Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXVI, No. 3, Article 19, 1950.

On the Two-dimensional Supersonic Free Gas Jet.*

By

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(Received September 25, 1950)

1. Introduction and summary

Theoretical studies on the two-dimensional supersonic free jet of a perfect gas were made by Prandtl (1) in 1904, and the ratio of the breadth of jet to the wave-length of the wavy motion has been obtained as a function of the Mach number associated with the mean-stream velocity. In his paper, however, the deformations of the free-streamline boundaries from the straight lines were neglected. Later, Riabouchinsky (2) treated the same problem by a different method under a similar assumption that the flow is almost uniform, and obtained the streamlines and the isobars. The approximation of his analysis is considered to be better than that of Prandtl. However, a hypothetical gas employed by him is essentially of the same approximate nature as Kármán-Tsien's hypothetical gas (3) whose pressure-density curve has a common tangent but different curvature with the adiabatic pressure-density curve.

In the present paper, a certain appropriate equation of state-change of gas is assumed so that the fundamental equations for the stream function and the velocity potential might easily be solved exactly. A hypothetical gas such introduced can approximate in fact the real gas obeying the adiabatic law better than Kármán-Tsien's hypothetical gas, since its pressure-density curve can be made to coincide up to its curvature with the corresponding curve for the real gas. Of course, the validity of the present theory is limited to some range of speeds over which both our hypothetical gas and the real gas have similar properties. But, in this range, once the fundamental equations of motion for our hypothetical gas be solved exactly, it will be possible to infer, from the results obtained, more reliably the general behaviours of the corresponding real gas flow, rather than from the approximate solutions

* Communicated by Professor S. Tomotika.

of the exact equations of motion of the real gas which obeys the adiabatic law.

It is shown that the fundamental equation for the stream function governing the flow of our hypothetical gas has a family of exact solutions of simple form, and combing some of appropriate solutions we can construct the solution for the supersonic free jet. In this case, we meet with a problem how to join two hodograph solutions on a certain curve. This is done, in our problem, on a characteristics of the fundamental hodograph equations. The special characteristics on which the hodograph solution branches are called the branch-line. When two solutions are joined on the branch-line, there occur in general discontinuities of the curvature of the streamline and of the velocity gradient in the physical plane. In the present paper, two solutions are joined on the branch-line in such a way that at least the curvature of the streamline and the velocity gradient are continuous there. It will be one of the important future problems to discuss how we must join two hodograph solutions so that they represent a perfectly regular flow in the physical plane.

A detailed numerical disccussion has been made in a special case, giving the Mach lines networks, the isobars and the curves of the constant direction of velocity vectors in the physical plane. In the appendix, a general relationship between the breadth, the wave-length, the flow quantity of jet and the Mach number of the mean-flow is given for a group of solutions representing the supersonic free jet. This relation is considered to be an extension of Prandtl's formula and, when the deformation of the free-streamline boundaries is neglected, it degenerates into his formula.

The writer wishes to express his cordial thanks to Professor Dr. S. Tomotika for his continued interest throughout this work, and also to Dr. K. Tamada for his kind suggestions.

2. Fundamental equations

It is well known that when the magnitude, q, and the angle of inclination, θ, of the velocity vector are taken as independent variables, the equations of steady irrotational motion of a perfect gas in two-dimensions are given by

$$\Phi_q = q\frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right)\Psi_\theta\,, \quad \Phi_\theta = \frac{\rho_0 q}{\rho}\Psi_q\,, \tag{2.1}$$

where Φ and Ψ are the velocity potential and the stream function for the flow respectively, and ρ and ρ_0 are the density at any point and at the stagnation point in the flow field respectively. Suffixes q, θ in Φ_q, Ψ_θ, Φ_θ, Ψ_q denote partial differentiation with respect to q or θ.

Instead of q, we introduce a new variable τ defined by

$$\tau = -\int^q (-K)^{\frac{1}{2}} \frac{\rho}{\rho_0 q} dq , \tag{2.2}$$

with

$$K = -\frac{\rho_0 q^2}{\rho} \frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right). \tag{2.3}$$

Now, let us assume that the pressure, p, of the gas is a function of the density ρ only. Then, by making use of Bernoulli's equation in differential form: $qdq + dp/\rho = 0$, we get $d\rho/dq = -\rho q/c^2$, where $c = \sqrt{dp/d\rho}$ is the local speed of sound. Thus, K is written as

$$K = \left(1 - \frac{q^2}{c^2}\right)\left(\frac{\rho_0}{\rho}\right)^2 . \tag{2.4}$$

If a certain equation of state of the gas is assumed, K becomes a definite function of q (and therefore, of τ) only.

