Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXVI, No. 3, Article 20, 1950.

On the Detached Shock Wave in front of a Body moving at Speeds greater than That of Sound.

By

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(Received September 25, 1950)

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1. Introduction and Summary

As is well known, when a body like a bullet moves through the air at speeds greater than that of sound, a shock wave is formed in front of the body, and if the velocity of the body is constant, the position of the wave relative to the body remains unaltered. The formation of such a shock wave implies that there is an abrupt change of density and velocity. Ahead of this surface of discontinuity the flow of air is stationary and behind it there is a continuous field of fluid flow which may contain further shock waves.

The equations governing the propagation of the shock wave were established by Rankine **(1)** and his equations determine the relationship between the conditions both in front of and behind a plane shock wave. Meyer has developed Rankine's relations **(2)** and studied the flow in the neighbourhood of an inclined plane or a wedge moving at high supersonic speeds. Later, his analysis was reproduced by Ackeret **(3)**, who gave a photograph of the flow in the neighbourhood of a wedge showing that the régime postulated by Meyer in which a shock wave attaches the tip of the body does in fact occur. The limitations of the solutions given by these writers were studied by Bourquard and G. I. Taylor **(4)**. Taylor has shown that for a given two-dimensional bullet with semi-vertical angle less than a certain critical value, the necessary condition for the existence of Meyer's régime is that the Mach number of the undisturbed stream should exceed a certain critical value. He has obtained a series of these critical Mach numbers for two-dimensional bullets with various semi-vertical angles, and gave a curve showing the relationship between the semi-vertical angle of the bullet and the said critical Mach number.

Recent developments of the high-speed aeroplanes have stimulated the studies on supersonic aerodynamics and various important investigations have been carried out. Almost all of them, however, are concerned with Meyer's régime, and owing to great mathematical difficulties, no complete theoretical investigations have yet been made so far on the régime in which a detached shock wave is formed in front of an obstacle.

The relationship between the stagnation pressure at the nose of a body and the undisturbed flow ahead of the normal shock wave was discussed by Lord Rayleigh (5) and his result is widely used in experimental researches.

Some time ago, interesting wind tunnel experiments and shooting tests were made by M. Sugimoto (6) and several excellent photographs showing detached shock wave in front of a circular cylinder and a sphere were obtained. Some of them are reproduced on Plate 1 of the present paper for the sake of reference. He also deduced a formula of somewhat empirical nature to locate the detached shock waves, by assuming that the flow behind the shock is equivalent to the subsonic flow of an incompressible fluid subject to appropriate boundary conditions. Thus, assuming a usual potential flow for the subsonic region, he calculated the pressure distribution on the surface of the body, and then comparing the values thus calculated with his own experimental data, he determined the Mach number of the subsonic flow in such a way that the point at which the local speed of sound is first attained is the same in the two cases. Thus, the velocity or pressure distribution in the subsonic region was uniquely determined, and the location of the shock wave was then determined by Rankine's shock wave equations. However, Sugimoto's formula requires empirical data and therefore the scope of its application is limited.

In a recent note, E. V. Laitone (7) has also attacked the problem. He has assumed that the flow behind a shock wave is a part of an incompressible potential flow with the appropriate boundary conditions. The undisturbed velocity of such a potential flow has been taken to be equal to that of the actual stream, while the flow pattern has been assumed to be the same as the potential flow pattern. This assumption determines the complete velocity distributions in the field of flow, while Rankine's relations determine the velocity drop immediately behind the normal portion of the shock wave for a given Mach number of the actual stream, and these conditions locate the normal portion of the detached shock wave. In Laitone's theory no empirical data are needed, but the assumed flow is not self-consistent, because the flow pattern is assumed to be the same as in the case of subsonic potential flow, notwithstanding that in conformity with the state of affairs in the actual flow the fluid flows with a supersonic speed at infinity upstream. These drawbacks restrict the scope of application of Laitone's theory, and especially it cannot be applied to the case when the Mach number of the actual flow is considerably great.

Quite recently, K. Tamada (8) has also discussed a detached shock wave in front of a circular cylinder and a sphere. He has assumed that a curved shock wave is composed of an infinite number of infinitesimal line-segments and that Rankine's relations should be satisfied at any portion of the curved shock wave. Generally speaking, the curved shock wave will give rise to different amounts of deflection to different streamlines as they pass through it and consequently a certain amount of vorticity will necessarily be generated behind the shock wave. In the close vicinity of the normal portion of the shock wave, however, deflections of the streamlines are small in magnitude and the generated vorticity may be small, too. Thus, if we confine our attentions to the close neighbourhood of the normal portion, the subsonic region behind the shock wave may be approximated by a potential flow. In Tamada's analysis, no unreasonable assumptions have been made on the flow behind the shock wave, because the Mach number of the undisturbed subsonic flow could be determined by means of Rankine's relations.

In the present paper an attempt is made to the discussion of a detached shock wave in front of a body, such as a circular cylinder, a sphere, and a two-dimensional bullet from somewhat different point of view. The flow behind the detached shock wave is approximated by a potential flow as in the papers of previous writers and the effect of the curvature of the shock is considered. In accordance with the results of observations, the shape of the detached shock wave is approximated by a kind of quadratic curve.

Strictly speaking, the application of the present theory is also restricted to the close vicinity of the normal portion, because the equation of continuity is not satisfied exactly. Nevertheless the numerical values given by the present theory are in fairly good agreement with the experimental evidences, and some interesting results are also obtained. Comparing the calculated shape of the detached shock waves in front of a circular cylinder and a sphere with the corresponding observed shock waves it is found that the present theory gives a satisfactory approximation to the actual state of affairs.

The theory is then extended to locate a detached shock wave in front of a two-dimensional bullet. The boundary conditions of the bullet are approximated by Bobyleff's discontinuous flow impinging symmetrically upon a bent lamina. Thus, the head of the bullet is approximated by the bent lamina, while its parallel portion is approximated by the region bounded by two free streamlines starting from the ends of the lamina.

It is found that when the semi-vertical angle of the bullet is greater than $67°48.9'$, the shock wave can never attach the nose of the bullet but remains always detached for any value of the Mach number of the undisturbed stream. For a bullet with smaller semi-vertical angle than $67°48.9'$, a detached shock wave is formed in front of the bullet, if the Mach number of the oncoming stream is smaller than a certain critical value, which itself depends upon the value of the semi-vertical angle of the bullet. These results are in good qualitative agreement with Taylor's results.

The present problem has been suggested by Professor S. Tomotika, to whom the writer wishes to express his cordial thanks for his suggestion and continued interest throughout the work. The writer's thanks are also due to Dr. K. Tamada for his invaluable discussions.

2. Assumptions

Referring to Fig. 1, let U be the velocity of the oncoming undisturbed stream and let q and (u, v) be the magnitude of the velocity at any point and its rectangular components respectively. We denote the pressure, the density of the fluid and the velocity of sound in the undisturbed stream by p_1, ρ_1 and c_1 respectively. As is well known, these three quantities are connected by the relation $c_1^2 = \gamma p_1/\rho_1$ where γ is the ratio of the two specific heats of the fluid and takes the value 1.405 for air. Also, we denote by p_2 and ρ_2 the pressure and the density of the fluid immediately behind the shock wave, and by p_0, ρ_0 the pressure and the density at zero velocity for the stream defined by (U, p_1, ρ_1). Further, let α denote the angle between the normal to the shock wave and the streamlines ahead of it and let α denote the angle of deflection from the undisturbed stream of the streamlines immediately behind the shock wave.

