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MICROCLIMATIC STUDY, III
A WORKING METHOD FOR FINDING AUSTAUSCH COEFFICIENTS FROM THE RECORDS OF TEMPERATURE FLUCTUATIONS

BY

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1. Introduction

In spite of tremendous development achieved in the realm of atmospheric turbulence, the modern meteorology has so far been prevented from further development by lack of quantitative studies on the Austausch coefficient. Especially in the field of the micro-meteorology concerning the lowest layer of the atmosphere, deficiency in the data has been preventing successful solution of many vital problems. Since Sverdrup (1) has started to solve the similar problem on the snow field, several reports have been published, but remarkable disagreement was seen in the values of the Austausch coefficient.

As is well known, the methods of finding the Austausch coefficient can be divided into the following two ways: the indirect method, in which the Austausch coefficient is calculated from the variation and distribution of mean state using the equation of eddy transfer, and the direct method, which deals with derivation from the mean state owing to the turbulent mixing. The latter seems to be better for calculating the Austausch coefficient during a certain period of time at a certain place. It seems quite natural to the present writer (2) who devised a new type self-recorder and obtained the records of small turbulent oscillation, to attempt to calculate the Austausch coefficient from these records. The writer originally wished to apply Ertel's method (3) which had looked most suitable for our purpose, but gave up the scheme, since some doubt arose in the principle of the method, and therefore decided to make a working method suitable for calculation of the coefficient from the records he had obtained, by improving the older method.

2. Discussion on Hesselberg's method

The principle of Hesselberg's method (4) is as follows:—

(i) The oscillating period of turbulence is at random, but there exists the mean period P which is almost constant. This P is significant.

(ii) The mixing length l can take any value between 0 and $\frac{1}{2}Pw$, but for convenience the mean value $\frac{1}{4}Pw$ is adopted for l, where w is the vertical component of the eddy velocity, namely:

$$l = -\frac{1}{4}Pw. \qquad (1)$$

(iii) Then the Austausch coefficient $A = -\rho\overline{wl}$, where ρ is the density of the air, becomes

$$A = \frac{1}{4} P \rho \overline{w^2}, \tag{2}$$

and when w obeys Maxwell's distribution law we have

$$A = \frac{1}{8} P \rho k, \tag{3}$$

where k is a certain unknown constant.

In order to use equation (2), it is necessary to know about $\overline{w^2}$. It is well known, however, that measurement of w is extremely difficult. While, equation (3) does not give directly the numerical value of the Austausch coefficient A because of the unknown constant k.

Although we may lay aside the fact that the assumption (1) for the mixing length l seems to be somewhat too bald, yet Hesselberg's method still suffers from a drawback that it is extremely difficult to obtain the mean period P in random fluctuation and in practice considerable personal error in obtaining it cannot be avoided.

3. Discussion on Ertel's method

The principle of Ertel's method (3) is as follows :—

(i) Let s be a fluctuating element, $\bar{s}$ its time-mean, l the mixing length and z the height. Then, we have

$$s = \bar{s} + l \frac{\partial \bar{s}}{\partial z}, \tag{4}$$

which gives

$$l = \frac{s - \bar{s}}{\partial \bar{s} / \partial z}.$$

Thus, the mixing length l can be calculated by using the values of $s - \bar{s}$ and $\partial \bar{s}/\partial z$ as obtained from the observed records.

(ii) As to the relationship between l and w, Ertel adopted Prandtl's assumptions $l \propto u$, $u \propto w$ and assumed that

$$l = -cw, \tag{5}$$

where c is a certain unknown constant.

