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A NOTE ON THE SKIN FRICTION OF A FLAT PLATE TO OSEEN'S APPROXIMATION

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SUMMARY

A second approximation for the skin friction at small Reynolds numbers of a flat plate placed edgeways along a steady uniform stream was first obtained by Piercy and Winny as early as 1933, by pursuing Bairstow, Cave, and Lang's general analysis based upon Oseen's equations of motion. Their analysis is now extended to a third approximation and an expansion formula correct to the fourth power of the Reynolds number is derived for the skin friction of the flat plate. The formula is found to be in perfect agreement with that obtained recently by Aoi and the present writer by developing quite a different analysis on the basis of Oseen's equations of motion.

1. Introduction

The skin friction at small Reynolds numbers of a flat plate placed edgeways along a steady uniform stream has been calculated by some writers. In 1933 Piercy and Winny (**1**), pursuing Bairstow, Cave, and Lang's analysis (**2**) based upon Oseen's equations of motion, obtained a second approximation (to the order of R^2) for the skin friction coefficient of a flat plate, where R is the Reynolds number defined in terms of the uniform-stream velocity U and the length, d, of the plate as $R = Ud/\nu$. In 1941 Davies (**3**), pursuing Lewis's work (**4**), also calculated a similar second approximation for the skin friction of the plate, on the basis of the extended Oseen's equations of motion due originally to Southwell and Squire (**5**). Recently, Sidrak (**6**) has made a study, on the basis of Oseen's equations, on the steady flow of an incompressible viscous fluid past an elliptic cylinder whose major-axis is parallel to a uniform stream, and has calculated a second approximation for the drag coefficient of such an elliptic cylinder. Further, he has derived, as its limiting case, an expansion formula similar to those given by Piercy and Winny, and by Davies for the skin friction coefficient of the flat plate.

No complete agreement can be found among these three expansion formulae, however; they differ slightly from each other. Thus, if we denote the skin friction coefficient of the flat plate by C_D, the three formulae take the following forms:

Piercy and Winny:

$$C_D = \frac{4\pi}{R(1+S_1)}\left[1 - \frac{1}{1+S_1}\left(\frac{1}{2}S_1^2 + \frac{1}{2}S_1 - 0.2083\right)\left(\frac{R}{8}\right)^2\right];$$

Davies:

$$C_D = \frac{4\pi}{R(1+S_1)}\left[1 - \frac{1}{1+S_1}\left(\frac{1}{2}S_1^2 + S_1 - 0.2083\right)\left(\frac{R}{8}\right)^2\right];$$

Sidrak:

$$C_D = \frac{4\pi}{R(1+S_1)}\left[1 - \frac{1}{1+S_1}\left(\frac{1}{2}S_1^2 + \frac{1}{4}S_1 + 0.3364\right)\left(\frac{R}{8}\right)^2\right],$$

where we have put for convenience†

$$S_1 = -\gamma - \log\frac{1}{16}R,$$

$\gamma = 0.57721\cdots\cdots$ being Euler's constant.

Davies attributes the discrepancy between his own formula and Piercy and Winny's to the difference of Southwell and Squire's equations of motion, upon which his analysis is based, from those of Oseen. But, it is seen without difficulty that in a special case of the flat plate edgeways along the uniform stream, Southwell and Squire's equations are quite equivalent to Oseen's equations, and therefore it may be expected that the discrepancy between their formulae might be due to incompleteness of, or some errors in, either or both of their analyses.

Quite recently, Yosinobu and the present writer (7) have examined carefully Davies's analysis and it has been found that an expansion formula in perfect agreement with Piercy and Winny's can be derived for the skin friction coefficient of the flat plate by pursuing Davies's original work, if, in place of $\mathrm{Cek}_n(\xi)$ and $\mathrm{Sek}_n(\xi)$ used by himself, we use $\mathrm{Fek}_n(\xi)$ and $\mathrm{Gek}_n(\xi)$ for solutions of the modified Mathieu equation, which are expressed in more rapidly convergent series in products of the modified Bessel functions.