Equations (2.1) are now transformed into

$$\left.\begin{aligned} \Phi_\tau = -\chi^2 \Psi_\theta , \quad \Phi_\theta = -\chi^2 \Psi_\tau , \\ \chi = (-K)^{\frac{1}{4}} . \end{aligned}\right\} \tag{2.5}$$

Further, in place of Φ and Ψ, we use, with Professor I. Imai (4), ϕ and ψ as dependent variables which are connected with Φ and Ψ as:

$$\Phi = \chi\phi , \qquad \Psi = \chi^{-1}\psi . \tag{2.6}$$

Then equations (2.5) become

$$\phi_{\tau\tau} - \phi_{\theta\theta} = \chi \frac{d^2 \chi^{-1}}{d\tau^2} \phi , \tag{2.7}$$

$$\psi_{\tau\tau} - \psi_{\theta\theta} = \chi^{-1} \frac{d^2 \chi}{d\tau^2} \psi . \tag{2.8}$$

As far as we confine ourselves to the adiabatic gas flow, no simple exact solutions can be obtained in general. Therefore, we now introduce such a hypothetical gas that enables the above equations to be solved simply. Thus, we assume that

$$\chi = c_1 \tau + c_2 \tag{2.9}$$

where c_1, c_2 are, for the present, arbitrary constants. Then, the equation for ψ takes the form:

$$\psi_{\tau\tau} - \psi_{\theta\theta} = 0,$$

and the general solution of this equation is given by

$$\psi = f(\tau + \theta) + g(\tau - \theta)$$

where f, g are both arbitrary functions. Thus the expression for the function Ψ becomes

$$\Psi = \chi^{-1}\{f(\tau + \theta) + g(\tau - \theta)\}, \tag{2. 10}$$

while the corresponding expression for Φ takes the form:

$$\Phi = \chi\{-f(\tau + \theta) + g(\tau - \theta)\} + c_1\int^{\theta}\{f(\tau + \theta) + g(\tau - \theta)\}d\theta. \tag{2. 11}$$

3. Equations of state of our hypothetical gas

The equation of state-change of our hypothetical gas can be obtained by a method used by Imai (5).

Now we put $\Psi = Q(q)\cos\theta$, where $Q(q)$ is a function of q only. Then, the equation for determining Q becomes linear and has two independent solutions: $1/q$ and p/q, where p is the pressure. Hence, if we write $\Psi = T(\tau)\cos\theta$, each of the two functions $1/q$ and p/q must be expressed as a linear combination of the solutions of the equation for determining $T(\tau)$, namely:

$$\frac{d^2T}{d\tau^2} + \frac{2c_1}{c_1\tau + c_2}\frac{dT}{d\tau} + T = 0. \tag{3. 1}$$

For convenience's sake, we use, instead of τ, another new variable t defined as:

$$t = \tau + \frac{c_2}{c_1}. \tag{3. 2}$$

Then, the general solution of equation (3. 1) is given by $At^{-1}\sin(t + \varepsilon)$, where A and ε are arbitrary constants of integration. Hence, we get

$$\frac{1}{q} = A\frac{\sin(t + \varepsilon)}{t}, \quad \frac{p}{q} = B\frac{\sin(t + \delta)}{t}, \tag{3. 3}$$

B and δ being also constants.

Assuming that $p = p_1$, $t = t_1$, $\tau = \tau_1$ at $q = q_1$, we get

$$\frac{q}{q_1} = \frac{t\,\sin(t_1+\varepsilon)}{t_1\sin(t+\varepsilon)}, \quad \frac{p}{p_1} = \frac{\sin(t_1+\varepsilon)\sin(t+\delta)}{\sin(t+\varepsilon)\sin(t_1+\delta)}, \tag{3.4}$$

with

$$t_1 = \tau_1 + \frac{c_2}{c_1}. \tag{3.5}$$

The density ρ can be obtained from (2. 2) as follows:

$$\frac{\rho}{\rho_1} = \frac{t_1\sin(t+\varepsilon)\sin(t_1+\varepsilon) - t_1\cos(t_1+\varepsilon)}{t\,\sin(t_1+\varepsilon)\sin(t+\varepsilon) - t\,\cos(t+\varepsilon)}, \tag{3.6}$$

where we have assumed that $\rho = \rho_1$ at $q = q_1$.

