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VECTORIAL STABILITY CRITERIA OF THE FEED-BACK PHYSICAL SYSTEM
PART II

BY

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1. Introduction.

The behaviour of the feed-back physical system is determined by its characteristic equation. In the preceding paper (1), the author has expressed the system with the characteristic vector which is composed of the coefficients of the characteristic equation and shown that the stability region $[\xi^n_\mu]$, $(\mu = 0, 1, 2, \ldots\ldots, n)$ of the n-dimensional system is related with the stability region $[\zeta^{n-2}_\nu]$, $(\nu = 0, 1, 2, \ldots\ldots, n-2)$ of the $(n-2)$-dimensional vector, as follows:

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \\ \xi_{n-2} \\ \xi_{n-1} \\ \xi_n \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ \beta & 1 & & & & & \\ \gamma & \beta & 1 & & & & \\ \ddots & \ddots & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \gamma & \beta & 1 & & \\ & & & \gamma & \beta & 1 & \\ & & & & \gamma & \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \\ \xi_{n-2} \\ 0 \\ 0 \end{pmatrix},$$

where $$\beta > 0, \quad \gamma > 0.$$

And it has also been shown that the stability criteria are introduced by putting $\beta = 0$ in the stable vectors, but no detailed descriptions about the construction of the stability criteria and the stability regions have been presented there.

In this paper, the geometrical properties and the constructions of the stability criteria and the stability regions are discussed. Further, the stabilizations by multiple feed-back are discussed and the theoretical results are ascertained by experiments.

2. $[\xi_{n-1}, \xi_n]$-stability criterion

From (21) of §6 in the preceding paper (1) (we shall denote it with (I. 6. 21) and use such notations in the following), the hypersurface of critical stability of an m-

dimensional vector is given by the following equations:

$$\left.\begin{aligned} \xi_{n-1} &= \gamma\xi_{n-3} - \gamma^2\xi_{n-5} + \gamma^3\xi_{n-7} - \cdots\cdots, \\ \xi_n &= \gamma\xi_{n-2} - \gamma^2\xi_{n-4} + \gamma^3\xi_{n-6} - \cdots\cdots. \end{aligned}\right\} \tag{1}$$

By fixing $\xi_{n-2}, \xi_{n-3}, \ldots, \xi_1, \xi_0$ at constant values in (1), the line of critical stability, i. e., $[\xi_{n-1}, \xi_n]$-curve of stability criterion is obtained.

This curve can be drawn both from $[\zeta_{n-3}, \zeta_{n-2}]$-curve of stability criterion of the $(n-2)$-dimensional vector and from $[\zeta_{n-2}, \zeta_{n-1}]$-curve of stability criterion of the $(n-1)$-dimensional vector.

i) Just as (1) has been derived, the hypersurface of critical stability of the $(n-2)$-dimensional vector is derived:

$$\left.\begin{aligned} \zeta_{n-3} &= \gamma\zeta_{n-5} - \gamma^2\zeta_{n-7} + \gamma^3\xi_{n-9} - \cdots\cdots, \\ \zeta_{n-2} &= \gamma\zeta_{n-4} - \gamma^2\zeta_{n-6} + \gamma^3\zeta_{n-8} - \cdots\cdots. \end{aligned}\right\} \tag{2}$$

By fixing $\zeta_{n-4}, \zeta_{n-5}, \ldots, \zeta_0$ at constant values

$$\zeta_{n-4} = \xi_{n-4}, \quad \zeta_{n-5} = \xi_{n-5}, \quad \ldots, \quad \zeta_0 = \xi_0 \tag{3}$$

in (2), $[\zeta_{n-3}, \zeta_{n-2}]$-curve of stability criterion is obtained, and from (1), (2) and (3), we derive

$$\left.\begin{aligned} \xi_{n-1} &= \gamma(\xi_{n-3} - \zeta_{n-3}), \\ \xi_n &= \gamma(\xi_{n-2} - \zeta_{n-2}). \end{aligned}\right\} \tag{4}$$

Since, in (4), (ξ_{n-3}, ξ_{n-2}) is a given point of the physical system and $(\zeta_{n-3}, \zeta_{n-2})$ is a varying point with γ on the curve of stability criterion of the $(n-2)$-dimensional vector, $[\xi_{n-1}, \xi_n]$-curve of stability criterion of the n-dimensional vector is simply obtained vectorially from the $[\zeta_{n-3}, \zeta_{n-2}]$-curve of stability criterion of the $(n-2)$-dimensional vector by use of (4) as shown in Fig. 1, in which

$\overrightarrow{OP} = [\zeta_{n-3}, \zeta_{n-2}]$: section of the $(n-2)$-dimensional vector of critical stability (variable),

$\overrightarrow{OQ} = [\xi_{n-3}, \xi_{n-2}]$: section of the fixed n-dimensional vector,

$\overrightarrow{QR} = [\xi_{n-1}, \xi_n] = \gamma\cdot\overrightarrow{PQ}$: section of the m-dimensional vector of critical stability (variable with PQ).

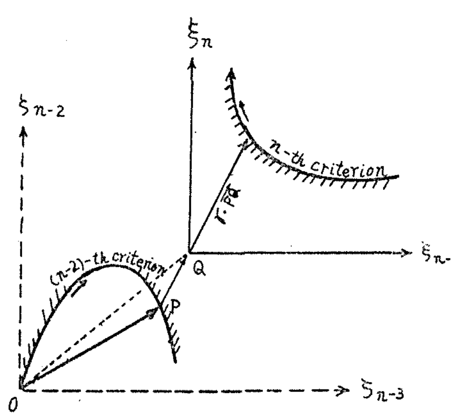


Fig. 1.

ii) The equation of the hypersurface of critical stability of the $(n-1)$-dimensional vector is given by

$$\left.\begin{aligned}\zeta_{n-2} &= \gamma\zeta_{n-4} - \gamma^2\zeta_{n-6} + \cdots\cdots,\\ \zeta_{n-1} &= \gamma\zeta_{n-3} - \gamma^2\zeta_{n-5} + \cdots\cdots.\end{aligned}\right\} \tag{5}$$

Fixing $\zeta_{n-3}, \zeta_{n-4}, \ldots, \zeta_0$ at constant values

$$\zeta_{n-3} = \xi_{n-3},\ \zeta_{n-4} = \xi_{n-4},\ \ldots,\ \zeta_0 = \xi_0$$

in (5) and combining with (1), we get

$$\left.\begin{aligned}\xi_{n-1} &= \zeta_{n-1},\\ \xi_n &= \gamma(\xi_{n-2} - \zeta_{n-2}),\end{aligned}\right\} \tag{6}$$

where $(\zeta_{n-2}, \zeta_{n-2})$ is a variable point on the line of critical stability of the $(n-1)$-dimensional vector, and ξ_{n-2} is a given component of the n-dimensional vector.