Further, a very careful examination on Sidrak's work has also been made by Yosinobu and the present writer (8) and it has been found that Sidrak makes several grave errors in the course of his analysis so that his results cannot be relied on. Developing correct analysis along the lines of Sidrak's work, we have shown that an expansion formula in complete agreement with Piercy and Winny's is again obtained.

On the other hand, Aoi and the present writer (9) have recently made, independently of Sidrak's work, a detailed study on the slow motion of a viscous liquid past an elliptic cylinder on the basis of Oseen's equations, and have obtained a general expression as well as an expansion formula correct to the fourth power of the Reynolds number for the drag of the elliptic cylinder. A third approximation for the skin friction coefficient of a flat plate has also been obtained as a limiting case of the expansion formula for the drag of the elliptic cylinder, and thus it has been found that up to the second approximation, our formula is in complete agreement with Piercy and Winny's.

Such a third approximation for the skin friction coefficient of the flat plate might have been alternatively obtained by extending Piercy and Winny's analysis, and this enquiry is the subject of the present note.

2. First and second approximations for the skin friction of the flat plate

Although both the first and second approximations for the skin friction of the flat plate have been obtained by Piercy and Winny (1), pursuing Bairstow, Cave, and Lang's general analysis, we shall first recapitulate their analysis for the sake of reference.

† This S_1 is equivalent to S in Sidrak's analysis. Strictly speaking, the figure 0.2083 should be replaced by 5/24.

We assume that the motion is two-dimensional and that a flat plate is placed edgeways along a steady uniform stream of velocity U of infinite extent. For convenience the length d of the plate is given the value 2, and ratio of U to ν, the kinematic coefficient of viscosity of the fluid, is denoted by $2k$. Then, the Reynolds number, R, as defined in terms of U and the length $d\,(=2)$ of the plate, becomes

$$R = Ud/\nu = 4k. \tag{2.1}$$

Taking the origin at the nose of the plate and the x-axis along the plate in the downstream direction of the uniform stream, let us denote by E and M two points which may occupy any positions on the plate and write ξ_E, ξ_M for kx_E, kx_M respectively, and put $\xi_M - \xi_E = w$.

The equation obtained by Bairstow, Cave, and Lang (2, equation E (13)) for the distribution χ_E of doublets along the plate appropriate to the problem is

$$\frac{\pi U}{2} = \int_0^{2k} \left\{ e^w K_0(w) - \frac{1}{2}\frac{\partial}{\partial w}\left[e^w K_0(w) + \frac{1}{2}\log w^2 \right] \right\} \chi'(\xi_E)\, d\xi_E\,, \tag{2.2}$$

where $\chi'(\xi_E) = \partial\chi_E/\partial\xi_E$ and K_0 is the modified Bessel function. These doublets satisfy the equation of viscous motion to Oseen's approximation, i. e.

$$\nu\nabla^4\psi = U\nabla^2(\partial\psi/\partial x)\,,$$

and χ_E as obtained from (2. 2) satisfies the boundary conditions at the surface of the plate.

A solution for χ_E gives immediately the skin friction D of unit span of the plate by the relation (2, formula F (8)):

$$D = 4\mu \int_{x_E=0}^{x_E=2} d\chi_E\,, \tag{2.3}$$

where $\mu = \rho\nu$ is the coefficient of viscosity of the fluid.

First approximation

With the origin changed to midway along the plate, equation (2. 2) may be written as:

$$\pi U = \int_{-k}^{k} \left\{ e^w \left[K_0(w) - \frac{\partial}{\partial w} K_0(w) \right] - \frac{1}{w} \right\} \chi'(\xi_E)\, d\xi_E\,. \tag{2.4}$$

Assuming w small and expanding the kernel, we find, retaining only first terms of the expansions,

$$\pi U = \int_{-k}^{k} \left(1 - \gamma - \log\frac{w}{2} \right) \chi'(\xi_E)\, d\xi_E\,, \tag{2.5}$$

where $\gamma = 0.57721\cdots$ is Euler's constant.

