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# THE DRAG ON AN ELLIPTIC CYLINDER MOVING IN A VISCOUS LIQUID AT SMALL REYNOLDS NUMBERS

BY

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SUMMARY

Pursuing Sidrak's lines of attack, the steady flow of an incompressible viscous fluid past an elliptic cylinder has been discussed, on the basis of Oseen's linearized equations of motion, in case when the major-axis of the ellipse is parallel to the undisturbed uniform stream. The exact solution of Oseen's equations has been first obtained and then the general expression for the drag experienced by the elliptic cylinder has been calculated. Also, an expansion formula correct to the second power of Reynolds number has been derived for the drag coefficient of the cylinder. The expansion formula thus derived is in complete agreement, to the second approximation, with a similar expansion formula which has been obtained recently by Tomotika and Aoi.

## 1. Introduction

The steady flow of an incompressible viscous fluid past an elliptic cylinder at small Reynolds numbers was studied so far by several writers. In 1924 Harrison (1), solving Oseen's equations of motion approximately, calculated a first approximation for the drag per unit span of an elliptic cylinder whose major-axis is parallel to a steady uniform flow. In 1937 Meksyn (2) discussed, on the basis of the same Oseen's equations, the flow past an elliptic cylinder set obliquely at an arbitrary angle of incidence in a uniform flow and computed, though only numerically, the drag and lift of the cylinder. In the next year 1938 Lewis (3) treated the flow of an incompressible viscous fluid past circular and elliptic cylinders and a flat plate, by using Oseen's extended equations of motion due originally to Southwell and Squire. After pointing out that in Meksyn's solution the circulation round the obstacle is infinite, Lewis obtained an improved solution which gives rise to finite circulation. However, no discussions have been made on the drag and lift of the elliptic cylinder. Further, Sidrak (4) has lately discussed the flow of a viscous liquid past an elliptic cylinder, on the basis of Oseen's equations, with the intention of obtaining approximate expansion formulae in powers of Reynolds number for the drag of an elliptic cylinder as well as of a flat plate, each placed parallel to the uniform stream. On examining his paper carefully it has been found, however, that he makes several grave errors in the course of his analysis so that his results are not reliable.

In a quite recent paper (5), Aoi and one of the present writers (S. T.) have also made, independently of Sidrak, detailed theoretical discussions based upon Oseen's equations of motion on the steady flow of a viscous liquid around an elliptic cylinder or a flat plate placed parallel or perpendicularly to a uniform stream, and the drag experienced by, and the flow patterns around, the obstacle have been discussed in detail by making use of the exact solution of Oseen's equations. Also, approximate expansion formulae for the drag have been obtained in each case correct to the fourth power of the Reynolds number.

Such expansion formulae should have been obtained as well by pursuing Sidrak's lines of attack, if correct analysis would have been developed; and to the discussion of the matter the present paper is directed.

## 2. The general solution of Oseen's equations of motion

We consider an infinitely long elliptic cylinder, with major-and minor-axes of length $2a$ and $2b$ respectively, set at right angles to a steady uniform stream of velocity $U$ of an incompressible viscous fluid in such a manner that its major-axis is parallel to the uniform stream. Let $(x, y)$ be the rectangular coordinates having the origin at the centre of the ellipse and the axes of $x$ and $y$ along the major- and minor-axes respectively. We denote by $U+u$, $v$ the components of the fluid velocity at any point, so that $u, v$ are the velocity of perturbation which become vanishingly small everywhere at a great distance from the cylinder. If squares of $u, v$ are omitted, we get, from the Navier-Stokes equations of motion, the well-known linearized equation of Oseen:

$$U\frac{\partial}{\partial x}(u,\ v) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right)p + \nu\nabla^2(u,\ v)\,, \tag{2.1}$$

where $p$ is the pressure, $\rho$ the density of the fluid, $\nu$ its kinematic coefficient of viscosity, and $\nabla^2$ stands for $\partial^2/\partial x^2+\partial^2/\partial y^2$. The fluid being assumed to be incompressible, the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{2.2}$$