By (6), the $[\xi_{n-1}, \xi_n]$-curve of critical stability of the n-dimensional vector can be graphically obtained as shown in Fig. 2, in which

P: moving point on the curve of stability criterion of the $(n-1)$-dimensional vector,

S: fixed point $(\xi_{n-2}, 0)$,

$\overrightarrow{OX} = \overrightarrow{SQ} = \xi_{n-1} = \zeta_{n-1}$,

$\overrightarrow{XR} = \gamma\cdot\overrightarrow{QP}$: $\xi_n = \gamma(\xi_{n-2} - \zeta_{n-2})$,

and the locus of the point R is the $[\xi_{n-1}, \xi_n]$-curve of critical stability of the n-dimensional vector.

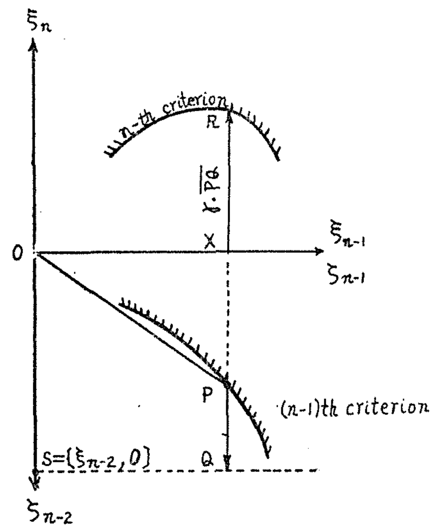


Fig. 2.

iii) As stated above, each line of critical stability of higher dimensions can be successively derived from either of the lines of critical stability of the 2- and 3-dimensional vectors shown in Fig. 3.

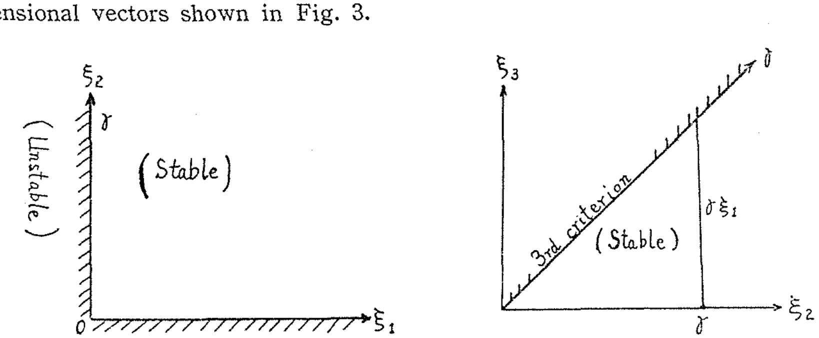


Fig. 3.

As stated in the preceding paper, the stability region can exist only on the right side of the curve of stability criterion thus obtained. In Fig. 4 are shown the curves of stability criterion of vectors of different dimensions which are obtained successively as well as the stability regions.

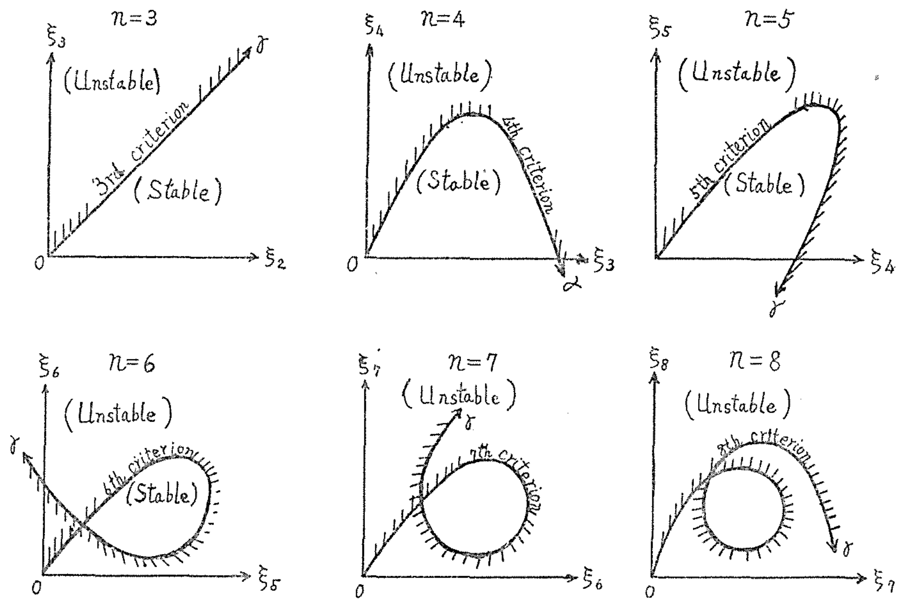


Fig. 4.