To solve this integral equation we assume

$$\chi'(\xi_E) = A/\sqrt{(k^2 - \xi_E^2)}\,. \tag{2.6}$$

Then, on performing integration we find

$$U = A(1 + S_1), \tag{2.7}$$

where we have put for brevity

$$S_1 = -\gamma - \log\frac{1}{4}k = -\gamma - \log\frac{1}{16}R. \tag{2.8}$$

Hence (2.6) is a solution of (2.5) if

$$A = U/(1 + S_1). \tag{2.9}$$

In this case the distribution χ_E of doublets along the plate is given by

$$\chi_E = \int_{-k}^{\xi_E} \chi'(\xi_E)\,d\xi_E = \frac{U}{1 + S_1}\left(\frac{\pi}{2} + \sin^{-1}x_E\right), \tag{2.10}$$

and the expression for the skin friction D of the plate becomes

$$D = 4\mu(\chi_1 - \chi_{-1}) = 4\mu\pi U/(1 + S_1). \tag{2.11}$$

If, as usual, we define the skin friction coefficient C_D by

$$C_D = D/(\rho U^2 d),$$

we obtain

$$C_D = \frac{1}{k}\frac{\chi_1 - \chi_{-1}}{U} = \frac{4\pi}{R(1 + S_1)}. \tag{2.12}$$

This is the first approximation for the skin friction coefficient of the flat plate.

Second approximation

Next, expanding the kernel of (2.4) as before, but retaining squares of small quantities, the integral equation for determining the distribution χ_E of doublets becomes

$$\pi U = \int_{-k}^{k}\left\{-\left(\gamma + \log\frac{1}{2}w\right)\left(1 + \frac{1}{2}w + \frac{1}{4}w^2\right) + \left(1 + \frac{1}{4}w + \frac{1}{6}w^2\right)\right\}\chi'(\xi_E)\,d\xi_E. \tag{2.13}$$

It is now assumed for the form of the solution that

$$\chi'(\xi_E) = \frac{A + B\xi_E + C\xi_E^2}{\sqrt{(k^2 - \xi_E^2)}}. \tag{2.14}$$

After substitution and reduction, we first obtain, by equating the coefficients of ξ_M,

$$\left(\frac{1}{4} - \frac{1}{2}S_1\right)A = \left\{1 + \left(\frac{1}{12} - \frac{1}{4}S_1\right)k^2\right\}B - \left(\frac{1}{4} - \frac{1}{4}S_1\right)k^2 C, \tag{2.15}$$

and next, by equating those of ξ_M^2,

$$\left(\frac{5}{12} - \frac{1}{2}S_1\right)A = \frac{1}{2}B + \left\{1 - \left(\frac{1}{3} - \frac{1}{4}S_1\right)k^2\right\}C. \tag{2.16}$$

Solving B and C from these two equations we find

$$\left.\begin{aligned} \frac{B}{A} &= \frac{1}{4} - \frac{1}{2}S_1 + O(k^2), \\ \frac{C}{A} &= \frac{7}{24} - \frac{1}{4}S_1 + O(k^2). \end{aligned}\right\} \tag{2.17}$$

On the other hand, on equating constant terms and taking account of (2.17), we have, to the order of k^2,

$$\frac{U}{A} = 1 + S_1 + \frac{1}{8}\left(\frac{3}{4} + \frac{7}{6}S_1\right)k^2. \tag{2.18}$$

These values of A, B, C make (2.14) to satisfy the integral equation (2.13) to the order of k^2, and to the same order the distribution χ_E of doublets along the plate is found to be given by

$$\chi_E = A\left(\frac{\pi}{2} + \sin^{-1}x_E\right) - Bk\sqrt{(1 - x_E^2)} + \frac{1}{2}Ck^2\left\{\frac{\pi}{2} + \sin^{-1}x_E - x_E\sqrt{(1 - x_E^2)}\right\}. \tag{2.19}$$

The skin friction D of the plate now becomes

$$D = 4\mu(\chi_1 - \chi_{-1}) = 4\mu\pi\left(A + \frac{1}{2}Ck^2\right), \tag{2.20}$$

and calculating the skin friction coefficient C_D as in the preceding case of the first approximation, we get finally

$$C_D = \frac{\pi}{k}\frac{1 + \frac{1}{8}\left(\frac{7}{6} - S_1\right)k^2}{1 + S_1 + \frac{1}{8}\left(\frac{3}{4} + \frac{7}{6}S_1\right)k^2}. \tag{2.21}$$

This result is due originally to Piercy and Winny.