Finally, the local Mach number M, defined as q/c, is given by

$$M^2 = 1 + \left\{\frac{t}{1 - t\cot(t+\varepsilon)}\right\}^2. \tag{3.7}$$

4. Determination of constants τ_1, c_1, c_2, ε, δ

For our hypothetical gas, χ and τ are related by (2. 9), while for the real gas obeying the adiabatic law, the corresponding quantities, denoted by χ_a and τ_a respectively, are related indirectly by the two equations:

$$\left.\begin{aligned} \chi_a &= (\alpha^2-1)^{-\frac{1}{4}(\alpha^2-1)}\zeta^{\frac{1}{2}}(\alpha^2+\zeta^2)^{\frac{1}{4}(\alpha^2-1)}, \\ \tau_a &= \tan^{-1}\zeta - \alpha\tan^{-1}\frac{\zeta}{\alpha}, \\ \text{where} \quad \zeta &= \left(\frac{\alpha^2 z - 1}{1-z}\right)^{\frac{1}{2}}, \quad z = \frac{q^2}{q_{\max}^2}, \quad \alpha^2 = \frac{\gamma+1}{\gamma-1}, \end{aligned}\right\} \tag{4.1}$$

$q_{\max}$ being the theoretically attainable maximum value of q, and γ the ratio of the specific heats of the real gas subject to the adiabatic law.

Now, we choose the curve of the state-change for our hypothetical gas as follows. First, we assume that τ is equal to τ_a at $q = q_1$, namely:

$$\tau_1 = \tan^{-1}\zeta_1 - \alpha\tan^{-1}\frac{\zeta_1}{\alpha}, \quad \zeta_1 = \left(\frac{\alpha^2 z_1 - 1}{1-z_1}\right)^{\frac{1}{2}}, \quad z_1 = \frac{q_1^2}{q_{\max}^2}. \tag{4.2}$$

Further, we assume that at the point $\tau = \tau_1$ the χ-τ curve coincides with the corresponding χ_a-τ_a curve for the real gas up to their tangents; i. e., we assume that

$$(\chi_a)_{\tau_a=\tau_1} = c_1\tau_1 + c_2\,, \qquad \left(\frac{d\chi_a}{d\tau_a}\right)_{\tau_a=\tau_1} = c_1\,.$$

Then we get

$$\left.\begin{aligned} c_1 &= -\frac{1}{2}\alpha^2(\alpha^2-1)^{-\frac{1}{4}(\alpha^2+3)}\zeta_1^{-\frac{5}{2}}(\alpha^2+\zeta_1^2)^{\frac{1}{4}(\alpha^2-1)}(1+\zeta_1^2)^2, \\ c_2 &= (\alpha^2-1)^{-\frac{1}{4}(\alpha^2-1)}\zeta_1^{\frac{1}{2}}(\alpha^2+\zeta_1^2)^{\frac{1}{4}(\alpha^2-1)} - c_1\tau_1\,, \end{aligned}\right\} \tag{4. 3}$$

and t_1 can be determined by (3. 5).

Next, we proceed to the determination of ε and δ. Since p, q and ρ must obey Bernoulli's equation in differential form: $q\,dq + dp/\rho = 0$, if we substitute (3. 3), (3. 4) and (3. 6) in this equation and put p_1 and ρ_1 equal to the corresponding quantities for the real gas respectively, we get following relation:

$$\sin(\varepsilon-\delta) + \frac{2\gamma}{\gamma-1}\,\frac{z_1}{1-z_1}\,\frac{\sin(t_1+\delta)}{t_1} \times \left\{\sin(t_1+\varepsilon) - t_1\cos(t_1+\varepsilon)\right\} = 0\,. \tag{4. 4}$$

Further, let the tangent to the ρ-q curve for our hypothetical gas at the point $q = q_1$, $\rho = \rho_1$, be coincident with the tangent to the ρ-q curve for the real gas. Then it follows that

$$\left\{\frac{t_1}{1-t_1\cot(t_1+\varepsilon)}\right\}^2 = \frac{2}{\gamma-1}\,\frac{z_1}{1-z_1} - 1\,. \tag{4. 5}$$

It will be seen that this condition is equivalent to assuming that for both our hypothetical gas and the real gas the respective Mach numbers at $q = q_1$ coincide with each other.