3. $[\xi_1, \xi_0]$-stability criterion

By (I. 5. 14′), the n-dimensional stable vector $[\xi]$ is derived from the $(n-2)$-dimensional vector $[\zeta]$ as follows:

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-2} \\ \xi_{n-1} \\ \xi_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \beta & 1 & & \\ \gamma & \beta & 1 & \\ & \gamma & \beta & \ddots \\ & & \gamma & \ddots & 1 \\ & & & \ddots & \beta \\ & & & & \gamma \end{pmatrix} \cdot \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{n-2} \end{pmatrix}. \tag{7}$$

By putting $\xi_\nu' = \xi_\nu/\xi_n$, $\zeta_\nu' = \zeta_\nu/\zeta_{n-2}$, (7) becomes

$$\begin{pmatrix} \xi_0' \\ \xi_1' \\ \xi_2' \\ \vdots \\ \xi_{n-1}' \\ \xi_n' \end{pmatrix} = \begin{pmatrix} \gamma' & & & & \\ \beta' & \gamma' & & & \\ 1 & \beta' & \ddots & & \\ & 1 & \ddots & \gamma' & \\ & & \ddots & \beta' & \\ & & & 1 & \end{pmatrix} \cdot \begin{pmatrix} \zeta_0' \\ \zeta_1' \\ \zeta_2' \\ \vdots \\ \zeta_{n-1}' \end{pmatrix}, \tag{8}$$

where $\gamma' = 1/\gamma$, $\beta' = \beta/\gamma$, or

$$\begin{pmatrix} \xi_n' \\ \xi_{n-1}' \\ \xi_{n-3}' \\ \vdots \\ \xi_1' \\ \xi_0' \end{pmatrix} = \begin{pmatrix} 1 & & & \\ \beta' & 1 & & \\ \gamma' & \beta' & \ddots & \\ & \gamma' & \ddots & 1 \\ & & \ddots & \beta' \\ & & & \gamma' \end{pmatrix} \cdot \begin{pmatrix} \zeta_{n-2}' \\ \zeta_{n-3}' \\ \zeta_{n-4}' \\ \vdots \\ \zeta_1' \\ \zeta_0' \end{pmatrix}. \tag{9}$$

Consequently, the n-dimensional vector $[\xi']$ of critical stability is derived from the $(n-2)$-dimensional vector $[\zeta']$ by the following equations:

$$\left.\begin{aligned} [\xi_{2\nu}'] &= [C_{\gamma'}] \cdot [\zeta_{2\nu}'], \\ [\xi_{2\nu+1}'] &= [C_{\gamma'}] \cdot [\zeta_{2\nu+1}'], \end{aligned}\right\} \tag{10}$$

where $\nu = 0, 1, 2, 3, \ldots,$ and

$$[C_{\gamma'}] = \begin{pmatrix} 1 & & & \\ \gamma' & 1 & & \\ & \gamma' & \ddots & \\ & & \ddots & 1 \\ & & & \gamma' \end{pmatrix}.$$

This is the case of n being odd, but the case of even n can be treated in a quite similar manner.

Therefore, if we take $\xi_\mu = \xi_{n-\mu}'$, $(\mu = 0, 1, 2, \ldots, n-2)$, the line of critical stability of ξ'-vector on the (ξ_1', ξ_0')-plane is quite alike that of ξ-vector on the (ξ_{n-1}, ξ_n)-plane, but the directions of increasing γ in these two lines are reverse to each other as shown in Fig. 5.

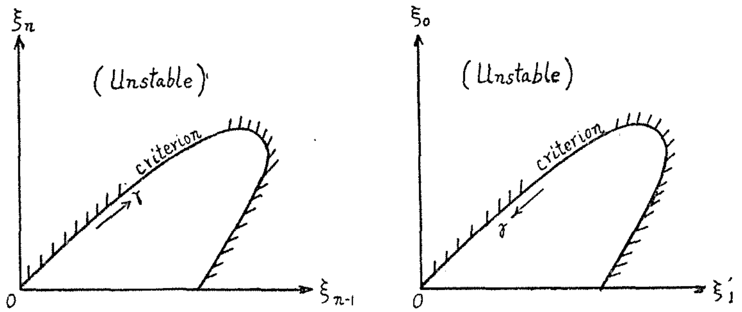


Fig. 5.

4. $[\xi_r,\ \xi_{r+1}]$-stability criterion

We assume that the components of the n-dimensional characteristic vector $[\xi]$ are fixed at constants, excepting ξ_r and ξ_{r+1}, and resolve this vector into the following two vectors, of which one is $(r+1)$-dimensional and the other, $(n-r)$-dimensional:

$$[\xi_0,\ \xi_1,\ \xi_2,\ \dots,\ \xi_{r-1},\ \xi_r',\ \xi_{r+1}'], \qquad [A']$$

$$[\xi_r'',\ \xi_{r+1}'',\ \xi_{r+2},\ \dots,\ \xi_{n-2},\ \xi_{n-1},\ \xi_n], \qquad [A'']$$

where ξ_r', ξ_{r+1}', ξ_r'' and ξ_{r+1}'' are variables.

The stability region of the n-dimensional characteristic vector is given from the $(n-2)$-dimensional stable vector $[\zeta]$, taking $\beta>0$ and $\gamma>0$ as parameters, by (I. 5. 14′) as follows:

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{r-1} \\ \hline \xi_r \\ \xi_{r+1} \\ \hline \xi_{r+2} \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \beta & 1 & & & & & & \\ \gamma & \beta & \ddots & & & & & \\ & \gamma & \ddots & 1 & & & & \\ & & \ddots & & & & & \\ \hline & & & \beta & 1 & & & \\ & & & \gamma & \beta & \ddots & & \\ \hline & & & & \gamma & \ddots & \ddots & \\ & & & & & \ddots & \ddots & 1 \\ & & & & & & \ddots & \beta \\ & & & & & & & \gamma \end{array}\right) \cdot \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{r-1} \\ \hline \zeta_r \\ \zeta_{r+1} \\ \hline \xi_{r+2} \\ \vdots \\ \zeta_{n-2} \end{pmatrix}.$$

On the other hand, the stable vectors of [A′] and [A″] are given by

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi'_r \\ \xi'_{r+1} \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \beta & 1 & & & \\ \gamma & \beta & \ddots & & \\ & \gamma & \ddots & 1 & \\ & & \ddots & \beta & \\ & & & \gamma & \end{pmatrix} \cdot \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{r-2} \\ \zeta_{r-1} \end{pmatrix}, \tag{11 a}$$

and

$$\begin{pmatrix} \xi''_r \\ \xi''_{r+1} \\ \xi_{r+1} \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \beta & 1 & & & \\ \gamma & \beta & \ddots & & \\ & \gamma & \ddots & 1 & \\ & & \ddots & \beta & \\ & & & \gamma & \end{pmatrix} \cdot \begin{pmatrix} \zeta_r \\ \zeta_{r+1} \\ \zeta_{r+2} \\ \vdots \\ \zeta_{n-3} \\ \zeta_{n-2} \end{pmatrix}. \tag{11 b}$$

From the relation:

$$\begin{pmatrix} \xi_r \\ \xi_{r+1} \end{pmatrix} = \begin{pmatrix} \xi'_r \\ \xi'_{r+1} \end{pmatrix} + \begin{pmatrix} \xi''_r \\ \xi''_{r+1} \end{pmatrix}, \tag{11c}$$

if $[\xi']$ and $[\xi'']$ are stable, $[\xi]$ is stable.