Correct to the order of $k^2\left(=\frac{1}{16}R^2\right)$, the above formula may also be written in the form:

$$C_D = \frac{4\pi}{R(1 + S_1)}\left\{1 - \frac{1}{1 + S_1}\left(\frac{1}{2}S_1^2 + \frac{1}{2}S_1 - \frac{5}{24}\right)\left(\frac{R}{8}\right)^2\right\}, \tag{2.22}$$

which has been quoted by Sidrak (**6**) as Piercy and Winny's second approximate formula.

If, in place of S_1, we use S such that

$$S = 1 + S_1 = 1 - \gamma - \log\frac{1}{16}R, \tag{2.23}$$

the two formulae (2.12) and (2.22) become respectively

$$C_D = \frac{4\pi}{RS}, \tag{2.24}$$

$$C_D = \frac{4\pi}{RS}\left\{1 - \frac{1}{S}\left(S^2 - S - \frac{5}{12}\right)\frac{R^2}{128}\right\} \tag{2.25}$$

3. A third approximation for the skin friction of the flat plate

In order to extend the preceding analysis to a third approximation, we must retain terms up to the order of w^4 in the expansion of the kernel in (2.4). Then, the integral equation for determining the distribution χ_E of doublets along the plate becomes

$$\pi U = \int_{-k}^{k} \left\{ -\left(\gamma + \log \frac{1}{2} w\right)\left(1 + \frac{1}{2}w + \frac{1}{4}w^2 + \frac{5}{48}w^3 + \frac{7}{192}w^4\right) + \left(1 + \frac{1}{4}w + \frac{1}{6}w^2 + \frac{17}{192}w^3 + \frac{71}{1920}w^4\right)\right\} \chi'(\xi_E)\, d\xi_E . \tag{3.1}$$

For the form of the solution we assume

$$\chi'(\xi_E) = \frac{A + B\xi_E + C\xi_E^2 + D\xi_E^3 + E\xi_E^4}{\sqrt{(k^2 - \xi_E^2)}} . \tag{3.2}$$

After substitution and reduction, we find, correct to the order of k^4, that

on equating constant terms,

$$\begin{aligned} U = &\left\{1 + S_1 + \left(\frac{1}{48} + \frac{1}{8}S_1\right)k^2 + \left(\frac{181}{30720} + \frac{7}{512}S_1\right)k^4\right\} A \\ &- \left\{\frac{1}{4}S_1 k^2 + \left(\frac{1}{96} + \frac{5}{128}S_1\right)k^4\right\} B \\ &+ \left\{\left(\frac{1}{4} + \frac{1}{2}S_1\right)k^2 + \left(\frac{1}{128} + \frac{3}{32}S_1\right)k^4\right\} C \\ &+ \left(\frac{1}{64} - \frac{3}{16}S_1\right)k^4 D + \left(\frac{5}{32} + \frac{3}{8}S_1\right)k^4 E ; \end{aligned} \tag{3.3}$$

on equating the coefficients of ξ_M,

$$\begin{aligned} &\left\{-\left(\frac{1}{4} - \frac{1}{2}S_1\right) + \left(\frac{1}{384} + \frac{5}{32}S_1\right)k^2\right\} A \\ &+ \left\{1 + \left(\frac{1}{12} - \frac{1}{4}S_1\right)k^2 - \left(\frac{19}{1920} + \frac{7}{128}S_1\right)k^4\right\} B \\ &- \left\{\left(\frac{1}{4} - \frac{1}{4}S_1\right)k^2 + \left(\frac{1}{128} - \frac{15}{128}S_1\right)k^4\right\} C \\ &+ \left\{\frac{1}{2}k^2 + \left(\frac{5}{64} - \frac{3}{16}S_1\right)k^4\right\} D - \left(\frac{13}{64} - \frac{3}{16}S_1\right)k^4 E = 0 ; \end{aligned} \tag{3.4}$$