In the present paper, a simple assumption is made to the solution of the equation of fluid motion near a solid body. We shall confine our attentions to the case when the body has an axis of symmetry and the direction of the undisturbed stream is parallel to it. In this case, a shock wave formed in front of the body, either attached or detached, is symmetrical with respect to the axis of symmetry. A particular streamline coinciding with the axis of symmetry experiences no deflection, while the deflection of streamlines from the undisturbed stream becomes greater as the inclination of the shock wave becomes greater. As a consequence of the deflection of streamlines, certain amount of vorticities will be generated in reality behind the curved shock wave and the flow behind the shock wave is no more irrotational. But, if we confine our attentions to the close vicinity of the axis of symmetry, the deflection of streamlines from the undisturbed stream is small and the generated vorticity behind the shock wave may be neglected, and therefore, the flow behind the curved shock wave may be approximated by a potential flow subject to suitable boundary conditions.

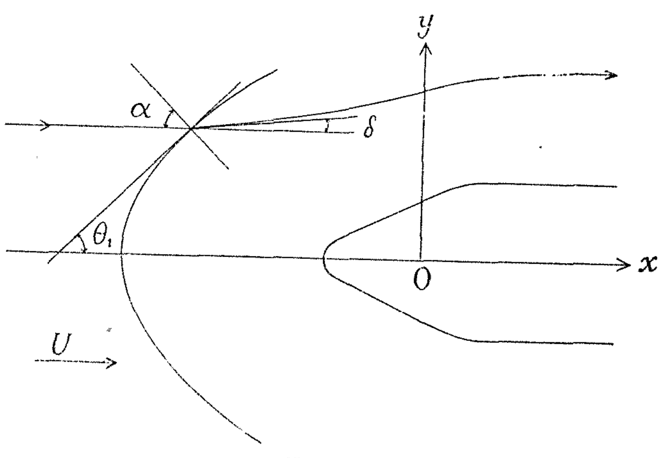

Fig. 1.

If, then, the velocity potential for such an irrotational flow be determined, the velocity distributions behind the shock wave is uniquely determined, and the angle of deflection from the undisturbed stream of the streamlines immediately behind the shock wave is determined by the relation:

$$\tan \delta = v/u. \qquad (1)$$

3. The shock wave equations

Meyer's equations governing the conditions on the two sides of an oblique shock wave are given by

$$\cos^2 \alpha = \frac{\{(\gamma - 1) + (\gamma + 1)\, p_2/p_1\}\,(\gamma - 1)}{4\gamma\{(p_0/p_1)^{(\gamma-1)/\gamma} - 1\}}, \tag{2}$$

$$\tan(\alpha + \delta) = \frac{(\gamma - 1) + (\gamma + 1)\, p_2/p_1}{(\gamma - 1)\, p_2/p_1 + (\gamma + 1)} \tan \alpha\,. \tag{3}$$

If we denote the Mach number of the undisturbed stream M so that $M = U/c_1$, Bernoulli's theorem gives immediately

$$M^2 = \frac{2}{\gamma - 1}\left\{\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right\}. \tag{4}$$

Making use of this and eliminating α and p_2/p_1 from the above two equations (2) and (3), we obtain an equation for determining the angle of deflection, δ, of the streamlines as they pass through the shock wave, namely:

$$\tan \delta = \frac{2}{\tan \theta_1} \frac{(M^2 - 1)\tan^2\theta_1 - 1}{(\gamma - 1) M^2 \tan^2\theta_1 + (\gamma + 1) M^2 + 2 + 2\tan^2\theta_1}, \tag{5}$$

where θ_1 is the angle which a tangent to the shock wave on its arbitrary point makes with the direction of the undisturbed stream and is evidently connected with α by the relation $\theta_1 + \alpha = \pi/2$.

Our present purpose is to find out whether a curved shock wave in front of an obstacle is capable of changing the uniform stream into a potential flow whose direction immediately behind the wave is given by equation (1).

In conformity with the results of observations, the detached shock wave ahead of a body can be approximated by a surface of simple form. Since the shape of the section of the detached shock wave by a plane containing the axis of symmetry is similar to a parabola, we shall assume that it can be expressed as:

$$y^2 = a(x + t), \tag{6}$$

where the coordinates x, y have been chosen as shown in Fig. 1. The two parameters a and t in this equation are determined by Meyer's equations (2) and (3) above referred to. As will readily be seen, the latter parameter t represents the distance of the normal portion of the shock wave from the origin of the coordinate-axes.

The coordinates of the normal portion of the shock wave at the intersection of the wave and the axis of symmetry are given by $(-t, 0)$. In the close vicinity of this normal portion, the coordinates of a point on the shock wave are $(-t + \Delta x, \Delta y)$. A simple calculation gives immediately

$$\tan\theta_1 = \frac{a}{2\Delta y}. \tag{7}$$

Substituting this in equation (5), $\tan\delta$ can be expressed in the form of a series in powers of Δy as:

$$\tan\delta = \frac{4(M^2-1)\Delta y}{a\{(\gamma-1)M^2+2\}}\left\{1 - \frac{4(\gamma+1)M^4}{a^2(M^2-1)\{(\gamma-1)M^2+2\}}(\Delta y)^2 + \mathrm{O}[(\Delta y)^4]\right\}. \tag{8}$$

Equation (1) can also be expressed in the form of a series similar to (8), but the coefficients in the series are different according to the form of the obstacle. Comparing the two series expansions for $\tan\delta$ and equating the coefficients of Δy and $(\Delta y)^3$, we obtain two equations for determining the two parameters a and t in equation (6).

A constant factor U_1 in the expression for the velocity potential is determined such that another Rankine's relation connecting the magnitude of the velocities in front of and immediately behind the shock wave is satisfied. If we denote by q_2 the magnitude of the velocity immediately behind the shock wave, the said Rankine's relation may be written as:

$$\frac{q_2}{U} = \frac{\sin\theta_1}{\sin(\theta_1-\delta)}\,\frac{(\gamma-1)p_2/p_1+(\gamma+1)}{(\gamma+1)p_2/p_1+(\gamma-1)}, \tag{9}$$

with

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M^2\sin^2\theta_1 - \frac{\gamma-1}{\gamma+1}.$$

At the normal portion of the shock wave we have

$$\sin\theta_1 = \sin(\theta_1 - 1) = 1$$

and equation (9) is reduced to

$$\frac{q_2}{U} = \frac{1}{\gamma+1}\left\{(\gamma-1) + \frac{2}{M^2}\right\}, \tag{10}$$

which is identical with one of the conditions used by Laitone (7).

The practical examples will be illustrated in the following sections.