The five equations (4. 2)—(4. 5) enable us to determine five constants τ_1, c_1, c_2, ε, δ involved in the equations of state of our hypothetical gas. Since we have assumed that at $q = q_1$ the χ-τ curve coincides with the χ_a-τ_a curve up to their tangents, it is readily proved that the p-ρ curves for both gases coincide with each other up to their curvatures. Therefore, our present hypothetical gas is a better approximation to the real gas near the point $q = q_1$ than Kármán-Tsien's hypothetical gas. Fig. 1 shows the curve of p/p_1 plotted against ρ_1/ρ in a special case when $M_1 = \sqrt{5}$. It will be seen that our hypothetical gas approximates excellently the real gas.

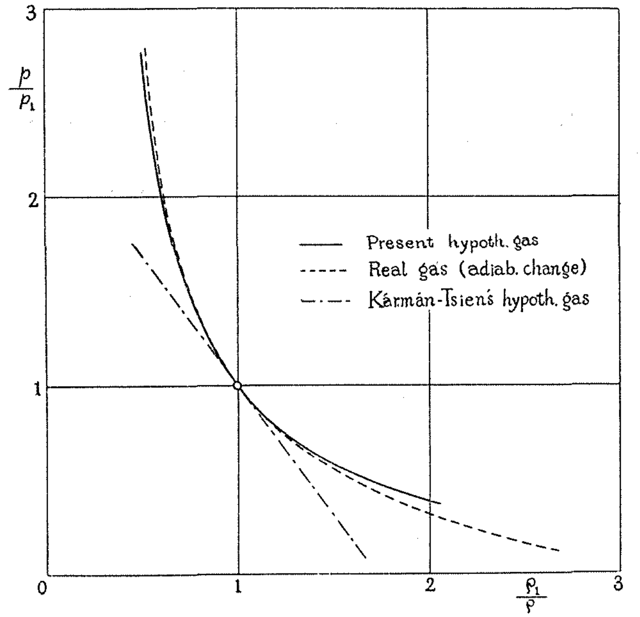

Fig. 1.

5. Solution for the supersonic free gas jet

The streamlines in the two-dimensional wavy flow of supersonic free jet are expressed, in the hodograph plane, as a family of closed curves round a point which corresponds to the mean velocity. It is well known that if a streamline touches one of the characteristics of the fundamental equation (2. 8), there appears in general a turn back of the streamline in the physical plane and a limit line is formed. It has been found, however, that when the envelope of the family of the streamlines in the hodograph plane coincides with one of the characteristics, the limit line does not appear in the physical plane. Therefore, we now search for a family of closed streamlines in the hodograph plane which satisfy the above condition.

For convenience, we introduce new variables ξ, η in place of τ, θ as follows:

$$\xi = \frac{1}{2\sigma}\left\{\tau_1 + \sigma - (\tau + \theta)\right\}, \quad \eta = \frac{1}{2\sigma}\left\{\tau_1 + \sigma - (\tau - \theta)\right\}. \quad (5.1)$$

As will be seen presently, σ is a constant which is connected with the deviation of the velocity vector from that of the mean-flow. The characteristics of the equation (2. 8) are expressed by the equations $\tau \pm \theta =$ const and accordingly also by $\xi =$ const and $\eta =$ const. Now, we consider in the hodograph plane a square region with centre at the point $\tau = \tau_1$, $\theta = 0$ whose sides are given by $\tau + \theta = \tau_1 \pm \sigma$ and $\tau - \theta = \tau_1 \pm \sigma$, namely by $\xi = 0, 1$ and $\eta = 0, 1$ (Fig. 2). We assume that all the streamlines in the hodograph plane are simply closed curves touching the sides of the square as shown in Fig. 3. Also we suppose

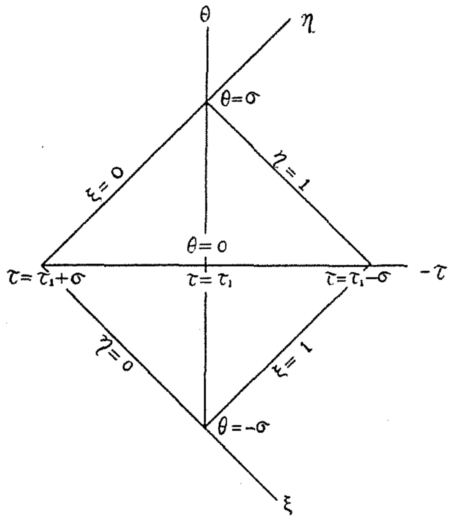

Fig. 2.