Consequently, by adding two stable vectors $[\xi']$ and $[\xi'']$ with reference to the same values of β and γ on the (ξ_r, ξ_{r+1})-plane, the stability region of $[\xi]$ and the stability criterion is obtained as shown in Fig. 6.

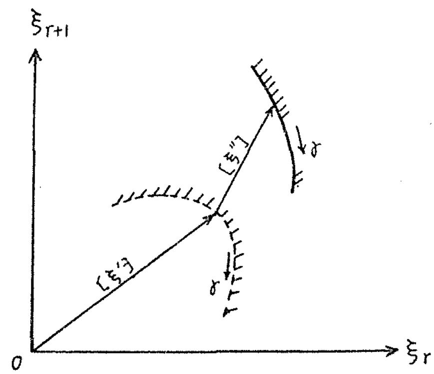


Fig. 6.

5. $[\xi_{n-2}, \xi_{n-1}, \xi_n]$-stability criterion

We put $\beta = 0$ in (7), and resolving it into the following two equations:

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-3} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \gamma & 0 & \ddots & \\ & \gamma & \ddots & 1 \\ & & \ddots & 0 \\ & & & \gamma \end{pmatrix} \cdot \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{n-3} \end{pmatrix}, \tag{12 a}$$

and

$$\begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \\ \xi_n \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 1 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} \zeta_{n-4} \\ \zeta_{n-3} \\ \zeta_{n-2} \end{pmatrix}. \tag{12 b}$$

Then (12b) gives the surface of critical stability when ξ_ν, ($\nu = 0, 1, 2, 3, \ldots, n-3$), are fixed at constant values. When ξ_ν, ($\nu = 0, 1, 2, 3, \ldots, n-3$), are fixed at constant values, ζ_ν ($\nu = 0, 1, 2, 3, \ldots, n-3$), are obtained as functions of γ only from (12a), and the vector of critical stability (12b) has its components ξ_{n-2}, ξ_n as functions of γ and ζ_{n-2}, and component ξ_{n-1} as a function of γ only.

Accordingly, the vector of critical stability (12b) lies on the plane parallel to ξ_{n-2}- and ξ_n-axes, having the gradient of $\tan^{-1}\gamma$ against ξ_{n-2}-axis, and as ξ_{n-1} depends only on the frequency, the oscillating frequency of the system will be determined by ξ_{n-1}.

If we assume $\zeta_{n-2} = 0$, equation (12b) becomes:

$$\begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} \zeta_{n-4} \\ \zeta_{n-3} \end{pmatrix}, \tag{13}$$

which is the $[\xi_{n-2}, \xi_{n-1}]$-stability criterion of the $(n-1)$-dimensional vector.

6. $[\xi_r, \xi_{r+2}]$- and $[\xi_{n-2}, \xi_n]$- stability criteria

As (I. 6. 19) in §6 of Part I, the vector of critical stability is resolved into two vectors, of which one is a vector which has all zero components in reference to the odd-number-axes, and the other is a vector which has all zero components referring to the even-number-axes.

Eliminating $\zeta_{2\nu}$ and $\zeta_{2\nu+1}$ from the two equations of (I. 6. 19) respectively, we obtain

$$\left.\begin{aligned} \sum_\nu (-1)^\nu \gamma^\nu \xi_{n-(2\nu+1)} &= 0\,, \\ \sum_\nu (-1)^\nu \gamma^\nu \xi_{n-2\nu} &= 0\,. \end{aligned}\right\} \tag{14}$$

Now, if we fix all the components excepting ξ_r and ξ_{r+2} which are obtained only in one of (14), for instance the latter, the values of γ will be determined from the other equation (in this case the former) of (14), which does not contain ξ_r and ξ_{r+2}. For each determined value of γ, the latter equation of (14) represents a straight line on (ξ_r, ξ_{r+2})-plane, and the set of these straight lines determine the stability criterion as shown in Fig. 7.

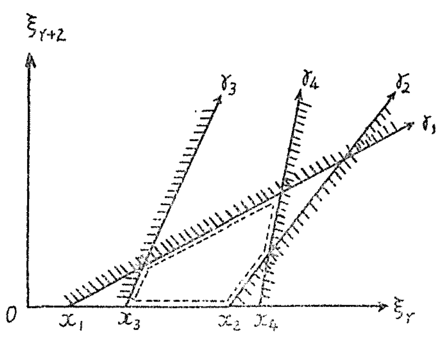


Fig. 7.

The latter equation of (14) is expressed for one determined value γ_λ, as follows:

$$\xi_{r+2} = \gamma_\lambda(\xi_r - x_\lambda),$$

where

$$x_\lambda = \gamma_\lambda \left\{ \sum_{\mu=1}^{\rho} (-\gamma_\lambda)^{\mu-1} \xi_{r-2\mu} + \sum_{\nu=2}^{n-\rho} (-\gamma_\lambda)^{-\nu-1} \xi_{r+2\nu} \right\},$$

with $\begin{cases} \rho = r/2 & \text{for even } r, \\ \rho = (r-1)/2 & \text{for odd } r, \end{cases}$ $\begin{cases} \lambda = 1, 2, \dots, n/2 & \text{for even } n, \\ \lambda = 1, 2, \dots, (n-1)/2 & \text{for odd } n. \end{cases}$

Putting especially $r = n-2$, we obtain $[\xi_{n-2}, \xi_n]$-stability criterion as follows:

$$\xi_n = \gamma_\lambda(\xi_{n-2} - x_\lambda),$$

where

$$x_\lambda = \sum_{\mu=1}^{\rho} (-1)^{\mu+1} \gamma_\lambda{}^{\mu} \xi_{n-2(\mu+1)},$$

with $\lambda = 1, 2, \dots, \rho+1$

and $\begin{cases} \rho = (n-2)/2 & \text{for even } n, \\ \rho = (n-3)/2 & \text{for odd } n. \end{cases}$