on equating the coefficients of ξ_M^2,

$$-\left\{\left(\frac{5}{24}-\frac{1}{4}S_1\right)+\left(\frac{29}{3840}-\frac{7}{64}S_1\right)k^2\right\}A$$

$$+\left\{\frac{1}{4}+\left(\frac{29}{384}-\frac{5}{32}S_1\right)k^4\right\}B$$

$$+\left\{\frac{1}{2}-\left(\frac{1}{6}-\frac{1}{8}S_1\right)k^2-\left(\frac{1}{80}-\frac{21}{256}S_1\right)k^4\right\}C$$

$$+\left\{\frac{1}{8}k^2+\left(\frac{17}{256}-\frac{15}{128}S_1\right)k^4\right\}D$$

$$+\left\{\frac{1}{4}k^2-\left(\frac{17}{128}-\frac{3}{32}S_1\right)k^4\right\}E=0; \tag{3. 5}$$

on equating the coefficients of $\xi_M{}^3$,

$$-\left(\frac{59}{576}-\frac{5}{48}S_1\right)A$$

$$+\left\{\frac{1}{12}+\left(\frac{239}{5760}-\frac{7}{96}S_1\right)k^2\right\}B$$

$$+\left\{\frac{1}{12}-\left(\frac{89}{1152}-\frac{5}{96}S_1\right)k^2\right\}C$$

$$+\left\{\frac{1}{3}+\frac{1}{24}k^2+\left(\frac{137}{3840}-\frac{7}{128}S_1\right)k^4\right\}D$$

$$+\left\{\frac{1}{24}k^2-\left(\frac{47}{768}-\frac{5}{128}S_1\right)k^4\right\}E=0; \tag{3. 6}$$

and finally on equating the coefficients of $\xi_M{}^4$,

$$-\left(\frac{449}{11520}-\frac{7}{192}S_1\right)A+\frac{5}{192}B$$

$$+\left\{\frac{1}{48}-\left(\frac{659}{23040}-\frac{7}{384}S_1\right)k^2\right\}C$$

$$+\left(\frac{1}{24}+\frac{5}{384}k^2\right)D$$

$$+\left\{\frac{1}{4}+\frac{1}{96}k^2-\left(\frac{347}{15360}-\frac{7}{512}S_1\right)k^4\right\}E=0. \tag{3. 7}$$

Solving the four equations (3. 4), (3. 5), (3. 6), and (3. 7) we obtain

$$\left.\begin{aligned}
\frac{B}{A} &= \frac{1}{4} - \frac{1}{2}S_1 - \left(\frac{7}{192} + \frac{1}{8}S_1 + \frac{1}{16}S_1^2\right)k^2 + O(k^4)\,, \\
\frac{C}{A} &= \frac{7}{24} - \frac{1}{4}S_1 + \left(\frac{131}{11520} - \frac{13}{128}S_1 - \frac{1}{16}S_1^2\right)k^2 + O(k^4)\,, \\
\frac{D}{A} &= \frac{11}{64} - \frac{1}{8}S_1 + O(k^2)\,, \\
\frac{E}{A} &= \frac{443}{5760} - \frac{5}{96}S_1 + O(k^2)\,.
\end{aligned}\right\} \quad (3.8)$$

Substituting these values in (3.3), we obtain, after some reductions,

$$\frac{U}{A} = 1 + S_1 + \frac{1}{8}\left(\frac{3}{4} + \frac{7}{6}S_1\right)k^2 + \frac{1}{64}\left(\frac{213}{144} + \frac{481}{720}S_1 - \frac{9}{4}S_1^2 - S_1^3\right)k^4. \quad (3.9)$$