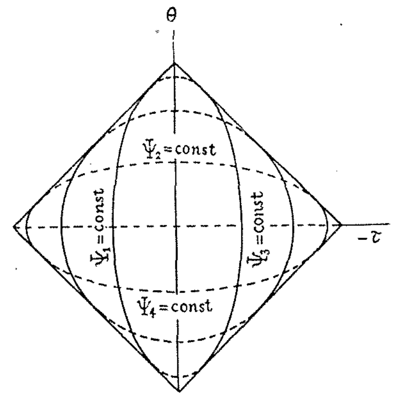

Fig. 3.

that these closed streamlines are transformed into the streamlines in a wavy flow of jet in the physical plane. Figs. 4 (a) and 4 (b) represent the correspondence between two planes. The origin of the physical plane is taken at the mid-point of the narrowest portion of the jet, and the x- and y-axes are taken parallel and perpendicular to the main-stream respectively. The line ABCD denotes the free-streamline boundary on which $q = q_1$, and it corresponds to the segment ABCD in the hodograph plane on which $\tau = \tau_1$. If we denote the points of the minimum and maximum velocities on the x-axis by O and O′ respectively, these correspond to the points $\tau = \tau_1 \pm \sigma$, $\theta = 0$ in the hodograph plane. The lines OB and BO′ in the xy-plane are the particular Mach lines passing through O and O′, and they correspond to the sides OB, BO′ of the square region in the $\tau\theta$-plane. The regions denoted by 1, 2, 3, 4 in the xy-plane correspond respectively to the triangular regions:

OBD ($\xi+\eta \leqq 1,\ \xi \geqq 0,\ \eta \geqq 0$), OBO′ ($\xi \leqq \eta,\ \xi \geqq 0,\ \eta \leqq 1$), BO′D ($\xi+\eta \geqq 1,\ \xi \leqq 1,\ \eta \leqq 1$) and O′DO ($\xi \geqq \eta,\ \xi \leqq 1,\ \eta \geqq 0$) in the hodograph plane. We denote the velocity potentials and the stream functions in these regions by Φ_1, Ψ_1; Φ_2, Ψ_2; Φ_3, Ψ_3; Φ_4, Ψ_4 respectively.

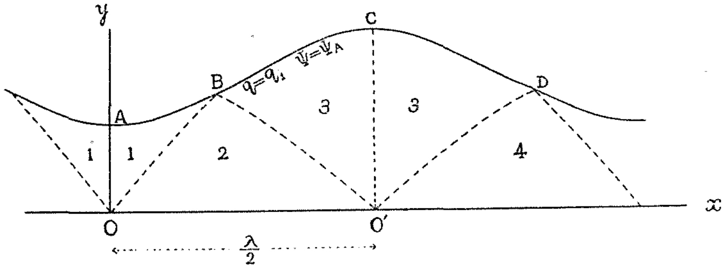

Fig. 4. (a)

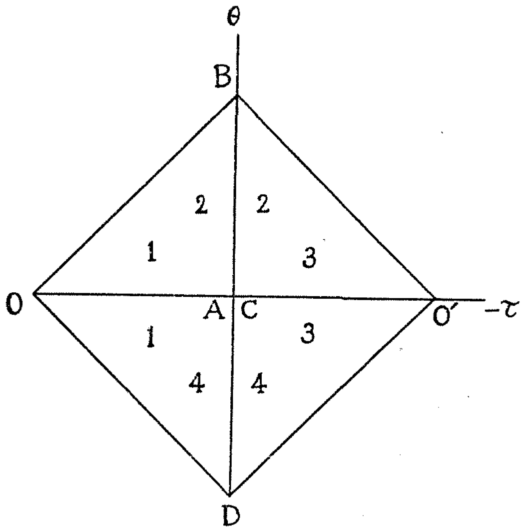

Fig. 4. (b)

