

$$\lambda_{m,n}(\xi_0) = I_{m-n}(\xi_0)\,K_{m-1}(\xi_0) + I_{m+n}(\xi_0)\,K_{m+1}(\xi_0)$$
$$+ I_{m-n+1}(\xi_0)\,K_m(\xi_0) + I_{m+n-1}(\xi_0)\,K_m(\xi_0)\,. \qquad (29)$$

Now, by (21) and (24), the pressure at any point in the fluid is given by

$$p = \rho U^2 \sum_{n=0}^{\infty} A_n \frac{\cos(n+1)\theta}{r^{n+1}}\,.$$

Obviously this represents the pressure relative to that at infinity, since $p \to 0$ as $r \to \infty$. If, therefore, we use p anew to denote the absolute pressure and also if we denote the corresponding pressure at infinity by p_∞, we have

$$\frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 2\sum_{n=0}^{\infty} A_n \frac{\cos(n+1)\theta}{r^{n+1}}\,. \qquad (30)$$

If, in particular, p_s denotes the pressure on the surface of the cylinder, we have

$$\frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = 2\sum_{n=0}^{\infty} A_n \frac{\cos(n+1)\theta}{a^{n+1}}\,. \qquad (31)$$

Putting the expressions for v_r and v_θ as given by (26) into the boundary conditions (27), we readily have

$$\sum_{n=0}^{\infty} A_n \frac{\cos n\theta}{a^{n+1}} = \cos\theta - \frac{1}{4}e^{ka\cos\theta}\sum_{m=0}^{\infty} B_m \Big\{K_{m-1}(ka)\cos m\theta$$
$$+ 2K_m(ka)\cos\theta\cos m\theta + K_{m+1}(ka)\cos m\theta\Big\}\,,$$

$$\sum_{n=1}^{\infty} A_n \frac{\sin n\theta}{a^{n+1}} = -\sin\theta + \frac{1}{4}e^{ka\cos\theta}\sum_{m=0}^{\infty} B_m \Big\{K_{m-1}(ka)\sin m\theta$$
$$+ 2K_m(ka)\sin\theta\cos m\theta - K_{m+1}(ka)\sin m\theta\Big\}\,,$$

and consequently

$$\sum_{n=0}^{\infty} A_n \frac{\cos(n+1)\theta}{a^{n+1}} = 1 - \frac{1}{4}e^{ka\cos\theta}\sum_{m=0}^{\infty} B_m \Big\{K_{m-1}(ka)\cos(m-1)\theta$$
$$+ 2K_m(ka)\cos m\theta + K_{m+1}(ka)\cos(m+1)\theta\Big\}\,.$$

Thus, the pressure coefficient on the surface of the cylinder becomes finally

$$\frac{p_s - p_\infty}{\frac{1}{2}\rho U^2} = 2 - \frac{1}{2} e^{ka\cos\theta} \sum_{m=0}^{\infty} B_m \Big\{ K_{m-1}(ka)\cos(m-1)\theta$$

$$+ 2K_m(ka)\cos m\theta + K_{m+1}(ka)\cos(m+1)\theta \Big\}, \quad (32)$$

and this exact formula can conveniently be used to compute the pressure distributions on the surface of the circular cylinder.

Numerical calculations have been carried out in two cases in which the Reynolds number of the flow $R = Ud/\nu$, where d is the diameter of the cylinder, is equal to 0.8 and 4 respectively. The results are shown in Table II and Fig. 2. In this figure, Thom's experimental results (3) for $R = 3.5$ are also shown by small black circles.*

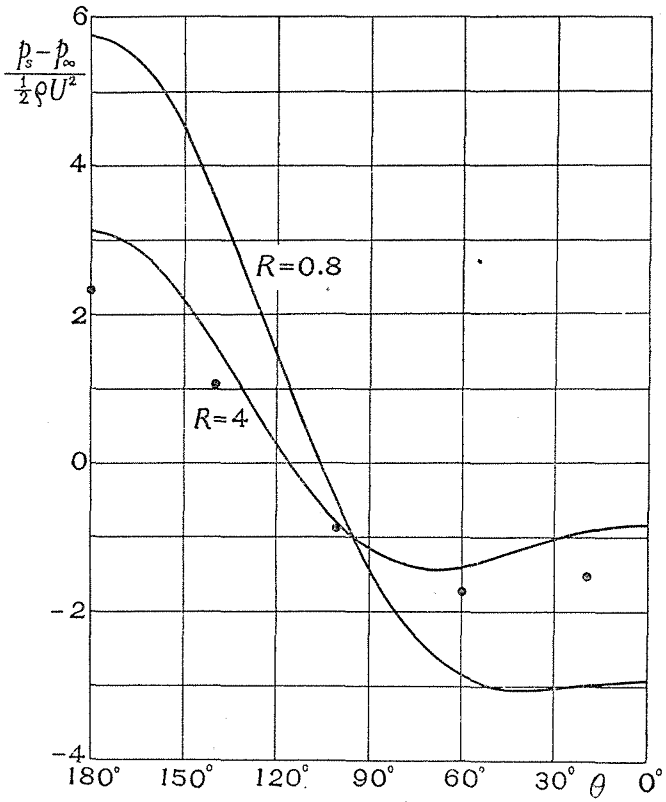

Fig. 2. Pressure distributions on a circular cylinder.

---

* Thom's experiments were conducted in an oil whose kinematic coefficient of viscosity was $\nu = 0.4\,\mathrm{cm^2/sec}$. The diameter d of the cylinder used by him was 0.318 cm and the undisturbed velocity U of the stream was 4.4 cm/sec.

TABLE II.

θ	$R=0.8$	$R=4$
0°	− 2.924	− 0.834
30°	− 3.028	− 1.021
60°	− 2.824	− 1.401
90°	− 1.416	− 1.158
120°	1.430	0.245
150°	4.448	2.194
180°	5.758	3.111

4. Summary and conclusion

In this paper, the computations of the pressure distributions on the surface of a sphere as well as on the surface of a circular cylinder have been made, by making use of the exact analytical solutions of Oseen's linearized equations of motion for the steady flows of an incompressible viscous fluid past these obstacles.

For this purpose, we have first obtained the exact formula for the pressure coefficient in each case, and then detailed numerical computations have been made. Thus, it has been found that the calculated pressure distributions are in good agreement with observation.

Comparing the results of the present paper with those in our previous paper (**1**), it may be concluded that, as we have expected, Oseen's linearized equations of motion are capable of representing satisfactorily the steady flow of an incompressible viscous fluid past an obstacle, provided that the Reynolds number of the flow is fairly small.

REFERENCES

1. S. Tomotika and T. Aoi, The steady flow of viscous fluid past sphere and circular cylinder at small Reynolds numbers. Quart. Journ. Mech. and Applied Math., Oxford, **3** (1950), 140–161.

2. S. Goldstein, The steady flow of viscous fluid past a fixed obstacle at small Reynolds numbers. Proc. Roy. Soc. London, A, **123** (1929), 225–235.

3. A. Thom, The flow past circular cylinders at low speeds. Proc. Roy. Soc. London, A, **141** (1933), 651–669.

Physical Institute,
Faculty of Science,
University of Kyoto.