4. The detached shock wave in front of a circular cylinder

We choose the coordinate-axes as shown in the inset of Fig. 2. For convenience, we take the radius of the circular cylinder as unity. The velocity potential ϕ for the continuous irrotational flow past the circular cylinder is given by

$$\phi = -U_1\left(r + \frac{1}{r}\right)\frac{x}{r}, \tag{11}$$

where $r^2 = x^2 + y^2$.

The rectangular components of velocity at any point are given by

$$u = U_1\left\{1 - \frac{x^2 - y^2}{(x^2+y^2)^2}\right\}, \quad v = -U_1\frac{2xy}{(x^2+y^2)^2}. \tag{12}$$

Substituting these in (1) we have

$$\tan\delta = \frac{-2xy}{(x^2+y^2)^2 - x^2 + y^2}. \tag{13}$$

At a point $((\Delta y)^2/a - t, \Delta y)$ on the shock wave near its normal portion, $\tan\delta$ can be expanded in powers of Δy in the form:

$$\tan\delta = \frac{2\Delta y}{t^2(t^2-1)}\left\{1 - \frac{(2t^2+1)a - 3t^2 + 1}{at^2(t^2-1)}(\Delta y)^2 + O[(\Delta y)^4]\right\}, \tag{14}$$

Comparing this with (8) and equating the coefficients of Δy and $(\Delta y)^3$, we obtain, after some calculations,

$$a = \frac{2(M^2-1)t(t^2-1)}{(\gamma-1)M^2+2} = \frac{t}{2t^2+1}\left\{\frac{2(\gamma+1)M^4}{(M^2-1)^2} + 3t^2 - 1\right\}, \tag{15}$$

and these two equations determine the values of the two parameters a and t.

Eliminating a from these equations, we get

$$t^2 = \frac{(3\gamma-1)M^2+4}{8(M^2-1)} + \frac{1}{8(M^2-1)^{\frac{3}{2}}}\left\{(41\gamma^2 - 22\gamma + 17)M^6 - 3(3\gamma^2 - 42\gamma + 35)M^4 - 40(\gamma-5)M^2 - 80\right\}^{\frac{1}{2}}, \tag{16}$$

and by this equation we can calculate the value of the distance of the normal portion of the shock wave from the centre of the circular cylinder.

If the value of t thus determined is substituted in either of the two equations (15), the value of a can be found, and further, if use is made of the value of a thus obtained, the whole shape of the detached shock wave can be found.

Before proceeding to the numerical discussions, we shall consider two limiting cases. In the first place, when the Mach number, M, of the undisturbed stream increases to infinity, equations (15) are reduced to

$$a = \frac{2t(t^2-1)}{\gamma - 1} = \frac{t(3t^2 + 2\gamma + 1)}{2t^2 + 1}, \tag{17}$$

and equation (16) takes a simple form as:

$$t^2 = \frac{3\gamma - 1}{8} + \frac{1}{8}(41\gamma^2 - 22\gamma + 17)^{\frac{1}{2}}, \tag{18}$$

which gives the minimum value of t.

In the case of flow of the air for which $\gamma = 1.405$, it can readily be shown that this minimum value of t is greater than unity, and this implies that even when the Mach number increases to infinity, the shock wave does never attach the circular cylinder and remains always detached.

In the other limiting case when the Mach number M of the undisturbed stream tends to unity, equations (15) give immediately the results that $a = 0$ and $t \to \infty$. Thus, in this case the detached shock wave recedes to infinity upstream and its shape becomes a straight line perpendicular to the direction of the undisturbed stream.

Numerical values of the distance, t, of the normal portion of the detached shock wave from the centre of the cylinder have been calculated by the formula (16) for various values of M and are shown in Table I, where Sugimoto's experimental values and Laitone's theoretical values are also given for comparison. Laitone's values have been calculated by the present writer with the aid of his formula. The results are also shown in Fig. 2, where a dotted-line curve gives Laitone's theoretical values and the crosses show Sugimoto's experimental values. It will be seen that the present theory gives a better approximation to the results of observation than Laitone's theory.

Fig. 3 shows the shape of the detached shock wave in two cases in which M is equal to 2.04 and 2.83 respectively. Observed shock waves are shown by dotted-line curve for comparison (cf. Plate 1). It is to be noted that although the approximation of the present theory is fairy good only in the close vicinity of the normal portion of the shock wave

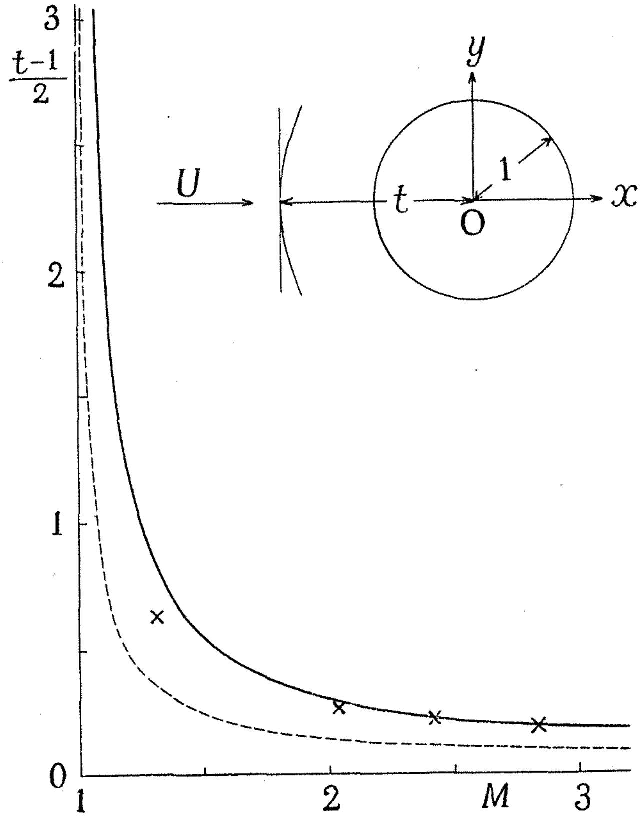

Fig. 2. Case of a circular cylinder. —— Present theory
······ Laitone's theory
× Experiments (Sugimoto)

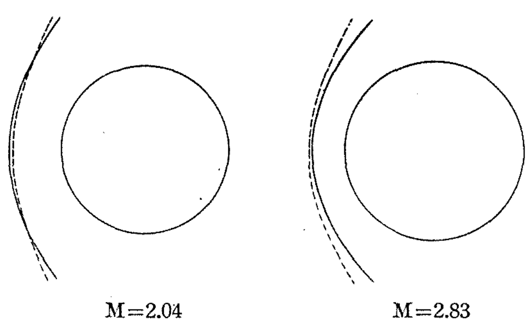

Fig. 3. Case of a circular cylinder.

but not so good at a finite distance from the normal portion, yet the agreement between the results of the present theory and those of observation is fairy satisfactory.