To determine $\Psi_i (i = 1, 2, 3, 4)$ we shall take account of the physical conditions that the flow in the physical plane is symmetrical with respect to both the x-, y-axes and that the flow is bounded by two free-stream-lines. These conditions, together with the continuity of the stream-lines, are stated as follows:

$$\left.\begin{array}{ll}\text{(i)} & \Psi_1(\xi,\eta)=\Psi_1(\eta,\xi),\ \Psi_1(\xi,1-\xi)=\text{const},\ \Psi_1(0,0)=0\,;\\ \text{(ii)} & \Psi_2(0,\eta)=\Psi_1(0,\eta),\quad \Psi_2(\xi,\xi)=0\,;\\ \text{(iii)} & \Psi_3(\xi,1)=\Psi_2(\xi,1),\quad \Psi_3(\xi,1-\xi)=\Psi_1(\xi,1-\xi)\,;\\ \text{(iv)} & \Psi_4(1,\eta)=\Psi_3(1,\eta),\quad \Psi_4(\xi,\xi)=0\,.\end{array}\right\}\tag{5.2}$$

These conditions determine Ψ_i's as follows:

$$\left.\begin{array}{ll}\Psi_1=\chi^{-1}\{F(\xi)+F(\eta)\}\,, & \Psi_2=\chi^{-1}\{-F(\xi)+F(\eta)\}\,,\\ \Psi_3=\chi^{-1}\{-F(\xi)-F(\eta)+2k\}\,, & \Psi_4=\chi^{-1}\{F(\xi)-F(\eta)\}\,,\end{array}\right\}\tag{5.3}$$

where

$$F(+0)=0\,,\quad F(\xi)+F(1-\xi)=\text{const}=k\quad(0\leqq\xi\leqq 1)\,.\tag{5.4}$$

Also, Φ_i's are obtained in the forms:

$$\left.\begin{array}{l}\Phi_1=\chi\{-F(\xi)+F(\eta)\}-2c_1\sigma\int_0^\xi F(\xi)\,d\xi+2c_1\sigma\int_0^\eta F(\eta)\,d\eta\,,\\ \Phi_2=\chi\{F(\xi)+F(\eta)\}+2c_1\sigma\int_0^\xi F(\xi)\,d\xi+2c_1\sigma\int_0^\eta F(\eta)\,d\eta\,,\\ \Phi_3=\chi\{F(\xi)-F(\eta)\}+2c_1\sigma\int_0^\xi F(\xi)\,d\xi-2c_1\sigma\int_0^\eta F(\eta)\,d\eta\\ \qquad\qquad -2c_1\sigma k(\xi-\eta)+2k(c_1\tau_1+c_2)\,,\\ \Phi_4=\chi\{-F(\xi)-F(\eta)\}-2c_1\sigma\int_0^\xi F(\xi)\,d\xi-2c_1\sigma\int_0^\eta F(\eta)\,d\eta\\ \qquad\qquad +c_1\sigma k+4k(c_1\tau_1+c_2)\,.\end{array}\right\}\tag{5.5}$$

6. Conditions for F

As for the conditions on the branch-lines which are the Mach lines passing through O and O′, we have only assumed that the stream function and the velocity potential are continuous. Therefore, although the velocity vector and accordingly the tangent to the stream-line are both continuous near the branch-line, it is not yet certain whether the velocity gradient and the curvature of the streamline are continuous or not, and whether the turn back of the streamline occurs or not.

If we denote the line element of a streamline by ds, the velocity gradient along the streamline and the curvature of the streamline are expressed respectively by

$$\frac{dq}{ds} = \frac{1}{2}\sigma q \frac{dq}{d\tau}\frac{1}{\chi^2}\frac{\Psi_\eta - \Psi_\xi}{\Psi_\xi \Psi_\eta},$$

$$\frac{d\theta}{ds} = \frac{1}{2}\sigma q \frac{1}{\chi^2}\frac{\Psi_\xi + \Psi_\eta}{\Psi_\xi \Psi_\eta}.$$