These values of A, B, C, D, E make (3.2) to satisfy the integral equation (3.1) to the order of k^4, and to the same order the distribution χ_E of doublets along the plate is obtained as:

$$\begin{aligned}
\chi_E = {} & A\left(\frac{\pi}{2} + \sin^{-1}x_E\right) - Bk\sqrt{(1 - x_E^2)} \\
& + \frac{1}{2}Ck^2\left\{\frac{\pi}{2} + \sin^{-1}x_E - x_E\sqrt{(1 - x_E^2)}\right\} \\
& - \frac{1}{3}Dk^3\left\{2\sqrt{(1 - x_E^2)} + x_E^2\sqrt{(1 - x_E^2)}\right\} \\
& + Ek^4\left\{\frac{3\pi}{16} + \frac{3}{8}\sin^{-1}x_E - \frac{3}{8}x_E\sqrt{(1 - x_E^2)} - \frac{1}{4}x_E^3\sqrt{(1 - x_E^2)}\right\}. \quad (3.10)
\end{aligned}$$

The skin friction coefficient C_D of the plate now becomes

$$C_D = \frac{\pi A}{kU}\left(1 + \frac{1}{2}\frac{C}{A}k^2 + \frac{3}{8}\frac{E}{A}k^4\right), \quad (3.11)$$

and if we substitute the values of C/A, E/A and U/A as given by (3.8) and (3.9), we ultimately obtain a third approximation for C_D in the form:

$$\begin{aligned}
C_D = \frac{4\pi}{R(1 + S_1)}\Bigg\{1 & - \frac{1}{1 + S_1}\left(S_1^2 + S_1 - \frac{5}{12}\right)\frac{R^2}{128} \\
& - \frac{1}{(1 + S_1)^2}\left(S_1^4 + \frac{49}{12}S_1^3 + \frac{381}{72}S_1^2 + \frac{707}{360}S_1 - \frac{301}{720}\right)\frac{R^4}{128^2}\Bigg\}. \quad (3.12)
\end{aligned}$$

If, as before, use is made of $S (= 1 + S_1)$ in place of S_1, the formula may be put in the form:

$$C_D = \frac{4\pi}{RS}\left\{1 - \frac{1}{S}\left(S^2 - S - \frac{5}{12}\right)\frac{R^2}{128} - \frac{1}{S^2}\left(S^4 + \frac{1}{12}S^3 - \frac{23}{24}S^2 - \frac{133}{360}S - \frac{25}{144}\right)\frac{R^4}{128^2}\right\}. \quad (3.13)$$

This is in complete agreement with the formula obtained recently by Aoi and the present writer (9) by developing quite a different analysis.

REFERENCES

1. N. A. V. Piercy and H. F. Winny, 'The skin friction of flat plates to Oseen's approximation', *Proc. Roy. Soc.* A, **140** (1933), 543.
2. L. Bairstow, B. M. Cave, and E. D. Lang, 'The resistance of a cylinder moving in a viscous fluid', *Phil. Trans. Roy. Soc.* A, **223** (1923), 383.
3. T. V. Davies, 'An investigation of the flow of a viscous fluid past a flat plate, using elliptic coordinates', *Phil. Mag.*, **31** (1941), 283.
4. T. Lewis, 'Solutions of Oseen's extended equations for circular and elliptic cylinders and a flat plate', *Quart. Journ. Math.*, Oxford Series, **9** (1938), 21.
5. R. V. Southwell and H. B. Squire, 'A modification of Oseen's approximate equation for the motion in two dimensions of a viscous incompressible fluid', *Phil. Trans. Roy. Soc.* A, **232** (1933), 27.
6. S. Sidrak, 'The flow of a viscous liquid past an elliptic cylinder', *Proc. Roy. Irish Acad.*, **53** (1950), 65.
7. S. Tomotika and H. Yosinobu, 'The flow of a viscous liquid past a flat plate at small Reynolds numbers'. (To be published shortly)
8. S. Tomotika and H. Yosinobu, 'The drag on an elliptic cylinder moving in a viscous liquid at small Reynolds numbers', *Memoirs Coll. Sci., Univ. of Kyoto,* A **27** (1953), 11.
9. S. Tomotika and T. Aoi, 'The steady flow of a viscous fluid past an elliptic cylinder and a flat plate at small Reynolds numbers'. To appear soon in *Quart. Journ. Mech. and Applied Math.*