(iii) Then the Austausch coefficient $A = -\rho \overline{wl}$ becomes

$$A = c \rho \overline{w^2} = \frac{\rho \overline{(s - \bar{s})^2}}{c (\partial \bar{s} / \partial z)^2}. \tag{6}$$

(iv) For the unknown constant c in this equation Hesselberg made use of the value $\frac{1}{4}P$, but Ertel attempted to eliminate it by assuming that s should be conserved. Then we have

$$\frac{ds}{dt} = 0. \tag{7}$$

Since, however, $\partial \bar{s}/\partial t = 0$, it follows from (7) that

$$-\frac{\partial s}{\partial t}=w\frac{\partial \bar{s}}{\partial z}, \tag{8}$$

which gives

$$\overline{|w|}=\overline{\left|\frac{\partial s}{\partial t}\right|}\Big/\frac{\partial \bar{s}}{\partial z}. \tag{9}$$

On the other hand, we have, from (4) and (5),

$$\overline{|w|}=\frac{\overline{|l|}}{c}=\frac{\overline{|s-\bar{s}|}}{c(\partial\bar{s}/\partial z)}. \tag{10}$$

If we assume that the turbulent velocity w obeys Gauss's distribution law, the following Cornu's relation may be used:

$$2\overline{w^2}=\pi\overline{|w^2|}. \tag{11}$$

Thus, the Austausch coefficient A becomes ultimately

$$A=\rho c\overline{w^2}=\rho c\frac{\pi}{2}\overline{|w|}^2=\rho\frac{\pi}{2}\frac{\overline{|s-\bar{s}|}\;\overline{|\partial s/\partial t|}}{(\partial\bar{s}/\partial z)^2}. \tag{12}$$

Ertel (3) has further made a refinement and has given the following expression for the Austausch coefficient. Thus,

$$A=\rho\frac{S_m(s)\,S_m(\partial s/\partial t)}{(\partial\bar{s}/\partial z)^2}, \tag{13}$$

where S_m means the mean deviation in the sense of the theory of errors. In practice, $S_m(s)$ and $S_m(\partial s/\partial t)$ can be obtained as follows by using observed values s_i $(i=1, 2,\cdots,n)$ taken at equal time-interval Δt from the records:

$$S_m(s)=\sqrt{\frac{\sum(s-\bar{s})^2}{n}},$$

$$S_m(\partial s/\partial t)=\frac{1}{\Delta t}\sqrt{\frac{\sum_{i=1}^{n-1}(s_{i+1}-s_i)^2}{n-1}}.$$

The above formula (13) can be applied in practice to calculating the Austausch coefficient by using acutual observed records, and indeed Lettau (5) tried to apply the formula, but no adequate records could be found.

The above Ertel's method is characteristic in that the unknown constant c is eliminated by introducing the value of $\partial s/\partial t$, but this very point is not entirely free from question.

Eq. (7) or eq. (8) conveys the same idea as eq. (4), and both are based upon the assumption of the conservation of the fluctuating physical element which is transported by moving eddies. But, differentiation of (4) with respect to the time gives

$$\frac{\partial s}{\partial t}=\frac{\partial l}{\partial t}\frac{\partial \bar{s}}{\partial z},$$

which is not in accord with (8), since $\partial l/\partial t \neq w$.

As an another example, let us consider a simple model such as $s = \bar{s} + a\sin\sigma t$. In this case, l is found to be proportional to $\sin\sigma t$, and since $\partial s/\partial t = -a\sigma\cos\sigma t$, w is proportional to $\cos\sigma t$. Thus, the relation (5) between l and w can no longer be always satisfied independently of the time. In other words, the two relations (4) and (8) contradict to each other, proving failure of Ertel's method.

4. A working method proposed by the present author

Following the priciple mentioned above and combining Ertel's intermediate formula (6), namely:

$$A = \frac{\rho\,\overline{(s-\bar{s})^2}}{c(\partial\bar{s}/\partial z)^2}$$

with Hesselberg's relation $c = \frac{1}{4}P$, the present author now proposes a working formula of the form:

$$A = \frac{4\rho}{P}\frac{\overline{(s-\bar{s})^2}}{(\partial\bar{s}/\partial z)^2}. \tag{14}$$

Then the coefficient of eddy transfer can be calculated by using our observed records:

(i) Acutually, the fluctuating curves appeared on the records are the result of superposition of the oscillations with various periods and amplitudes, and the shorter the period, the larger the mechanical decaying factor of the amplitude, so that the variation with extremely short period disappears from our records. Hence we must examine the influence of the existence of such an extremely short periodic oscillation upon the Austausch coefficient.