Substituting (5. 3) in these equations, we get, on the branch-line $\xi = 0$,

$$\left(\frac{dq}{ds}\right)_{\xi=+0} = \frac{1}{2}\sigma\left(q\frac{dq}{d\tau}\right)_{\xi=0}\frac{(\chi)_{\xi=0}[F'(\eta) \mp F'(+0)]}{[c_1\sigma F(\eta) + (\chi)_{\xi=0}F'(\eta)][c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F'(+0)]},$$

$$\left(\frac{d\theta}{ds}\right)_{\xi=+0} = \frac{1}{2}\sigma(q)_{\xi=0}\frac{2c_1\sigma F(\eta) + (\chi)_{\xi=0}[F'(\eta) \pm F'(+0)]}{[c_1\sigma F(\eta) + (\chi)_{\xi=0}F'(\eta)][c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F'(+0)]},$$

where the upper of the double signs denotes the limit from the region 1, while the lower the limit from the region 2. Accordingly, in order that the velocity gradient and the curvature of the streamline may be continuous, it must be that

$$F'(+0) = 0 \qquad \text{or} \qquad F'(+0) = \infty.$$

The same conditions are also obtained when we consider the other branch-lines: $\eta = 1$, $\xi = 1$ and $\eta = 0$.

Next, along a streamline we get

$$\left(\frac{dx}{d\xi}\right)_{\xi=+0} = -2\left(\frac{\cos\theta}{q}\right)_{\xi=0}[c_1\sigma F(\eta) \pm (\chi)_{\xi=0}F'(+0)],$$

the double signs being of the same meaning as before. Thus, when $F'(+0) = 0$, $dx/d\xi$ is finite and of equal amount when approached to $\xi = 0$ from either region, and therefore the streamline turns back in the physical plane. On the other hand, when $F'(+0) = +\infty$, we have $(dx/d\xi)_{\xi\to+0} \to \mp\infty$, and therefore the streamline does not turn back and a smooth flow is obtained. Hence, we impose upon the function F the following conditions:

$$F(+0) = 0,\ F'(+0) = +\infty,\ F(\xi) + F(1-\xi) = k\ (0 \leqq \xi \leqq 1). \quad (6.\ 1)$$

If we adopt an appropriate one-valued, continuous and smooth function $F(\xi)$ satisfying these conditions, we get, by starting with the stream functions Ψ_i's given in (5. 3), a supersonic free jet flow which is continuous up to the velocity gradient and the curvature of the streamline.

7. An example

As an example, we assume that

$$F(\xi) = \sqrt{\xi} - \sqrt{1-\xi} + 1\,, \tag{7.1}$$

which satisfies the conditions (6.1). Substituting this into (5.3) and (5.5), we obtain the expressions for Ψ_i's and Φ_i's. Transformation into the physical plane is made by integrating the following equations:

$$dx = \frac{1}{q}\cos\theta\, d\Phi - \frac{\rho_0}{\rho q}\sin\theta\, d\Psi\,,$$

$$dy = \frac{1}{q}\sin\theta\, d\Phi + \frac{\rho_0}{\rho q}\cos\theta\, d\Psi\,.$$

It is found that the flow thus obtained expresses in fact a supersonic free jet changing its state of motion periodically in the direction of the x-axis. We denote the wave length of the jet, its mean breadth, the deviation of the breadth from its mean value by λ, b and Δ respectively. Then, they are defined by

$$\lambda = 2x_{O'}\,, \qquad b = y_A + y_C\,, \qquad \Delta = y_C - y_A\,, \tag{7.2}$$

where suffixes O′, A, C denote the points at which the values of the coordinates are to be taken (Fig. 4(a)). For the flow deduced from (7.1), it follows that when σ is small,

$$\left.\begin{aligned} \frac{2b}{\lambda} &= \frac{1}{\sqrt{M_1^2-1}}\left\{1 + \frac{7(\gamma+1)M_1^4}{120(M_1^2-1)}\sigma^2 + O(\sigma^4)\right\}, \\ \frac{2\Delta}{\lambda} &= \frac{1}{3}(2\sqrt{2}-1)\,\sigma + \frac{1}{315}(25\sqrt{2}-11)\,\sigma^3 + O(\sigma^5)\,, \end{aligned}\right\} \tag{7.3}$$

M_1 being the Mach number associated with the free-streamline on which $q = q_1$.