7. Composition of stability criteria

Let $[\xi_\nu']$, $(\nu = 0, 1, 2, 3, \dots, n)$, denote an n-dimensional stable vector and $[\xi_\mu'']$, $(\mu = 0, 1, 2, \dots, m)$, an m-dimensional stable vector. When all components of these vectors excepting ξ_{n-1}', ξ_n', ξ_0'' and ξ_1'' are fixed at constant values, $[\xi_{n-1}, \xi_n]$-stability region of the $(n+m-1)$-dimensional characteristic vector $[\xi_\nu]$, $(\nu = 0, 1, 2, 3, \dots, n+m-1)$, is obtained as follows.

By the methods in §2 and §3, the $[\xi_{n-1}', \xi_n']$-stability region of the $[\xi_\nu']$-vector and $[\xi_0'', \xi_0'']$-stability region of the $[\xi_\mu'']$-vector are readily obtained, and corresponding stable vectors are expressed as (15 a, b) and (16 a, b) respectively:

$$\begin{pmatrix} \xi_0' \\ \xi_1' \\ \xi_2' \\ \vdots \\ \xi_{n-3}' \\ \xi_{n-2}' \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ \beta & 1 & & & & \\ \gamma & \beta & 1 & & & \\ & \gamma & \beta & \ddots & & \\ & & \gamma & \ddots & \ddots & \\ & & & \gamma & \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta_0' \\ \zeta_1' \\ \zeta_2' \\ \vdots \\ \zeta_{n-3}' \\ \zeta_{n-2}' \end{pmatrix}, \tag{15 a}$$

$$\begin{pmatrix} \xi_{n-1}' \\ \xi_n' \end{pmatrix} = \begin{pmatrix} \gamma & \beta \\ 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} \zeta_{n-3}' \\ \zeta_{n-2}' \end{pmatrix}, \tag{15 b}$$

$$\begin{pmatrix} \xi_0'' \\ \xi_1'' \end{pmatrix} = \begin{pmatrix} \gamma'' & 0 \\ \beta'' & \gamma'' \end{pmatrix} \cdot \begin{pmatrix} \zeta_0'' \\ \zeta_1'' \end{pmatrix}, \tag{16 a}$$

$$\begin{pmatrix} \xi_2'' \\ \xi_3'' \\ \xi_4'' \\ \vdots \\ \xi_{m-1}'' \\ \xi_m'' \end{pmatrix} = \begin{pmatrix} 1 & \beta'' & \gamma'' & & & \\ & 1 & \beta'' & \ddots & & \\ & & 1 & \ddots & \gamma'' & \\ & & & \ddots & \beta'' & \gamma'' \\ & & & & 1 & \beta'' \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta_0'' \\ \zeta_1'' \\ \zeta_2'' \\ \vdots \\ \zeta_{n-1}'' \\ \zeta_{m-1}'' \end{pmatrix}. \tag{16 b}$$

From (15 a, b) and (16 a, b), the $(n+m-1)$-dimensional stable vector $[\xi_\sigma]$, $(\sigma = 0, 1, 2, \ldots, n+m-1)$, is obtained as follows:

$$\begin{aligned} \xi_\sigma &= \xi_\sigma' && (\sigma = 0, 1, 2, \ldots, n-2), \\ \xi_{n-1} &= \xi_{n-1}' + \xi_0'', \\ \xi_n &= \xi_n' + \xi_1'', \\ \xi_\sigma &= \xi_{\sigma-n+1}'' && (\sigma = n-1, n-2, n-3, \ldots, n+m-1). \end{aligned}$$

When the components $\xi_\sigma (\sigma = 0, 1, 2, \ldots, n+m-1)$ in the above equations are fixed at constant values, the stability region of the $(m+n-1)$-dimensional characteristic vector is given by

$$\begin{pmatrix} \xi_{n-1} \\ \xi_n \end{pmatrix} = \begin{pmatrix} \xi_{n-1}' \\ \xi_n' \end{pmatrix} + \begin{pmatrix} \xi_0'' \\ \xi_1'' \end{pmatrix} = \begin{pmatrix} \gamma & \beta & \gamma'' & 0 \\ 0 & \gamma & \beta'' & \gamma'' \end{pmatrix} \begin{pmatrix} \zeta_{n-3}' \\ \zeta_{n-2}' \\ \zeta_{n-1}'' \\ \zeta_n'' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \beta & 1 & 0 \\ 0 & \gamma & \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta_{n-3} \\ \zeta_{n-2} \\ \zeta_{n-1} \\ \zeta_n \end{pmatrix},$$

where

$$\begin{aligned} &\zeta_{n-3} = \zeta_{n-3}', \quad \zeta_{n-1} = \zeta_0''/\gamma, \quad \beta'' = \beta/\gamma, \\ &\zeta_{n-2} = \zeta_{n-2}', \quad \zeta_n = \zeta_1''/\gamma, \quad \gamma'' = 1/\gamma. \end{aligned}$$

Putting $\beta = 0$ in these equations, the stability criterion of the $(m+n-1)$-dimensional characteristic vector will be obtained.

For example, we shall show the composition of the stability criterion of the 5-dimensional vector from the two stability criteria of 3-dimensional vectors $[\xi']$ and $[\xi'']$.