TABLE I. Values of $(t-1)/2$

Mach number	1.00	1.10	1.34	2.04	2.41	2.83	3.00	∞
Present theory	∞	2.099	0.760	0.283	0.221	0.182	0.172	0.097
Laitone's theory	∞	0.816	0.324	0.129	0.103	0.086	0.082	0.048
Sugimoto's experiments	—	—	0.622	0.261	0.219	0.191	—	—

5. The detached shock wave in front of a sphere

In this section a similar discussion will be made on the location of a detached shock wave in front of a sphere. Taking the centre of the sphere of unit radius as the origin of the coordinate-axes, let the cylindrical coordinates be denoted by x, y and ϑ, instead of usual z, r and θ, the axis of x being taken parallel to the positive direction of the undisturbed stream as shown in the inset of Fig. 4.

In this case the flow is evidently symmetrical about the axis of x, so that the flow pattern is independent of ϑ and it is sufficient to discuss the matter in the xy-plane only. Since the shape of a detached shock wave is approximated by a paraboloid of revolution about the axis of x and hence the section of the wave in the xy-plane is approximated by equation (6).

The velocity potential for the continuous irrotational flow of an incompressible fluid past a sphere is given by

$$\phi = -U_1\left(r + \frac{1}{2r^2}\right)\frac{x}{r}, \tag{19}$$

where $r^2 = x^2 + y^2$.

The components of velocity at any point are

$$u = \frac{U_1}{2r^5}(2r^5 - 2x^2 + y^2), \quad v = -\frac{3U_1}{2r^5}xy, \tag{20}$$

and therefore the angle of deflection, δ, of a streamline at any point from the undisturbed stream is given by

$$\tan\delta = -\frac{3xy}{2r^5 - 2x^2 + y^2}. \tag{21}$$

When the point under consideration lies on the shock wave, x and y must satisfy the relation (6). In the close vicinity of the normal portion of the shock wave, $\tan\delta$ can be expanded in a power series of Δy in the form:

$$\tan\delta = \frac{3\Delta y}{2t(t^3-1)}\left\{1 - \frac{(5t^3+1)a - 8t^4 + 2t}{2at^2(t^3-1)}(\Delta y)^2 + O[(\Delta y)^4]\right\}. \quad (22)$$

Thus, comparing the two expansions (8) and (22) for $\tan\delta$ and equating the coefficients of Δy and $(\Delta y)^3$, we get, after some calculations,

$$a = \frac{8(M^2-1)t(t^3-1)}{3\{(\gamma-1)M^2+2\}} = \frac{t}{5t^3+1}\left\{\frac{3(\gamma+1)M^4}{(M^2-1)^2} + 8t^3 - 2\right\}. \quad (23)$$

Eliminating a from these equations, we have

$$t^3 = \frac{(3\gamma+1)M^2+2}{10(M^2-1)} + \frac{3}{10\sqrt{2}(M^2-1)^{\frac{3}{2}}}\{(7\gamma^2-2\gamma+3)M^6 - 2(\gamma-1)(\gamma-8)M^4 - 6(\gamma-5)M^2 - 12\}^{\frac{1}{2}}, \quad (24)$$

and the distance, t, of the normal portion of the shock wave from the centre of the sphere can be calculated by this equation.

In a limiting case when the Mach number of the undisturbed stream tends to infinity, equations (23) and (24) are reduced respectively to

$$a = \frac{8t(t^3-1)}{3(\gamma-1)} = \frac{t(8t^3+3\gamma+1)}{5t^3+1}, \quad (25)$$

and

$$t^3 = \frac{3\gamma+1}{10} + \frac{3}{10\sqrt{2}}(7\gamma^2-2\gamma+3)^{\frac{1}{2}}, \quad (26)$$

the latter giving the minimum value of t.

In the case of flow of the air for which the value of γ is 1.405, the minimum value of t as given by (26) is easily found to be larger than unity. Thus, it is found that as in the case of a circular cylinder, the detached shock wave is always formed in front of the sphere for any value of the Mach number M of the undisturbed stream.

Numerical values of the distance, t, of the normal portion of the detached shock wave have been calculated and are shown in Table II.

The similar values have also been calculated on the basis of Laitone's theory, and they, together with Sugimoto's experimental values, are shown in the table for comparison.

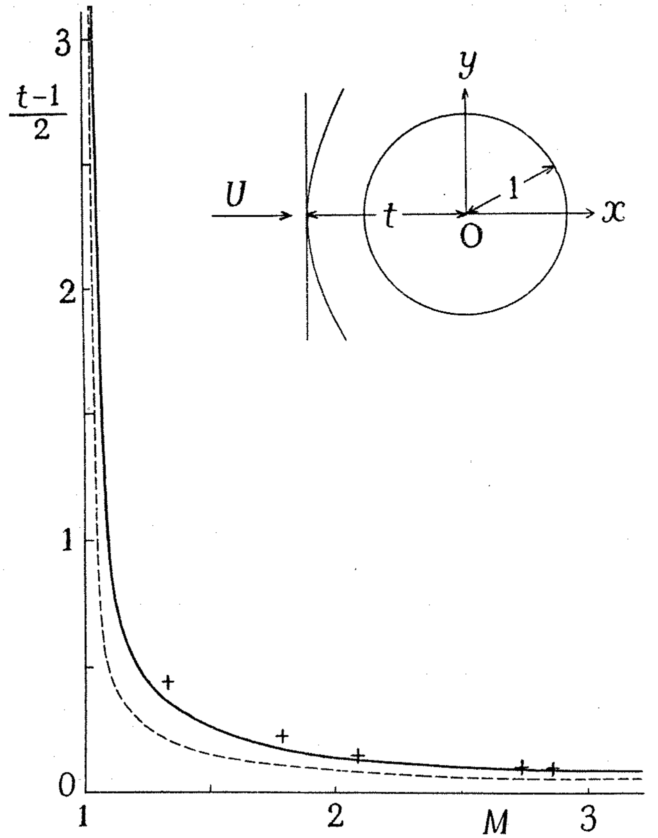

Fig. 4. Case of a sphere. —— Present theory
······ Laitone's theory
$+$ Experiments (Sugimoto)

TABLE II. Value of $(t-1)/2$.

Mach number	1.00	1.10	1.32	1.78	2.08	2.40	2.83	∞
Present theory	∞	0.835	0.359	0.171	0.131	0.107	0.088	0.048
Laitone's theory	∞	0.453	0.207	0.103	0.080	0.067	0.056	0.032
Sugimoto's experiments	——	——	0.433	0.206	0.150	0.130	0.103	——

The results are also shown in Fig. 4. It will be seen that the results of the present theory are in good agreement with those of Sugimoto's observation.

Making use of the value of a as calculated by either of the two equations (23), the form of the detached shock wave has been calculated in two cases in which the value of the Mach number M is 2.08 and 2.63 respectively. They are shown in Fig. 5, where the dotted curves indicate the location of the shock waves found experimentally.

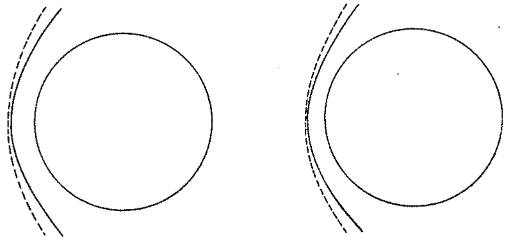

Fig. 5. Case of a sphere.