Numerical calculations have been worked out for a particular case when

$$M_1 = \sqrt{5} \qquad \text{with} \qquad \gamma = 1.4\,,$$

i. e., when

$$\zeta_1 = 2\,, \qquad z_1 = \frac{1}{2} \qquad \text{and} \qquad \alpha = \sqrt{6}\,.$$

In this case, constants τ_1 etc. are obtained as follows:

$$\tau_1 = -0.570077\,, \qquad t_1 = -0.533333\,,$$
$$c_1 = -6.306724\,, \qquad c_2 = -0.231732\,,$$
$$\epsilon = -0.095463\,, \qquad \delta = 0.973171\,.$$

The constant σ, which is a measure of the deviation of the velocity from the mean velocity, has been chosen as

$$\sigma = 0.2\,,$$

when the speed of flow varies in the range $0.884 < q/q_1 < 1.097$.

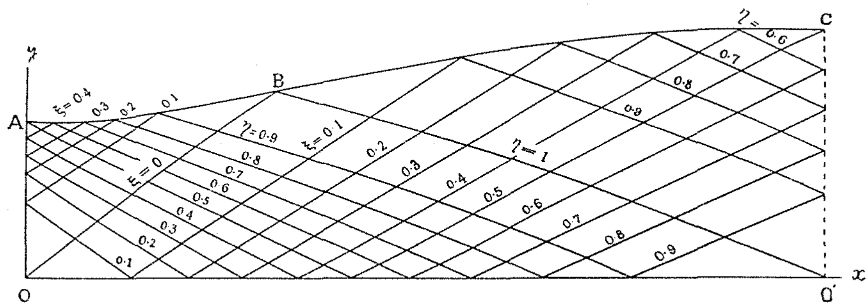

Fig. 5. Two families of Mach lines.

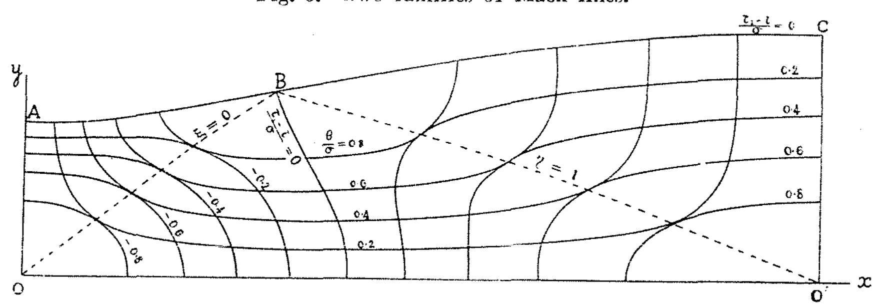

Fig. 6. Curves of $\tau = $ const and $\theta = $ const.

Fig. 5 shows the two families of the Mach lines. The Mach lines marked with $\xi = 0$ and $\eta = 1$ represent the branch-lines. Fig. 6 shows the curves of the constant speed ($\tau = $ const) and those of the constant direction ($\theta = $ const). These two families contact with each other on the branch-lines.

8. Appendix

Starting with the solution obtained in § 5, we get the relations between λ, b and Δ as follows:

$$
\left.\begin{aligned}
\frac{2b}{\lambda} &= \frac{1}{\sqrt{M_1^2-1}} + \frac{(\gamma+1)M_1^4}{(M_1^2-1)^{\frac{3}{2}}}\frac{\sigma S}{k-4\sigma S}, \\
\frac{2\Delta}{\lambda} &= \frac{4\sigma C}{k-4\sigma S},
\end{aligned}\right\} (8.1)
$$

where

$$
S = \int_0^{\frac{1}{2}} F(\xi)\sin\sigma(1-2\xi)d\xi\,, \quad C = \int_0^{\frac{1}{2}} F(\xi)\cos\sigma(1-2\xi)d\xi\,.
$$

Also, we denote by Q the quantity of mass carried by the jet in unit time, which may be defined by $Q = 2\rho_0\Psi_A$ with the value Ψ_A of the stream function on the free streamline ABCD. Then we obtain, after some reductions,

$$
Q = bq_1\rho_1\left\{1 - \frac{4(M_1^2-1)-(\gamma+1)M_1^4}{(\gamma+1)M_1^4\sqrt{M_1^2-1}}\left(\frac{\lambda}{2b} - \sqrt{M_1^2-1}\right)\right\}. \quad (8.2)
$$

When σ tends to zero, it follows that

$$
\frac{2\Delta}{\lambda} \to 0\,, \qquad Q \to bq_1\rho_1\,, \qquad \frac{\lambda}{2b} \to \sqrt{M_1^2-1}\,,
$$

which are the well-known results of Prandtl.

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