If we put $k=U/2\nu$, these equations are satisfied by

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} + \frac{1}{2k}\frac{\partial\chi}{\partial x} - \chi\,,\\ v &= -\frac{\partial\phi}{\partial y} + \frac{1}{2k}\frac{\partial\chi}{\partial y}\,, \end{aligned}\right\} \tag{2.3}$$

and

$$p = \rho U\frac{\partial\phi}{\partial x}\,, \tag{2.4}$$

provided that

$$\nabla^2\phi = 0\,, \tag{2.5}$$

and

$$\nabla^2\chi - 2k\frac{\partial\chi}{\partial x} = 0\,. \tag{2.6}$$

We now introduce the elliptic coordinates $(\xi,\ \eta)$ defined as:

$$x + iy = c\cosh(\xi + i\eta)\,,$$

with $c=\sqrt{a^2-b^2}$.

Then, equation (2.5) becomes

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}=0\,, \tag{2.7}$$

and, if we put $\chi=e^{kx}\chi_1$ in (2.6), the equation for $\chi_1$ becomes

$$\frac{\partial^2\chi_1}{\partial\xi^2}+\frac{\partial^2\chi_1}{\partial\eta^2}-k^2c^2(\cosh^2\xi-\cos^2\eta)\chi_1=0\,. \tag{2.8}$$

Bearing in mind that the flow past the cylinder is symmetrical about the axis of $x$ and that the perturbation due to the presence of the body must vanish at infinity, the appropriate general solution of equation (2.7) is given by

$$\phi=cUb_0\xi-cU\sum_{n=1}^{\infty}\frac{1}{n}b_ne^{-n\xi}\cos n\eta\,, \tag{2.9}$$

where the $b_n$'s are constants of integration.

Next, in order to solve equation (2.8) we put $\chi_1=F(\xi)G(\eta)$ and separate the variables. Then we are led to the modified Mathieu equation for $F(\xi)$ and the Mathieu equation for $G(\eta)$, namely:

$$\frac{d^2F}{d\xi^2}-(\lambda+k^2c^2\cosh^2\xi)F=0\,, \tag{2.10}$$

$$\frac{d^2G}{d\eta^2}+(\lambda+k^2c^2\cos^2\eta)G=0\,. \tag{2.11}$$

Solutions, periodic in $\eta$ with period $2\pi$, of equation (2.11) are obtained for a discrete set of characteristic values of $\lambda$. They fall into the four groups:

$$\left.\begin{aligned}
\mathrm{ce}_{2n}(\eta)&=\sum_{r=0}^{\infty}a_{2r}^{(2n)}\cos 2r\eta\,,\\
\mathrm{ce}_{2n+1}(\eta)&=\sum_{r=0}^{\infty}a_{2r+1}^{(2n+1)}\cos(2r+1)\eta\,,\\
\mathrm{se}_{2n+1}(\eta)&=\sum_{r=0}^{\infty}b_{2r+1}^{(2n+1)}\sin(2r+1)\eta\,,\\
\mathrm{se}_{2n+2}(\eta)&=\sum_{r=0}^{\infty}b_{2r+2}^{(2n+2)}\sin(2r+2)\eta\,.
\end{aligned}\right\} \tag{2.12}$$

The corresponding pairs of solutions of equation (2.10) which tend to zero when $\xi\to\infty$, are

$$\begin{aligned}
\mathrm{Fek}_{2n}(\xi)&=\frac{p'_{2n}}{\pi a_0^{(2n)}}\sum_{r=0}^{\infty}(-1)^r a_{2r}^{(2n)}I_r(v_1)K_r(v_2)\,,\\
\mathrm{Fek}_{2n+1}(\xi)&=\frac{s'_{2n+1}}{\pi a_1^{(2n+1)}}\sum_{r=0}^{\infty}(-1)^r a_{2r+1}^{(2n+1)}[I_r(v_1)K_{r+1}(v_2)-I_{r+1}(v_1)K_r(v_2)]\,,
\end{aligned}$$