If

$$\left.\begin{array}{l}\xi_0' = \zeta_0' = \xi_0\\ \xi_1' = \zeta_1' = \xi_1\end{array}\right\} : \left\{\begin{array}{l}\text{fixed components of the 5-dimensional}\\ \text{vector } [\xi],\end{array}\right.$$

$$\left.\begin{array}{l}\xi_2' = \gamma\zeta_0'\\ \xi_3' = \gamma\zeta_1'\end{array}\right\} : \left\{\begin{array}{l}[\xi_2', \xi_3']\text{-stability criterion of the 3-}\\ \text{dimensional vector } [\xi'],\end{array}\right.$$

$$\left.\begin{array}{l}\xi_0'' = \zeta_0''/\gamma\\ \xi_1'' = \zeta_1''/\gamma\end{array}\right\} : \left\{\begin{array}{l}[\xi_0'', \xi_1'']\text{-stability criterion of the 3-}\\ \text{dimensional vector } [\xi''],\end{array}\right.$$

$$\left.\begin{array}{l}\xi_2'' = \zeta_0'' = \xi_4\\ \xi_3'' = \zeta_1'' = \xi_5\end{array}\right\} : \left\{\begin{array}{l}\text{fixed components of the 5-dimensional}\\ \text{vector } [\xi],\end{array}\right.$$

the $[\xi_2, \xi_3]$-stability criterion of the 5-dimensional characteristic vector can be obtained by the following equations, as shown graphically in Fig. 8,

$$\xi_2 = \xi_2' + \xi_0'' = \gamma\zeta_0' + \zeta_0''/\gamma ,$$
$$\xi_3 = \xi_3' + \xi_1'' = \gamma\zeta_1' + \zeta_1''/\gamma ,$$

where all the components of the 5-dimensional characteristic vector $[\xi]$ excepting ξ_2 and ξ_3, are fixed at constant values.

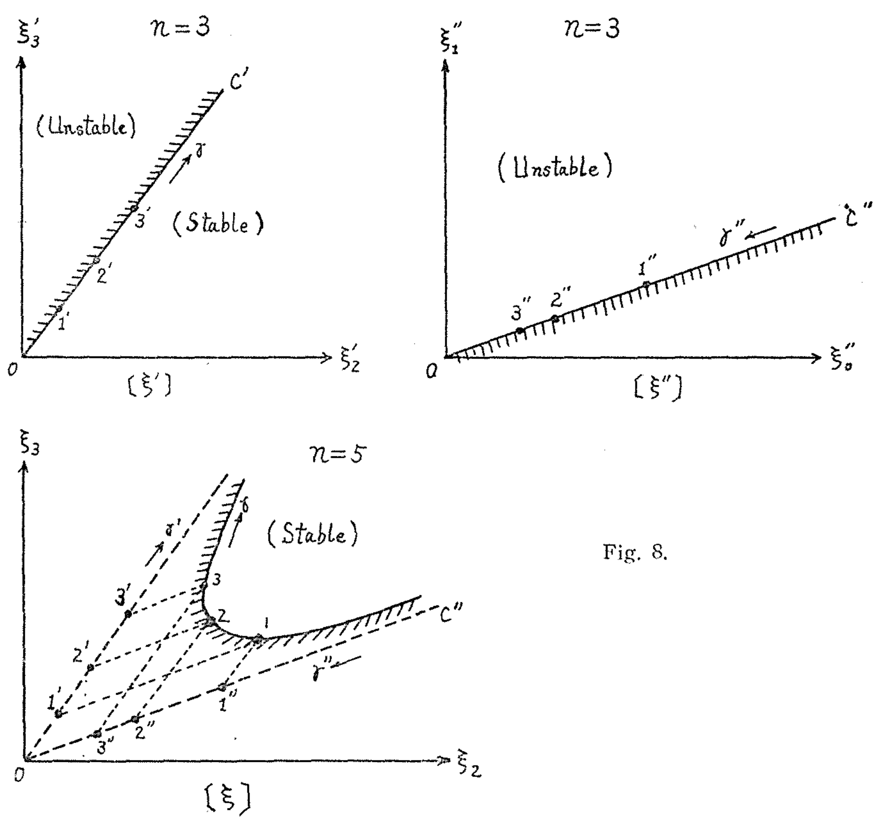


Fig. 8.

8. γ-scale on the line of critical stability

In order to construct the stability criterion of higher dimensions from those of lower dimensions, it is necessary to scale the value of γ as a function of frequency on the line of critical stability. In the following, some properties of γ-scale is described.

As stated in §7 of the preceding paper, in order that the feed-back physical system may be stable, it is necessary that both equations of the hypersurface of critical stability have all real roots of γ.

At one point of γ-scale, the characteristic equation has the factor $(p^2+\gamma)$, and has the roots $\pm j\sqrt{\gamma}$. Consequently, the oscillating frequency of the system is given by:

$$\omega = \sqrt{\gamma}\,.$$

The $[\xi_0, \xi_1]$-line of critical stability has the scale of $\gamma' = 1/\gamma$ instead of γ, and in this case, the oscillating frequency is given by

$$\omega = 1/\sqrt{\gamma'}\,.$$

In the case of the normalized characteristic vector, $\xi_\nu' = \xi_\nu/(\xi_1)^\nu$, $(\nu = 1, 2, \ldots, n)$, the characteristic equation has the factor $\{(\xi_1 p)^2+\gamma\}$. So the oscillating frequency is given by

$$\omega = \sqrt{\gamma}/\xi_1\,.$$

9. Stabilization by multiple feed-back

The additional feed-back is often used for stabilizing the feed-back physical system as shown in Fig. 9, in which the output signal of Z_s is fed back to the input of Z_r through the feed-back element Z_{rs}.

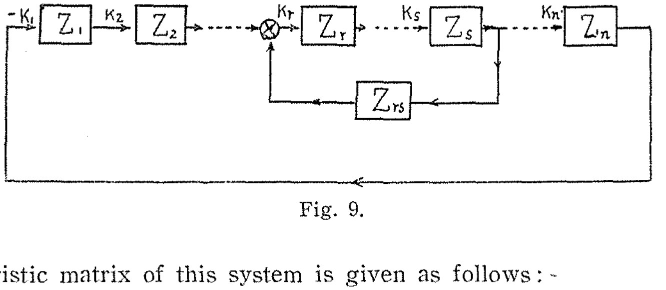


Fig. 9.