6. The shock wave in front of a two-dimensional bullet

In the present section the writer will discuss a detached shock wave in front of a two-dimensional bullet. The similar problem has been discussed by Laitone and Taylor. Laitone has approximated a bullet by an appropriate distribution of line sources along the axis of the bullet, while Taylor has approximated a bullet by a two-dimensional wedge.

The writer now approximates the tip portion of a two-dimensional bullet by a bent lamina and the parallel portion of it by a pair of free streamlines starting from the two edges of the lamina and extending to infinity downstream. Such a discontinuous flow past a bent lamina was discussed by Bobyleff as early as 1881 and short account of his theory is given several text-books on hydrodynamics (**9**). Recently, a detailed discussion on Bobyleff's problem has been made by S. Tomotika and Z. Hasimoto (**10**). Before proceeding further, we shall recapitulate here the essential part of Bobyleff's theory for the sake of later use.

We assume that a uniform flow impinges symmetrically upon a bent lamina $A_1 C A_2$ whose section consists of two equal straight lines forming an angle (Fig. 6 (a)). Let 2β be the angle measured on the downstream side, and also let λ_1 and λ_2 denote a pair of free streamlines starting from the two edges A_1 and A_2 of the lamina and extending to infinity downstream. Taking the plane of fluid motion as the z-plane, we take the x- and y-axes as shown in Fig. 6 (a).

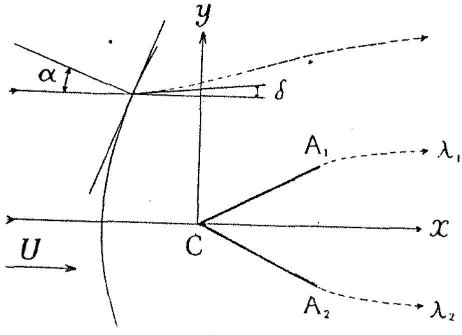

Fig. 6. (a) z-plane.

Let $f = \phi + i\psi$ be the complex velocity potential for the discontinuous flow under consideration. The f-plane is shown in Fig. 6 (b).

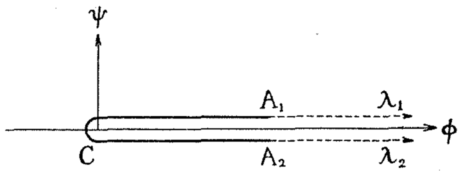

Fig. 6. (b) f-plane.

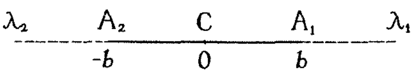

Fig. 6. (c) s-plane.

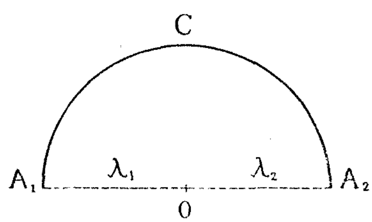

Fig. 6. (d) ζ-plane.

This f-plane is transformed into the upper-half of an s-plane by the transformation :

$$f = s^2 . \tag{27}$$

The two points A_1 and A_2 corresponding to the two edges of the lamina are then transformed into the points on the real axis of the s-plane whose coordinates are $(b, 0)$ and $(-b, 0)$ respectively.

The upper-half of the s-plane is transformed into the interior of a semi-circle of unit radius in a ζ-plane by the equation :

$$s = -\frac{b}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{28}$$

The bent lamina is transformed into the circumference of the semi-circle, while the free streamlines are transformed into a diameter which coincides with the real axis of the ζ-plane.

The conjugate complex velocity at any point in the flow is given by

$$\frac{df}{dz} = e^{-i\Omega}, \tag{29}$$

with

$$\Omega = \delta + i \log q , \tag{30}$$

where δ denotes the inclination to the axis of x of a streamline at any point and q is the magnitude of the velocity there.

If we can find Ω as a function of ζ, we can completely determine df/dz or the velocity distributions in the stream. The conditions which should be satisfied by the function $\Omega(\zeta)$ are as follows: (i) it must be an analytic function of ζ regular in the interior of the semi-circle $A_1C\,A_2$; (ii) on the circumference of the semi-circle the real part of $\Omega(\zeta)$ must satisfy the conditions:

$$\left.\begin{aligned}\Re\,[\Omega(\zeta)] &= -\beta\,(0 < \vartheta < \pi/2)\,,\\ &= \beta\,(\pi/2 < \vartheta < \pi)\,;\end{aligned}\right\} \tag{31}$$

and lastly, (iii) the imaginary part of $\Omega(\zeta)$ must vanish on the diameter $A_1C\,A_2$.

Such a function can be determined by Poisson's formula. Omitting the details of analysis for brevity, the final results only will be given. We have

$$\Omega(\zeta) = \frac{2\beta}{i\pi}\log\frac{i+\zeta}{i-\zeta}. \tag{32}$$

If we eliminate s and f from equations (27), (28), (29) and (32), we obtain

$$dz = \frac{b^2}{2}\left\{\frac{i+\zeta}{i-\zeta}\right\}^{2\beta/\pi}\left(\zeta^2 - \frac{1}{\zeta^2}\right)\frac{1}{\zeta}\,d\zeta\,, \tag{33}$$

and integration of this equation would give z as a function of ζ. Since, however, the analytical integration is almost hopeless, we have carried out numerical integration.

The half-length L of the bent lamina is given by

$$L = A_1C = 2b^2\left\{\frac{1}{2} + \frac{\beta}{\pi} + \frac{\beta^2}{\pi^2}\psi\left(\frac{2-\beta/\pi}{2}\right)\right.$$
$$\left. - \frac{\beta^2}{\pi^2}\psi\left(\frac{1-\beta/\pi}{2}\right)\right\}, \qquad (34)$$

where $\psi(x) = \dfrac{d}{dx}\log\Gamma(x)$. When the value of L is given, the value of the parameter b in equation (28) can be determined by equation (34).

Next, we shall proceed to the discussion of the location of a detached shock wave in front of a two-dimensional bullet, by applying the above results. In the close vicinity of the axis of x, the direction of a streamline is approximated as:

$$\tan\delta = \delta + \frac{1}{3}\delta^3 + \cdots\cdots, \qquad (35)$$

with
$$\delta = \Re\left[\Omega\left(\zeta\right)\right]. \qquad (36)$$

Referring to the coordinate-axes in Fig. 6 (a), let the coordinates of the normal portion of the shock wave be denoted by $(-t, 0)$. In the close vicinity of this normal portion, the coordinates of a point on the shock wave are given by $(-t + \Delta x, \Delta y)$, and since x and y must satisfy (6), $\Delta x = (\Delta y)^2/a$. Thus, if we write $\Delta z = \Delta x + i\Delta y$, we have

$$\Delta z = \frac{1}{a}(\Delta y)^2 + i\Delta y\,. \qquad (37)$$

It is readily found that the coordinates of a point in the ζ-plane which corresponds to the normal portion of the shock wave are given by $(0, i\eta_1)$. The increment $\Delta\zeta$ corresponding to Δz given by (37) is

$$\Delta\zeta = \left(\frac{d\zeta}{dz}\right)_{i\eta_1}\Delta z + \frac{1}{2}\left(\frac{d^2\zeta}{dz^2}\right)_{i\eta_1}(\Delta z)^2 + \frac{1}{6}\left(\frac{d^3\zeta}{dz^3}\right)_{i\eta_1}(\Delta z)^3 + \cdots\cdots$$
$$= \left(\frac{d\zeta}{dz}\right)_{i\eta_1}\left[\Delta z + \frac{1}{2}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1}(\Delta z)^2\right.$$
$$\left. + \frac{1}{6}\left\{\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} + \left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)^2_{i\eta_1}\right\}(\Delta z)^3 + \cdots\cdots\right]. \qquad (38)$$