$$\left.\begin{aligned}\mathrm{Gek}_{2n+1}(\xi) &= \frac{p'_{2n+1}}{\pi b_1^{(2n+1)}}\sum_{r=0}^{\infty}(-1)^r b_{2r+1}^{(2n+1)}[I_r(v_1)K_{r+1}(v_2)+I_{r+1}(v_1)K_r(v_2)],\\ \mathrm{Gek}_{2n+2}(\xi) &= \frac{s'_{2n+2}}{\pi b_2^{(2n+2)}}\sum_{r=0}^{\infty}(-1)^r b_{2r+2}^{(2n+2)}[I_r(v_1)K_{r+2}(v_2)-I_{r+2}(v_1)K_r(v_2)],\end{aligned}\right\} \tag{2.13}$$

where $v_1=\frac{1}{2}kce^{-\xi}$, $v_2=\frac{1}{2}kce^{\xi}$ and $p'_{2n}$, $p'_{2n+1}$, $s'_{2n+1}$, $s'_{2n+2}$ are all constants independent of $\xi$.

Since, in the present case, the flow is symmetrical about the axis of $x$, $\chi$ must be an even function with respect to $\eta$. Therefore, the appropriate general solution for $\chi$ is given by

$$\chi = Ue^{kc\cosh\xi\cos\eta}\sum_{m=0}^{\infty}a_m\mathrm{FEK}_m(\xi)\,\mathrm{ce}_m(\eta), \tag{2.14}$$

where the $a_m$'s are constants of integration, and

$$\left.\begin{aligned}\mathrm{FEK}_{2n}(\xi) &= \frac{\pi}{p'_{2n}}\mathrm{Fek}_{2n}(\xi),\\ \mathrm{FEK}_{2n+1}(\xi) &= \frac{\pi}{s'_{2n+1}}\mathrm{Fek}_{2n+1}(\xi).\end{aligned}\right\} \tag{2.15}$$

Inserting the above general expressions for $\phi$ and $\chi$ into (2.3) and (2.4) we can readily obtain the expressions for the components $u$, $v$ of the velocity of perturbation and the pressure $p$ at any point. Thus we have

$$\begin{aligned}u = {}& \frac{U}{2h^2}\left[b_0e^{-\xi}\cos\eta + b_1e^{-2\xi} - \sum_{n=1}^{\infty}\{b_{n-1}e^{-(n-2)\xi} - b_{n+1}e^{-(n+2)\xi}\}\cos n\eta\right]\\ &+\frac{U}{2kch^2}e^{kc\cosh\xi\cos\eta}\sum_{m=0}^{\infty}a_m\Big[\mathrm{FEK}_m'(\xi)\sinh\xi\,\mathrm{ce}_m(\eta)\cos\eta\\ &\qquad-\mathrm{FEK}_m(\xi)\Big\{\cosh\xi\,\mathrm{ce}_m'(\eta)\sin\eta+\frac{1}{2}kc\,\mathrm{ce}_m(\eta)(\cosh2\xi-\cos2\eta)\Big\}\Big],\end{aligned} \tag{2.16}$$

$$\begin{aligned}v = {}& -\frac{U}{2h^2}\left[b_0e^{-\xi}\sin\eta + \sum_{n=1}^{\infty}\{b_{n-1}e^{-(n-2)\xi} - b_{n+1}e^{-(n+2)\xi}\}\sin n\eta\right]\\ &+\frac{U}{2kch^2}e^{kc\cosh\xi\cos\eta}\sum_{m=0}^{\infty}a_m\Big[\mathrm{FEK}_m'(\xi)\cosh\xi\,\mathrm{ce}_m(\eta)\sin\eta\\ &\qquad+\mathrm{FEK}_m(\xi)\sinh\xi\,\mathrm{ce}_m'(\eta)\cos\eta\Big],\end{aligned} \tag{2.17}$$

$$p = \frac{\rho U^2}{2h^2}\left[-(b_0e^{-\xi}\cos\eta + b_1e^{-2\xi}) + \sum_{n=1}^{\infty}\{b_{n-1}e^{-(n-2)\xi} - b_{n+1}e^{-(n+2)\xi}\}\cos n\eta\right], \tag{2.18}$$

where $\mathrm{FEK}_m'(\xi)=\frac{d}{d\xi}\mathrm{FEK}_m(\xi)$, $\mathrm{ce}_m'(\eta)=\frac{d}{d\eta}\mathrm{ce}_m(\eta)$ and $h^2=\frac{1}{2}(\cosh2\xi-\cos2\eta)$.