If we assume that $s - \bar{s} = a\sin\sigma t$ with $\sigma = 2\pi/P$, we have, from (14),

$$A = \frac{4\rho}{P}\frac{a^2P^{-1}\int_0^P \sin^2\sigma t\,dt}{(\partial\bar{s}/\partial z)^2} = \frac{2a^2\rho}{P}\frac{1}{(\partial\bar{s}/\partial z)^2},$$

or

$$A = \frac{R^2\rho}{2P}\frac{1}{(\partial\bar{s}/\partial z)^2}, \tag{15}$$

where R $(=2a)$ is the range of the oscillation.

Thus, it is found that the Austausch coefficient A is proportional to R^2/P, so that if $R_1^2/P_1 \ll R_2^2/P_2$ the coefficient A is not affected by the existence of the oscillation with R_1^2/P_1.

(ii) The time scale of our recorder is of about 1 millimetres per 10 seconds, so that the mean value and range over 20 seconds were calculated.

(a) Let s be the mean value over 20 seconds and also let $\bar{s}$ be the mean value of s over 10 minutes. Using these values we calculate the value of A_l as defined by

$$K_l = \frac{A_l}{\rho} = \frac{4\,\overline{(s-\bar{s})^2}}{P(\partial\bar{s}/\partial z)^2}. \tag{16}$$

This formula can legitimately be used, since there are more than thirty values of $s - \bar{s}$ taken from the records. If n is the number of maximum or minimum points of the variation curve of $s - \bar{s}$, the mean period P is given by $(10\times 60)/n$ sec, although not always clear-cut as desired.

(b) Our recorder is not fast enough to be capable of reading several points within 20 seconds, and a more rapidly rolling recorder must be employed. Thus, we are obliged to assume $s-\bar{s}=R\sin(2\pi t/P)$ with $P=20$ sec for the variations within 20 seconds, where R is the mean value over 10 minutes of the ranges (i. e. maximum minus minimum) in 20 seconds interval. Then we calculate the value of A_s as defined by

$$K_s=\frac{A_s}{\rho}=\frac{1}{40}\frac{R^2}{(\partial\bar{s}/\partial z)^2}. \tag{17}$$

(iii) If $A_l \gg A_s$, then A_l may be regarded as the Austausch coefficient. But, if $A_l \doteqdot A_s$ or $A_l < A_s$, some difficulties arise. Since the Austausch coefficient A is believed to depend upon the time scale, the difference between A_l and A_s may be caused by the scale differences. Further discussion on this point will be made in due time.

(iv) In applying the author's working method of calculating the Austausch coefficient, it is necessary to know the value of $\partial\bar{s}/\partial z$. It often occurs that the value of A depends more upon $(\partial\bar{s}/\partial z)^2$ than upon $\overline{(s-\bar{s})^2}$, so that the more precise value of $\partial\bar{s}/\partial z$ is requested. In order to obtain such a precise value of $\partial\bar{s}/\partial z$, it is desirable to make use of the junctions made specially for this purpose. Since, however, the number of junctions is limited in the actual observations, such a value may be obtained, in most cases, graphically from the values of temperatures at several heights. Therefore, considerable personal error cannot be avoided. It is to be noted here that the above method of calculating the value of A is by no means applicable to all cases, since it is not seldom that the value of $\partial\bar{s}/\partial z$ cannot be obtained directly.

5. An indirect method of calculating A from the mean state

As an addendum, an indirect method of calculating the Austausch coefficient A from the mean state is given. Let us assume that the following relation holds good:

$$\frac{\partial\bar{s}}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial\bar{s}}{\partial z}\right), \qquad (K=A\rho).$$

(When $\bar{s}$ represents the temperature, this relation does not always hold good in case when the radiation effects as well as other effects are taken into account.) Also we take for $\partial\bar{s}/\partial t$ the value of $\Delta s/\Delta t$ in 5 or 10 minutes. We estimate the values of $\partial^2\bar{s}/\partial z^2$ and $\partial\bar{s}/\partial z$ graphically from the vertical distribution curve of $\bar{s}$.