The inclination δ of a streamline passing through a point in the close vicinity of the normal portion is given, by means of (36), in the form:

$$\delta = \Re\left\{\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1} \Delta\zeta + \frac{1}{2}\left(\frac{d^2\Omega}{d\zeta^2}\right)_{i\eta_1} (\Delta\zeta)^2 + \frac{1}{6}\left(\frac{d^3\Omega}{d\zeta^3}\right)_{i\eta_1} (\Delta\zeta)^3 + \cdots\cdots\right\}. \quad (39)$$

Thus, if use is made of (37) and (38), we easily obtain the series expansion of δ in powers of Δy, and if the series for δ thus obtained be inserted in the right-hand side of (36), $\tan\delta$ can be expressed in the form of a series in powers of Δy as:

$$\begin{aligned}\tan\delta = i\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} \Delta y\Bigg[1 + \Bigg\{&\frac{1}{a}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} - \frac{2i\eta_1}{a(1-\eta_1^2)}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &+ \frac{1+3\eta_1^2}{3(1-\eta_1^2)^2}\left(\frac{d\zeta}{dz}\right)_{i\eta_1}^2 + \frac{i\eta_1}{1-\eta_1^2}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &- \frac{1}{6}\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1}\left(\frac{d\zeta}{dz}\right)_{i\eta_1} - \frac{1}{6}\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} \\ &- \frac{1}{3}\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1}^2\left(\frac{d\zeta}{dz}\right)_{i\eta_1}^2\Bigg\}(\Delta y)^2 + \mathrm{O}[(\Delta y)^4]\Bigg]. \quad (40)\end{aligned}$$

Various terms in the coefficients of Δy and $(\Delta y)^3$ can be evaluated immediately. Thus, writing $\sigma = 2\beta/\pi$ for shortness, we have

$$\begin{aligned}&\left(\frac{d\Omega}{d\zeta}\right)_{i\eta_1} = -\frac{2\sigma}{1-\eta_1^2}, \qquad \left(\frac{d\zeta}{dz}\right)_{i\eta_1} = \frac{2i}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta^3}{1-\eta_1^4}, \\ &\left(\frac{d}{d\zeta}\frac{d\zeta}{dz}\right)_{i\eta_1} = \frac{2}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^2}{1-\eta_1^4}\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{3+\eta_1^4}{1-\eta_1^4}\right), \\ &\left(\frac{d^2}{d\zeta^2}\frac{d\zeta}{dz}\right)_{i\eta_1} = -\frac{2i}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1}{1-\eta_1^4} \\ &\qquad \times\left[\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{2(1+\eta_1^4)}{1-\eta_1^4}\right)\left(-\frac{2\sigma\eta_1}{1-\eta_1^2} + \frac{3+\eta_1^4}{1-\eta_1^4}\right) \right. \\ &\qquad \left. - \frac{2\sigma\eta_1(1+\eta_1^2)}{(1-\eta_1^2)^2} + \frac{7\eta_1^4}{(1-\eta_1^4)^2}\right]. \quad (41)\end{aligned}$$

Finally, comparing the above series for $\tan\delta$ with (8) and equating the coefficients of Δy and $(\Delta y)^3$, we obtain, after some reductions,

$$a = \frac{M^2-1}{(\gamma-1)M^2+2}\frac{b^2}{\sigma}\left\{\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^3}{(1-\eta_1^2)(1-\eta_1^4)}\right\}^{-1}, \quad (42)$$

and

$$\frac{16}{3}\frac{(M^2-1)^2}{a^2\{(\gamma-1)M^2+2\}^2}+\frac{4(\gamma+1)M^4}{a^2(M^2-1)\{(\gamma-1)M^2+2\}}$$

$$=-\frac{2}{ab^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{\sigma}\frac{\eta_1^2}{(1-\eta_1^4)^2}\left\{3+2\eta_1^2+3\eta_1^4-2\sigma\eta_1(1-\eta_1^2)\right\}$$

$$+\frac{2}{3}\frac{1}{b^2}\left(\frac{1-\eta_1}{1+\eta_1}\right)^{2\sigma}\frac{\eta_1^4}{(1-\eta_1^4)^4}$$

$$\times\left\{15+20\eta_1^2+49\eta_1^4+20\eta_1^6+15\eta_1^8+8\sigma^2\eta_1^2(1+\eta_1^2)^2\right.$$

$$\left.-8\sigma\eta_1(1+\eta_1^2)(3+2\eta_1^2+3\eta_1^4)\right\}. \qquad (43)$$

These two equations determine the values of the two parameters η_1, a.

In the close vicinity of the tip of the bullet, η_1 can be written as:

$$\eta_1=1-\varepsilon\,, \qquad (44)$$

where ε is an infinitesimal quantity. Equations (42) and (43) are then reduced respectively to

$$a=\frac{8b^2(M^2-1)}{(\gamma-1)M^2+2}\,\sigma^{-1}\,\varepsilon^{2-\sigma}2^{\sigma}\,, \qquad (45)$$

and

$$\frac{24(\gamma M^4+2M^2-1)}{(M^2-1)^2}\sigma^2+\left\{\frac{128(M^2-1)}{(\gamma-1)M^2+2}+48\right\}\sigma$$

$$-\frac{119(M^2-1)}{(\gamma-1)M^2+2}=0\,. \qquad (46)$$

These two equations determine a certain critical Mach number, M_{crit}, for a bullet with a given semi-vertical angle less than a certain value, such that when the Mach number of the undisturbed stream is greater than M_{crit} the shock wave attaches the tip of the bullet.