### 3. The boundary conditions

We shall now proceed to determine the constants of integration $a_m$'s and $b_n$'s by the boundary conditions. Since, however, the conditions at infinity are satisfied automatically by the preceding expressions (2. 16) and (2. 17) for the components of the velocity of perturbation, we have only to consider the conditions at the surface of the elliptic cylinder.

Assuming that the elliptic cylinder under consideration is defined by $\xi=\xi_0$, the said conditions become

$$u=-U, \quad v=0 \tag{3. 1}$$

at $\xi=\xi_0$.

Making use of the Fourier expansions:

$$e^{kc\cosh\xi\cos\eta}\Big[\mathrm{FEK}_m'(\xi)\sinh\xi\,\mathrm{ce}_m(\eta)\cos\eta - \mathrm{FEK}_m(\xi)\Big\{\cosh\xi\,\mathrm{ce}_m'(\eta)\sin\eta+\frac{1}{2}kc\,\mathrm{ce}_m(\eta)(\cosh 2\xi-\cos 2\eta)\Big\}\Big] = \sum_{n=0}^{\infty} G_{m,n}(\xi)\cos n\eta,$$

$$e^{kc\cosh\xi\cos\eta}\Big[\mathrm{FEK}_m'(\xi)\cosh\xi\,\mathrm{ce}_m(\eta)\sin\eta+\mathrm{FEK}_m(\xi)\sinh\xi\,\mathrm{ce}_m'(\eta)\cos\eta\Big] = \sum_{n=1}^{\infty} H_{m,n}(\xi)\sin n\eta,$$

the expressions (2.16) and (2. 17) for $u$ and $v$ can be rewritten as:

$$h^2u=\frac{U}{2}\Big[b_0e^{-\xi}\cos\eta+b_1e^{-2\xi}-\sum_{n=1}^{\infty}\{b_{n-1}e^{-(n-2)\xi}-b_{n+1}e^{-(n+2)\xi}\}\cos n\eta\Big] + \frac{U}{2kc}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_mG_{m,n}(\xi)\cos n\eta, \tag{3. 2}$$

and

$$h^2v=-\frac{U}{2}\Big[b_0e^{-\xi}\sin\eta+\sum_{n=1}^{\infty}\{b_{n-1}e^{-(n-2)\xi}-b_{n+1}e^{-(n+2)\xi}\}\sin n\eta\Big] + \frac{U}{2kc}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_mH_{m,n}(\xi)\sin n\eta. \tag{3. 3}$$

Thus, substituting these expressions into the boundary conditions (3. 1) and equating the coefficients of $\cos n\eta$ and $\sin n\eta$ ($n=0, 1, 2,\ldots$) on both sides of the equations, we get

$$b_1e^{-2\xi_0}+\frac{1}{kc}\sum_{m=0}^{\infty}a_mG_{m,0}(\xi)=-\cosh 2\xi_0, \tag{3. 4}$$

$$\Big(b_0\sinh\xi_0-\frac{1}{2}b_2e^{-2\xi_0}\Big)-\frac{1}{2kc}\sum_{m=0}^{\infty}a_mG_{m,1}(\xi_0)=0, \tag{3. 5}$$

$$(b_1 - b_3 e^{-4\xi_0}) - \frac{1}{kc}\sum_{m=0}^{\infty} a_m G_{m,2}(\xi_0) = -1\,, \tag{3.6}$$