Let a first approximation of K be K_1 which is given by

$$K_1=\frac{\partial\bar{s}}{\partial t}\bigg/\frac{\partial^2\bar{s}}{\partial z^2}.$$

Then, making use of the value of $\partial K_1/\partial z$ as obtained by plotting the value of K_1 against height, the value of K can be calculated approximately by the following equation:

$$K=\left(\frac{\partial\bar{s}}{\partial t}-\frac{\partial K_1}{\partial z}\frac{\partial\bar{s}}{\partial z}\right)\bigg/\frac{\partial^2\bar{s}}{\partial z^2}.$$

6. An example

(i) Data: The survey over the frozen lake of Suwa, from $15^h 05^m$ to 10^m on Feb. 1,

1943. The original record is given on Plate I.

(ii) The variation of 20 sec mean temperatures and the vertical distribution of mean temperature are shown in Fig. 1.

(iii) The results of observation for 10 minutes from $15^h 05^m$ to 15^m are shown in Table 1.

Table 1.

Height	θ (°C)	R (°C)	$\frac{\partial\theta}{\partial z}$	$\overline{(s-\bar{s})^2}$	K_l	K_s
500 cm	−0.91	0.25	2.8×10^{-3}	0.074	630	199
100	−2.25	0.20	4.1 〃	0.066	262	59.5
50	−2.48	0.27	5.2 〃	0.098	242	67.4
20	−2.68	0.25	9.0 〃	0.066	54.3	19.3
5	−3.07	0.20	26.0 〃	0.035	3.42	1.48

(iv) The results for 5 minutes as obtained by applying the indirect method are shown in Table 2 below.

Table 2.

Height	mean	temp.	$\Delta\theta$	$\frac{\Delta\theta}{\Delta t}$
	5^m—10^m	10^m—15^m		
500 cm	−1.09°	−0.73°	0.36°	1.2×10^{-3}
100	−2.38	−2.11	0.27	0.9 〃
50	−2.62	−2.35	0.27	0.9 〃
20	−2.80	−2.57	0.23	0.77 〃
5	−3.09	−3.04	0.05	0.17 〃

Height	$\frac{\Delta\theta}{\Delta z}$		$\frac{\Delta^2\theta}{\Delta^2 z}$	
	5^m—10^m	10^m—15^m	5^m—10^m	10^m—15^m
500 cm	2.12×10^{-3}	2.54×10^{-3}	4.2×10^{-6}	3.2×10^{-6}
100	4.4 〃	4.1 〃	1.3×10^{-5}	0.9×10^{-5}
50	5.4 〃	5.6 〃	4.0×10^{-5}	7.4×10^{-5}
20	9.0 〃	11.0 〃	2.1×10^{-4}	4.3×10^{-4}
5	28.0 〃	42.0 〃	1.0×10^{-3}	1.8×10^{-3}

Height	$K_1 = \frac{\partial\theta}{\partial t} \Big/ \frac{\partial^2\theta}{\partial z^2}$		$K = \dfrac{\frac{\partial\theta}{\partial t} - \frac{\partial K_1}{\partial z}\frac{\partial\theta}{\partial z}}{\frac{\partial^2\theta}{\partial^2 z}}$	
	5^m—10^m	10^m—15^m	5^m—10^m	10^m—15^m
500 cm	285	375	40.5	78.2
100	69.1	100	192	406
50	22.5	12.2	81.0	13.1
20	3.65	1.78	7.23	2.81
5	0.17	0.09	0.95	1.7