The characteristic matrix of this system is given as follows:-

$$[\boldsymbol{Z}] = \begin{pmatrix} Z_1 & & & & & & K_1 \\ -K_2 & Z_2 & & & & & \\ & \ddots & \ddots & & & & \\ & & -K_r & Z_n & Z_{rs} & & \\ & & & \ddots & \ddots & & \\ & & & & -K_s & Z_s & \\ & & & & & \ddots & \ddots \\ & & & & & -K_n & Z_n \end{pmatrix}. \tag{17}$$

By expanding the determinant of (17) with respect to the r–th row, the characteristic equation of the system is expressed in the following form :

$$|[\boldsymbol{Z}]| = |[\boldsymbol{Z}]_0| + |[\boldsymbol{Z}]_{\mathrm{I}}| = 0\,,$$

where $[\boldsymbol{Z}]_0$: the characteristic matrix of the system without any additional feed-back,

$[\boldsymbol{Z}]_{\mathrm{I}}$: the feed-back matrix which is produced by substituting Z_{rs} for the $(r,\ s)$–element and making zero all other elements of the r–th row in $[\boldsymbol{Z}]_{\mathrm{I}}$, so that $|[\boldsymbol{Z}]_{\mathrm{I}}|$ is the co-factor of Z_{rs} of $|[\boldsymbol{Z}]|$ multiplied by Z_{rs}, namely :

$$[\boldsymbol{Z}]_0 = \begin{pmatrix} Z_1 & & & & & & K_1 \\ -K_2 & Z_2 & & & & & \\ & \ddots & \ddots & & & & \\ & & -K_r & Z_r & & & \\ & & & \ddots & \ddots & & \\ & & & & -K_s & Z_s & \\ & & & & & \ddots & \ddots \\ & & & & & -K_n & Z_n \end{pmatrix},$$

and

$$[\boldsymbol{Z}]_{\mathrm{I}} = \begin{pmatrix} Z_1 & & & & & & K_1 \\ -K_2 & Z_2 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 0 & Z_{rs} & & \\ & & & \ddots & \ddots & & \\ & & & & -K_s & Z_s & \\ & & & & & \ddots & \ddots \\ & & & & & -K_n & Z_n \end{pmatrix}.$$

Accordingly, the change in the characteristic equation by the additional feed-back is

$$
|[\boldsymbol{Z}]_{\mathrm{I}}| = (-1)^{r+s} Z_{rs}
\left|\begin{array}{cccccc:cccc}
Z_1 & & & & & & & & & K_1 \\
-K_2 & Z_2 & & & & & & & & \\
& \ddots & \ddots & & & & & & & \\
& & -K_{r-1} & Z_{r-1} & & & & & & \\
\hdashline
& & & -K_{r+1} & Z_{r+1} & & & & & \\
& & & & \ddots & \ddots & & & & \\
& & & & -K_{s-1} & Z_{s-1} & & & & \\
& & & & & -K_s & & & & \\
& & & & & & Z_{s+1} & & & \\
& & & & & & -K_{s+2} & Z_{s+2} & & \\
& & & & & & & \ddots & \ddots & \\
& & & & & & & & -K_n & Z_n
\end{array}\right|
$$

$$
= (-1)^{r+s} Z_{rs} Z_1 Z_2 \cdots\cdots Z_{r-1} (-1)^{s-r} K_{r+1} K_{r+2} \cdots\cdots K_s Z_{s+1} Z_{s+2} \cdots\cdots Z_n
$$

$$
= (K_{r+1} K_{r+2} \cdots\cdots K_s)\ (Z_1 Z_2 \cdots\cdots Z_{r-1} Z_{s+1} \cdots\cdots Z_n) Z_{rs} \,. \tag{18}
$$

Thus, the change of the characteristic vector due to the additional feed-back depends on all other elements and connection coefficients of the system excepting Z_r, Z_{r+1}, ..., Z_s, K_1, K_2, ..., K_r and K_{s+1}, ..., K_n. This change is called the feed-back vector and is easily obtained from $|[\boldsymbol{Z}]_{\mathrm{I}}|$.

The stabilization by an additional feed-back is actually performed by feeding back the time rate of change of the manipulator output to the detector input, as shown in Fig. 10.

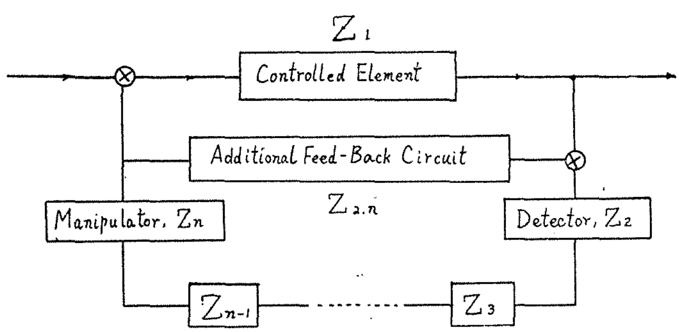


Fig. 10.

The feed-back matrix of the system as shown in Fig. 10, is given by

$$
[\boldsymbol{Z}]_{\mathrm{I}} = \left(\begin{array}{cccccc}
Z & & & & & 0 \\
0 & 0 & & & & Z_{2,n} \\
& -K_3 & Z_3 & & & \\
& & \ddots & \ddots & & \\
& & & -K_{n-1} & Z_{n-1} & \\
& & & & -K_n & Z_n
\end{array}\right),
$$

and since $Z_{2,n} = xp = xd/dt$, the determinant of this matrix becomes:

$$|[Z_I]| = K_3 K_4 \cdots\cdots K_n x p Z_1 ,$$

where x is a proportional constant called the feed-back coefficient. In the case of $Z_1 = L_1 p + R_1$, it becomes:

$$|[Z]_I| = Kx(L_1 p^2 + R_1 p) ,$$

where $K = K_3 K_4 \cdots\cdots K_n$.

Thus, the feed-back vector is obtained as follows:

$$\left.\begin{aligned} &\xi_\nu = 0, \quad (\nu = 0, 1, 2, \ldots, n-3, n) \\ &\begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \end{pmatrix}_I = Kx \begin{pmatrix} L_1 \\ R_1 \end{pmatrix}. \end{aligned}\right\}$$

Therefore, the characteristic vector is changed in two components only, by the additional feed-back element connected, that is:

$$\begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \end{pmatrix} = \begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \end{pmatrix}_0 + Kx \begin{pmatrix} L_1 \\ R_1 \end{pmatrix}.$$

Thus, the magnitude of the feed-back vector is proportional to this feed-back coefficient x, and its direction is determined by the time constant $T = L_1/R_1$ of the element $Z_1 = L_1 p + R_1$, independent of x, as shown in Fig. 11, in which

$\overrightarrow{OP} = [\xi_\nu]_0$: characteristic vector without any additional feed-back,

$\overrightarrow{PQ} = [\xi_\nu]_I$: feed-back vector,

$\tan\theta = 1/T_1$,

$T_1 = L_1/R_1$: time constant of the controlled element.