$$(b_{n-1}e^{-(n-2)\xi_0} - b_{n+1}e^{-(n+2)\xi_0}) - \frac{1}{kc}\sum_{m=0}^{\infty} a_m G_{m,n}(\xi_0) = 0\,; \tag{3.7}$$

$$(n = 3,\ 4,\ 5, \cdots)$$

$$\left(b_0 \cosh\xi_0 - \frac{1}{2} b_2 e^{-3\xi_0}\right) - \frac{1}{2kc}\sum_{m=0}^{\infty} a_m H_{m,1}(\xi_0) = 0\,, \tag{3.8}$$

$$(b_1 - b_3 e^{-4\xi_0}) - \frac{1}{kc}\sum_{m=0}^{\infty} a_m H_{m,2}(\xi_0) = 0\,, \tag{3.9}$$

$$(b_{n-1}e^{-(n-2)\xi_0} - b_{n+1}e^{-(n+2)\xi_0}) - \frac{1}{kc}\sum_{m=0}^{\infty} a_m H_{m,n}(\xi_0) = 0\,. \tag{3.10}$$

$$(n = 3,\ 4,\ 5, \cdots)$$

Subtracting (3.6) from (3.9) and (3.7) from (3.10), we obtain

$$\sum_{m=0}^{\infty} \lambda_{m,n} a_m = \begin{cases} kc\,, & (n=2) \\ 0\,, & (n = 3,\ 4,\ 5, \cdots) \end{cases} \tag{3.11}$$

where

$$\lambda_{m,n} = G_{m,n}(\xi_0) - H_{m,n}(\xi_0)\,, \quad \begin{pmatrix} m = 0,\ 1,\ 2, \cdots \\ n = 1,\ 2,\ 3, \cdots \end{pmatrix} \tag{3.12}$$

and the $a_m$'s can be determined by solving this system of simultaneous linear algebraic equations.

If use is made of the $a_m$'s thus determined, the value of $b_0$ can be found by the following equation as given by eliminating $b_2$ between (3.5) and (3.8):

$$b_0 = -\frac{e^{\xi_0}}{2kc}\sum_{m=0}^{\infty} \lambda_{m,1} a_m\,, \tag{3.13}$$

and also the value of $b_1$ can be obtained by (3.4). The remaining $b_n$'s can be determined successively from either of (3.7) and (3.10).

## 4. Transformation of the expressions for $\lambda_{m,n}$

Before proceeding to the calculation of the drag experienced by the elliptic cylinder under consideration, we shall now transform the expressions (3.12) for the $\lambda_{m,n}$'s into more convenient forms. To do this, we introduce functions $p_{m,n}(t)$ defined as:

$$p_{m,n}(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{t\cos\eta}\, \mathrm{ce}_m(\eta) \cos n\eta\, d\eta\,. \tag{4.1}$$

$$(m,\ n = 0,\ 1,\ 2, \cdots)$$

Substituting the series for $\mathrm{ce}_m(\eta)$ as given by (2.12) in the right-hand side and carrying out integrations, it is readily found that these functions $p_{m,n}(t)$ can be expressed as series of the modified Bessel functions*. Thus, we have

$$\left.\begin{aligned} p_{2m,n}(t) &= \sum_{r=0}^{\infty} a_{2r}^{(2m)}\{I_{n-2r}(t) + I_{n+2r}(t)\}, \\ p_{2m+1,n}(t) &= \sum_{r=0}^{\infty} a_{2r+1}^{(2m+1)}\{I_{n-2r-1}(t) + I_{n+2r+1}(t)\}. \end{aligned}\right\} \tag{4.2}$$

In what follows we take

$$t = kc\cosh\xi_0 = ka, \tag{4.3}$$

where $a=c\cosh\xi_0$ is the half-length of the major-axis of the ellipse $\xi=\xi_0$.