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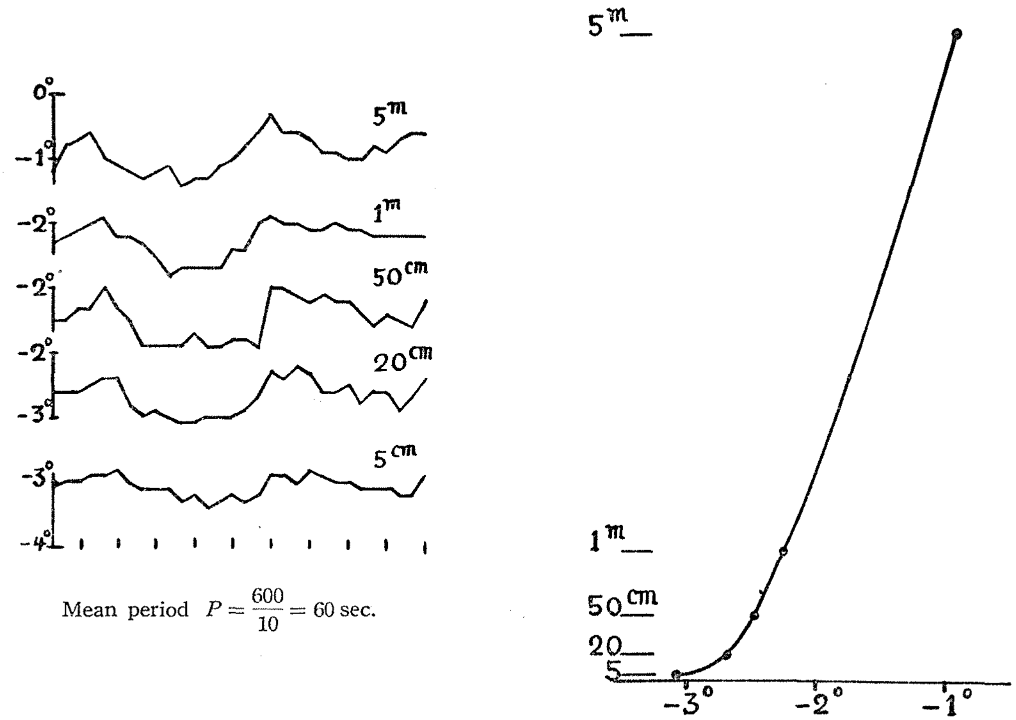
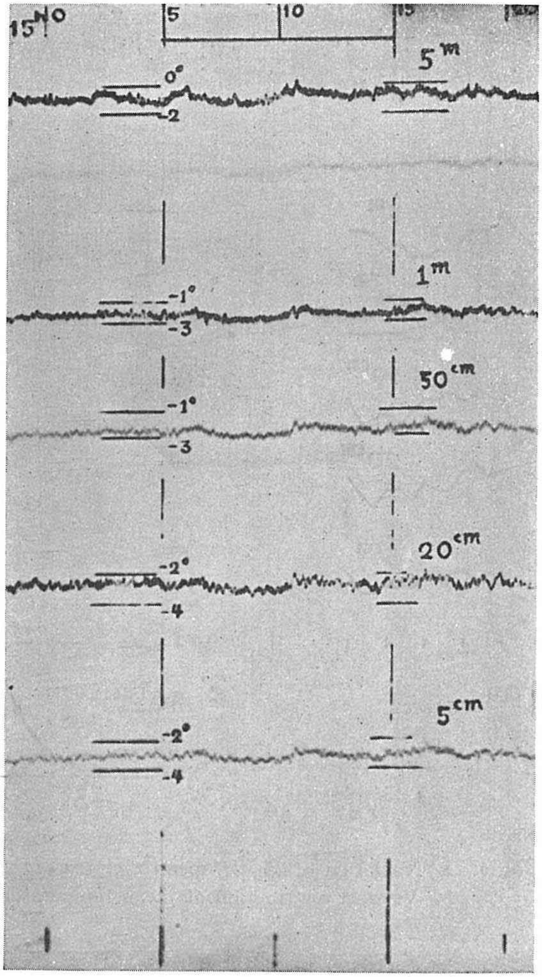


(a) (b)

FIG. 1. a) Variation of 20 sec. mean temperature,
b) Vertical distribution of mean temperature.

PLATE I



The record of temperatures over the frozen lake of Suwa.
at $15^h\,05^m$–10^m, Feb. 1 1943.
(Weather: Clear and calm)