If, however, the semi-vertical angle of a bullet becomes greater than a certain critical value, the shock wave does never attach the tip of the bullet for any large value of the Mach number, and there is formed a detached shock wave in front of the bullet. Such a critical semi-vertical angle $\beta_{max}\,(=\frac{1}{2}\pi\sigma_{max})$ can be found by solving an equation to which equation (46) is reduced in the limit $M\to\infty$, namely:

$$24\gamma\sigma_{max}^2+\left(\frac{128}{\gamma-1}+48\right)\sigma_{max}-\frac{119}{\gamma-1}=0\,. \qquad (47)$$

For the case of air ($\gamma = 1.405$), we have found that

$$\sigma_{max} = 0.7535\,, \qquad \beta_{max} = 67^\circ\, 48.9'. \tag{48}$$

Numerical values of the critical Mach number M_{crit} for bullets with various values of σ have been calculated and they are tabulated in Table III. Similar values of M_{crit} for wedges are given in Table IV for comparison. These latter values for wedges have been calculated by the present writer with the aid of the following two equations, which have been given by D. C. Pack in his recent paper (**11**).

$$2\tan^2 m \tan^3\alpha + \tan\beta \tan^2\alpha \{(\gamma + 1) + (\gamma + 3)\tan^2 m\}$$
$$- 2\tan\alpha + \tan\beta\{(\gamma - 1) + (\gamma + 1)\tan^2 m\} = 0\,, \tag{49}$$

and

$$(\gamma + 1)\tan\beta\tan^4\alpha - 4\tan^3\alpha + 2(\gamma - 3)\tan\beta\tan^2\alpha$$
$$- 4\tan^2\beta\tan\alpha + (\gamma + 1)\tan\beta = 0\,, \tag{50}$$

where $\operatorname{cosec} m = M$ and β denotes the semi-vertical angle of the wedge.

These equations are the analytical expressions for Taylor-Maccoll's analysis (**12**) where they have obtained their results by a graphical method.

TABLE III. Values of M_{crit} for our bullets.

β	2° 7.8′	5° 8.2′	11°24.9′	24°24.9′	29°29.7′	33°44.7′	45°44.3′	67°48.9′
M_{crit}	1.050	1.100	1.200	1.400	1.500	1.600	2.000	∞

TABLE IV. Values of M_{crit} for wedge.

β	5°	10°	15°	20°	30°	40°	45° 23′
M_{crit}	1.240	1.417	1.616	1.843	2.530	4.520	∞

The results for both our bullet and wedge are shown in Fig. 7. It will be seen that our results for the two-dimensional bullet are in good qualitative agreement with those for the wedge.

Numerical values of η_1, which specifies the location of the normal portion of the detached shock wave, have been calculated by the formulae (45) and (46) for various values of M, and the corresponding values of the distance, t, of the normal portion of the shock wave from the tip of the bullet have been obtained by a graphical method.

The results are shown in Fig. 8 in four cases in which the semi-vertical angle β of the bullet is equal to 15°, 30°, 60° and 67° 38.9′ respectively. In this figure are also shown Laitone's values for comparison, which have been calculated by the present writer by means of Laitone's formula. The two series of curves deviate considerably from each other especially in the close vicinity of the horizontal axis.

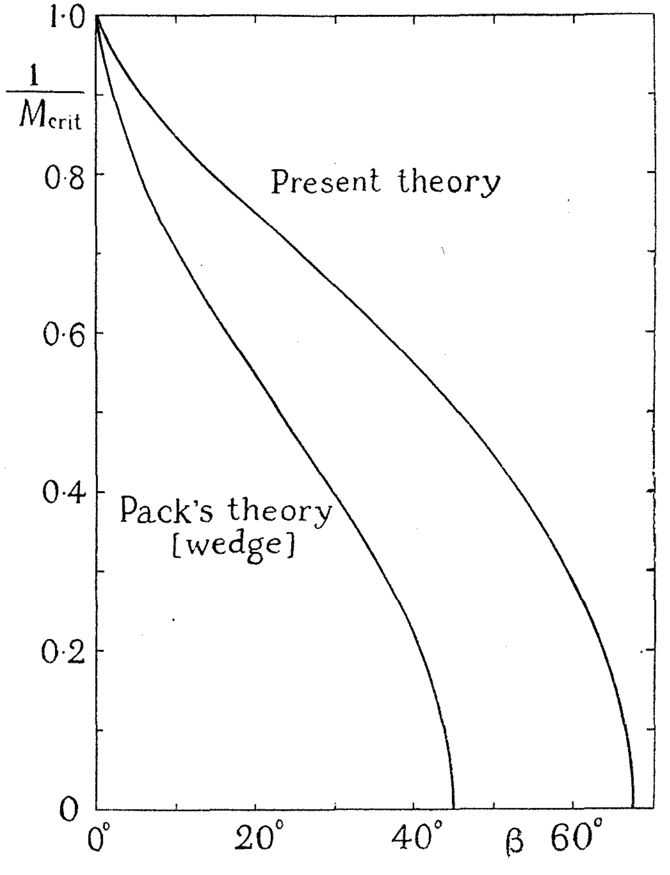

Fig. 7. Values of the critical Mach number plotted against the semi-vertical angle of a bullet or a wedge.

When the semi-vertical angle of the bullet is smaller than 67° 48.9′, each of our curves cuts the horizontal axis at a finite angle. On the other hand, any of Laitone's curves can never touch the horizontal axis and the least value of the distance of the normal portion of the detached shock wave from the tip of the bullet is always different from zero.

This implies evidently that according to his theory the shock wave remains always detached for any values of the semi-vertical angle and of the Mach number of the undisturbed flow. In fact, we cannot determine the critical values of M_{crit} and β_{max} on the basis of Laitone's theory. These results are not only in contradiction to Taylor's theoretical results but also to experimental evidences.

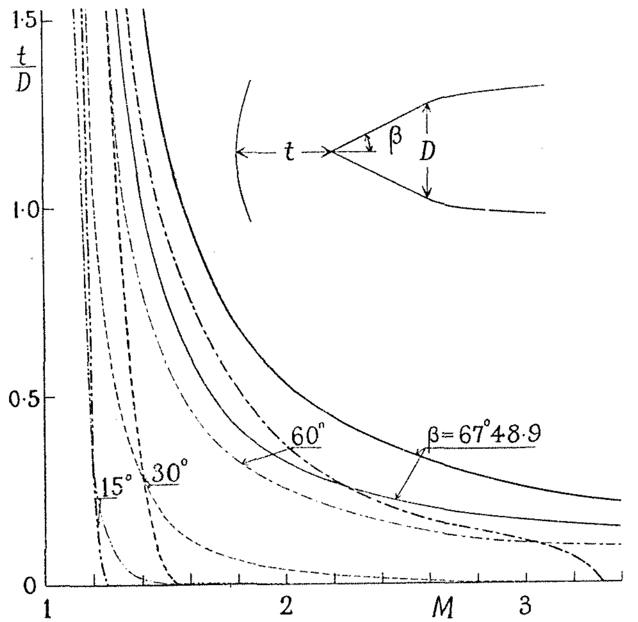

Fig. 8. Case of two-dimensional bullets. Thick-line curves give the values obtained by the present theory, while thin-line curves give the corresponding values obtained by Laitone's theory.

Fig. 9 is a reproduction of a figure in Kármán's recent paper **(13)**, which shows the experimental values of the ratio of the distance of the detached shock wave from the tip of a bullet to its diameter as a function of the Mach number of the undisturbed flow. These experimental data have been obtained by shooting tests of a conical headed three-dimensional bullet with semi-vertical angle of 20°.

In Fig. 10 is given a similar curve showing our theoretical results for a two-dimensional bullet with semi-vertical angle of 15°, and this may be compared with Fig. 9. It will be seen that there is a good qualitative agreement between two figures.