Then, writing for convenience's sake

$$\frac{1}{2}kc = q, \quad e^{\xi_0} = \sigma, \quad e^{-\xi_0} = \tau, \quad \omega = \sigma + \tau, \tag{4.4}$$

and bearing in mind that

$$\sigma\tau = 1, \quad t = \omega q, \tag{4.5}$$

the expressions (3.12) for the $\lambda_{m,n}$'s can be transformed, after some calculations, into the following forms:

$$\lambda_{m,n} = \mathrm{FEK}_m'(\xi_0)\Phi_{m,n} - \mathrm{FEK}_m(\xi_0)\Psi_{m,n}, \tag{4.6}$$
$$(m = 0,\ 1,\ 2,\cdots;\ n = 1,\ 2,\ 3,\cdots)$$

where

$$\left.\begin{aligned} \Phi_{m,n} &= \frac{1}{2}\Big\{\sigma p_{m,n+1}(\omega q) - \tau p_{m,n-1}(\omega q)\Big\}, \\ \Psi_{m,n} &= -\frac{1}{4}\Big\{(\sigma^2+3)q p_{m,n+2}(\omega q) + 2(n+1)\sigma p_{m,n+1}(\omega q) \\ &\quad - (3\sigma^2 + 2 + 3\tau^2)q p_{m,n}(\omega q) - 2(n-1)\tau p_{m,n-1}(\omega q) \\ &\quad + (3+\tau^2)q p_{m,n-2}(\omega q)\Big\}. \end{aligned}\right\} \tag{4.7}$$

## 5. The general expression for the drag on the elliptic cylinder

Next we shall proceed to the discussion on the drag experienced by the elliptic cylinder under consideration. As is well known, two different methods are commonly used in calculating the drag on a solid body immersed in a stream of viscous fluid. Thus, in one method the drag is calculated by summing up the viscous stresses exerted

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* Here we make use of the fact that when $s$ is any positive integer, the modified Bessel function $I_s(x)$ can be expressed as:

$$I_s(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\cos s\theta\, d\theta.$$

(Cf. G. N. Watson, '*Theory of Bessel functions*,' (Cambridge Univ. Press), 2nd ed. (1948), p. 181.)

by the fluid upon the surface of the obstacle, while in the other, it is calculated by applying the theorem of momentum to an infinite mass of fluid surrounding the body.

As shown in a paper (**6**) by Aoi and one of the present writers (S. T.), when the analysis is based upon Oseen's equations, the former direct method gives the general formula for the drag $D$ per unit span of a cylindrical obstacle in the form:

$$D = -\rho U \int \frac{\partial \phi}{\partial n} ds , \tag{5. 1}$$

where integration is taken round the circumferential curve $s$ of the cylinder and $\partial/\partial n$ means differentiation along the outward normal $n$ to $s$.

On the other hand, the latter indirect method gives

$$D = -\rho U \int \frac{\partial \phi}{\partial n'} ds' , \tag{5. 2}$$

where integration is taken round a large closed contour $s'$ everywhere at a great distance from the cylinder and $\partial/\partial n'$ means differentiation along the outward normal $n'$ to $s'$.

In the case of the elliptic cylinder $\xi=\xi_0$ under consideration, the formula (5. 1) becomes

$$D = -\rho U \int_0^{2\pi} \left(\frac{\partial \phi}{\partial \xi}\right)_{\xi=\xi_0} d\eta , \tag{5. 3}$$

while the formula (5. 2) takes the form:

$$D = -\rho U \int_0^{2\pi} \left(\frac{\partial \phi}{\partial \xi}\right)_{\xi\to\infty} d\eta , \tag{5. 4}$$

provided that a large ellipse confocal with the circumferential curve of the cylinder is taken as the closed contour $s'$.

Thus it is easily found that if use is made of the expression (2. 9) for $\phi$, the above two formulae (5. 3) and (5. 4) lead to one and the same result for the drag on the cylinder, namely:

$$D = -2\pi \rho c U^2 b_0 , \tag{5. 5}$$

which, by the aid of equation (3. 13), can also be written in the form:

$$D = \frac{\pi \rho U^2}{k} \sigma \sum_{m=0}^{\infty} \lambda_{m,1} a_m , \tag{5. 6}$$

where $R=2aU/\nu=4ka$ is the Reynolds number. This gives the general expression for the drag on the elliptic cylinder placed parallel to the uniform stream.