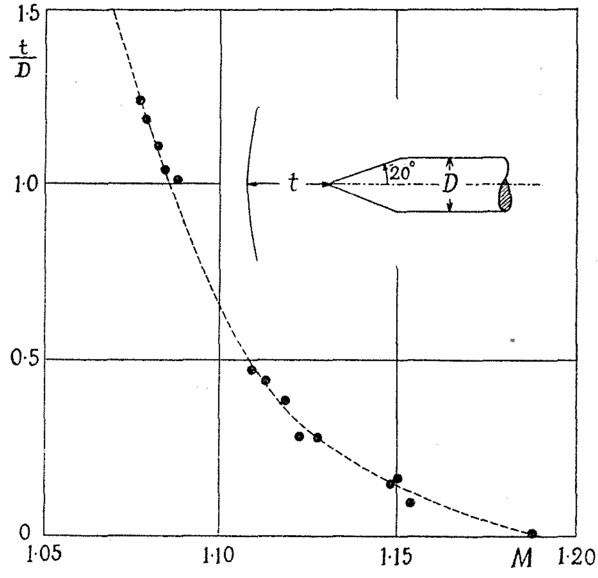

Fig. 9. Experimental values of t/D plotted against M, for a projectile with semi-vertical angle of 20° (Kármán (**13**)).

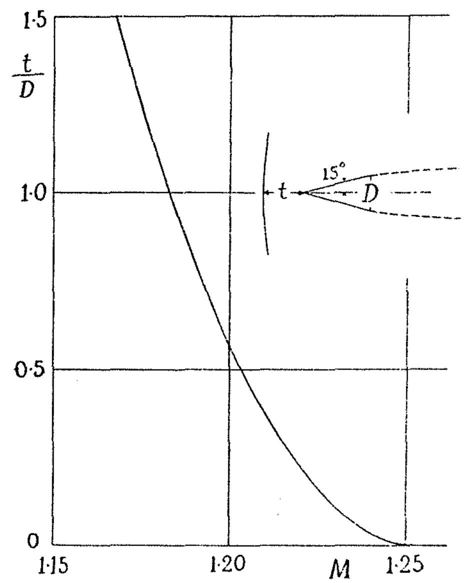

Fig. 10. Calculated values of t/D for a two-dimensional bullet with semi-vertical angle of 15°.

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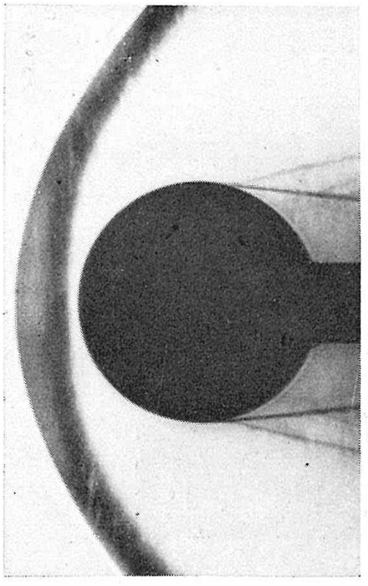
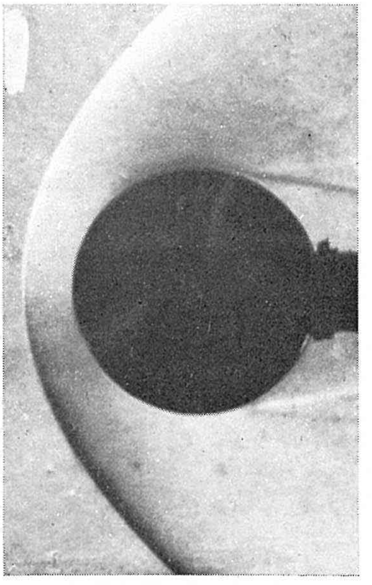
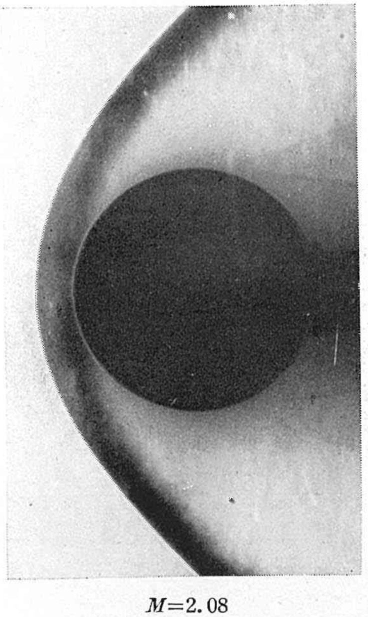


M=2.04

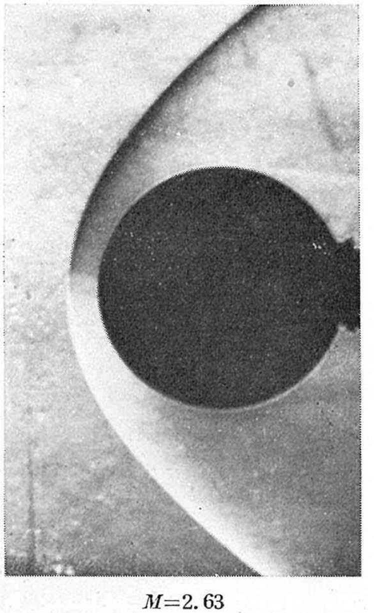


M=2.83

Circular cylinder

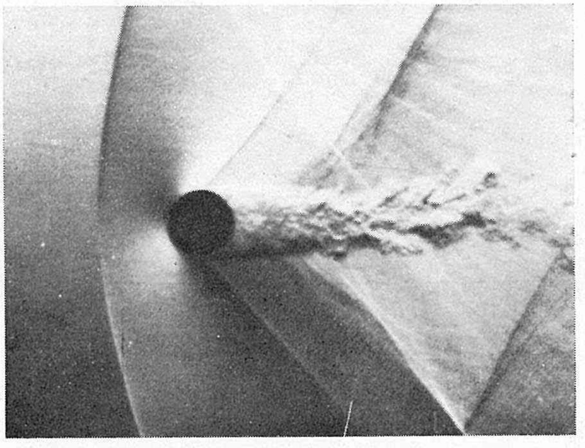


M=2.08

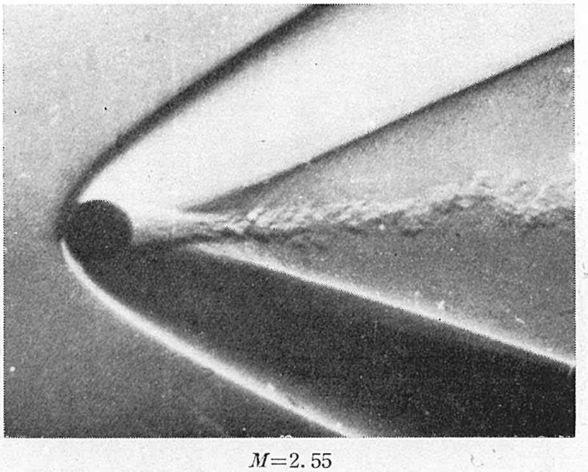


M=2.63

Sphere



$M=1.15$



$M=2.55$

Sphere (shooting test)