If we define the drag coefficient $C_D$ as $C_D=D/(\rho U^2 \cdot 2a)$, we have

$$C_D = \frac{2\pi}{R} \sigma \sum_{m=0}^{\infty} \lambda_{m,1} a_m . \tag{5. 7}$$

## 6. An expansion formula for the drag

Starting from the general expression (5.7) for the drag coefficient of the elliptic cylinder, we shall derive an expansion formula in powers of the Reynolds number which may be conveniently used for numerical computations of the values of the drag coefficient.

To this end, we have first to determine the $a_m$'s by solving the system of simultaneous linear algebraic equations (3.11). Theoretically the solution of this system of equations has to be obtained by means of infinite determinants, but for the purpose of deriving an expansion formula in powers of the Reynolds number $R$ for the drag coefficient it is necessary and sufficient to find the values of the first few of the $a_m$'s. In fact, it is readily justified that if we confine ourselves to the derivation of an expansion formula correct to the order of $R^2$, it is necessary and sufficient to determine the first two of the $a_m$'s, namely $a_0$ and $a_1$, by solving the following system of three linear algebraic equations:

$$\left.\begin{aligned} \lambda_{0,2}a_0 + \lambda_{1,2}a_1 + \lambda_{2,2}a_2 &= 2q\,, \\ \lambda_{0,3}a_0 + \lambda_{1,3}a_1 + \lambda_{2,3}a_2 &= 0\,, \\ \lambda_{0,4}a_0 + \lambda_{1,4}a_1 + \lambda_{2,4}a_2 &= 0\,, \end{aligned}\right\} \tag{6.1}$$

and in this case the drag coefficient $C_D$ is given by

$$C_D = \frac{2\pi}{R}\sigma(\lambda_{0,1}a_0 + \lambda_{1,1}a_1)\,. \tag{6.2}$$

Using the series expansions for the modified Bessel functions $I_s$ and $K_s$, the expansions in powers of $q\left(=\frac{1}{2}kc\right)$ have been calculated for the nine $\lambda_{m,n}$'s in (6.1). If we denote for brevity

$$S_1 = -\left\{\gamma + \log\left(\frac{1}{2}\sigma q\right)\right\} = -\left\{\gamma + \log\left(\frac{a+b}{a}\frac{R}{16}\right)\right\}, \tag{6.3}$$

where $\gamma=0.57721\cdots$ is Euler's constant, the results are as follows:

$$\lambda_{0,1} = \tau - \frac{1}{8}\left\{(\sigma^3 + 4\sigma - 3\tau) + 4\sigma^3 S_1\right\}q^2 + \cdots,$$

$$\lambda_{0,2} = \frac{1}{2}\left\{(1+\tau^2) + 2S_1\right\}q - \frac{1}{48}\left\{(\sigma^4 + 4\tau^2 + 3\tau^4) + 6(\sigma^4 - \tau^4)S_1\right\}q^3 + \cdots,$$

$$\lambda_{0,3} = \frac{1}{8}\left\{(\sigma + 4\tau + \tau^3) + 4\sigma S_1\right\}q^2 + \cdots,$$

$$\lambda_{0,4} = \frac{1}{48}\left\{(\sigma^2 + 9 + 9\tau^2 + \tau^4) + 6(\sigma^2 + 2 - \tau^2)S_1\right\}q^3 + \cdots;$$

$$\lambda_{1,1} = -\frac{1}{2}(\sigma - 2\tau) + \cdots,$$

$$\lambda_{1,2} = \frac{1}{4}\left\{-(\sigma^2 - 2 - \tau^2) + 2S_1\right\}q + \cdots,$$

$$\lambda_{1,3} = \frac{1}{2}\tau + \cdots,$$

$$\lambda_{1,4} = \frac{1}{8}(2 + \tau^2)q + \cdots;$$

$$\lambda_{2,2} = -(1 + \tau^4)\frac{1}{q} + \cdots,$$

$$\lambda_{2,3} = -\frac{1}{6}(3\sigma - 4\tau + 2\tau^5) + \cdots,$$

$$\lambda_{2,4} = \tau^2\frac{1}{q} + \cdots.$$

Making use of these expansions, the system of simultaneous linear algebraic equations (6. 1) has been solved for $a_0$ and $a_1$, and substituting the values of $a_0$ and $a_1$ thus obtained into the right-hand side of (6. 2) and bearing in mind that

$$q^2 = \frac{k^2c^2}{4} = \frac{R^2}{64}\frac{c^2}{a^2}, \quad R = 4ka,$$

and

$$c\sigma = ce^{\xi_0} = a + b, \quad c\tau = ce^{-\xi_0} = a - b, \quad c^2 = a^2 - b^2,$$

we have calculated an expansion formula for the drag coefficient $C_D$ correct to the order of $R^2$. The result is given by

$$C_D = \frac{4\pi}{R[S_1 + a/(a+b)]}\left[1 - \frac{1}{S_1 + a/(a+b)}\left\{\left(\frac{a+b}{a}\right)^2 S_1^2 + \frac{a^2+b^2}{a^2}S_1 - \frac{5a^4 - 8a^3b - 30a^2b^2 - 24ab^3 - 3b^4}{12a^2(a+b)^2}\right\}\frac{R^2}{128}\right]. \tag{6. 4}$$

This formula differs from the corresponding Sidrak's formula in the coefficient of $R^2$, but is in complete agreement with the second approximation of Tomotika and Aoi's expansion formula (5).

The above expansion formula may be put in a somewhat simpler form. Thus, denoting

$$\varepsilon = \frac{a-b}{a+b}, \tag{6. 5}$$

and

$$S = S_1 + \frac{a}{a+b} = S_1 + \frac{1}{2}(1 + \varepsilon), \tag{6. 6}$$

we have

$$C_D = \frac{4\pi}{RS}\left[1 - \frac{1}{(1+\varepsilon)^2 S}\left\{S^2 + \frac{1}{2}(\varepsilon^2 - 2\varepsilon - 1)S - \frac{1}{48}(\varepsilon^4 + 12\varepsilon^3 + 18\varepsilon^2 + 4\varepsilon - 15)\right\}\frac{R^2}{32}\right], \quad (6.7)$$

where

$$S = \frac{1}{2}(1+\varepsilon) - \left\{\gamma + \log\frac{R}{8(1+\varepsilon)}\right\}, \quad \varepsilon = \frac{a-b}{a+b}. \quad (6.8)$$

As the limiting cases of the above formula, we can immediately obtain the expansion formulae correct to the order of $R^2$ for the drag coefficient of a circular cylinder and of a flat plate placed edgeways along the uniform stream.

(*a*) *The case of a circular cylinder*

Putting $a=b$ in (6.4) or $\varepsilon=0$ in (6.7), we have

$$C_D = \frac{4\pi}{RS}\left[1 - \frac{1}{S}\left(S^2 - \frac{1}{2}S + \frac{5}{16}\right)\frac{R^2}{32}\right], \quad (6.9)$$

where

$$S = \frac{1}{2} - \gamma - \log\frac{1}{8}R. \quad (6.10)$$

This is in accord with the second approximation of Tomotika and Aoi's expansion formula (**7**, **8**) for the drag coefficient of the circular cylinder.

(*b*) *The case of a flat plate*

Putting $b=0$ in (6.4) or $\varepsilon=1$ in (6.7), we obtain the result that

$$C_D = \frac{4\pi}{RS}\left[1 - \frac{1}{S}\left(S^2 - S - \frac{5}{12}\right)\frac{R^2}{32}\right], \quad (6.11)$$

where

$$S = 1 - \gamma - \log\frac{1}{16}R. \quad (6.12)$$

This coincides exactly with what has been given by Piercy and Winny (**9**, **10**).

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