As will be seen from Fig. 11, the characteristic vector $\overrightarrow{OP}$ of the control system which is unstable without an additional feed-back, can be changed to the stable vector $\overrightarrow{OQ}$ by adjusting the feed-back coefficient x.

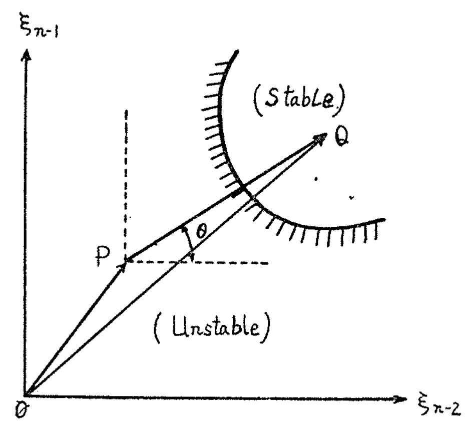


Fig. 11.

10. Improvement of characteristics by an additional feed-forward

The additional feed-forward which is performed such that through the element Z_{sr}, the output signal of Z_r is fed to the input of Z_s as shown in Fig. 12, has

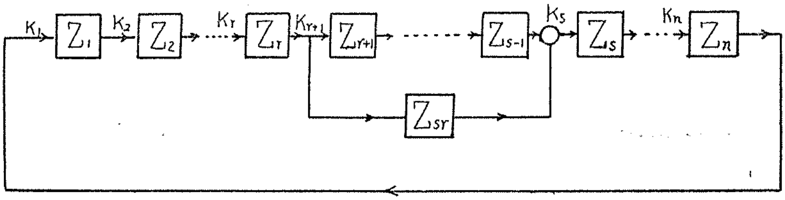


Fig. 12.

similar effects to the additional feed-back and is used to improve the stability and the response of the control system.

The characteristic matrix of this system is given by

$$[Z] = \begin{pmatrix} Z_1 & & & & & & & K_1 \\ -K_2 & Z_2 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & K_r & Z_r & & & & \\ & & & \ddots & \ddots & & & \\ & & Z_{sr} & & -K_s & Z_s & & \\ & & & & & \ddots & \ddots & \\ & & & & & -K_{n-1} & Z_{n-1} & \\ & & & & & & -K_n & Z_n \end{pmatrix}.$$

So, the characteristic equation becomes:

$$|[Z]| = |[Z]_0| + |[Z]_{II}| = 0 ,$$

where $[Z]_0$ is the characteristic matrix of the system without any feed-forward, and $[Z]_{II}$ is the feed-forward matrix, which is produced by substituting Z_{sr} for the (s, r) element, and making zero all other elements of the s-th row in $[Z]_0$.

Thus, we have:

$$|[Z]_{II}| = (-1)^{r+s} Z_{s,r} \left| \begin{array}{cccc|cccc|ccc} Z_1 & & & & & & & & & & K_1 \\ -K_2 & Z_2 & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & -K_{r-1} & Z_{r-1} & & & & & & & \\ & & & -K_r & & & & & & & \\ & & & & Z_{r+1} & & & & & & \\ & & & & -K_{r+2} & Z_{r+2} & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & -K_{s-1} & Z_{s-1} & & & \\ \hline & & & & & & & & -K_{s+1} & Z_{s+1} & \\ & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & -K_n & Z_n \end{array} \right|$$

$$= K' Z_{s,r} (Z_{r+1} Z_{r+2} \cdots\cdots Z_{s-1}) ,$$

where $K' = K_1 K_2 \cdots\cdots K_r K_{s+1} \cdots\cdots K_n$.

Hence, the change of the characteristic vector due to an additional feed-forward depends upon $Z_{r+1}, Z_{r+2}, \ldots, Z_{s-1}, Z_{s,r}$ and $K_1, K_2, \ldots, K_r, K_{s+1}, \ldots, K_n$. This change is called the "feed-forward vector".

If the time rate of change of the output of Z_{r-1} is fed forward to the input of Z_{r+1}, as it is often used, the feed-forward vector is given by:

$$\begin{pmatrix} \xi_{n-2} \\ \xi_{n-1} \end{pmatrix} = K'x \begin{pmatrix} L_r \\ R_r \end{pmatrix},$$

where $Z_r \equiv L_r p + R_r$.

Thus, quite similarly to the case of the additional feed-back, the stabilization of the system is performed and the magnitude of the feed-forward vector varies with the feed-forward coefficient x, and the direction is determined by the time constant $T_r = L_r/R_r$ of the element Z_r, as shown in Fig. 13, in which

$\overrightarrow{OP}$: characteristic vector without any additional feed-forward,

$\overrightarrow{PQ}$: feed-forward vector, and $\tan\theta = 1/T_r$.

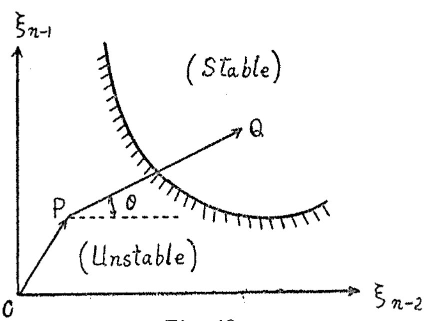


Fig. 13.

11. Experimental confirmation

i) Experimental apparatus

In order to confirm experimentally the author's stability criteria of the feed-back physical system, the stabilizing experiments of the speed control of a D. C. shunt motor by means of an additional feed-back (antihunting device) have been carried out.

The experimental apparatus consists of Ward-Leonard system with feed-back elements, whose connection diagram and elements are given in Fig. 14, and Table I, respectively.

The speed of the D. C. shunt motor in Ward-Leonard system, that is, the controlled variable, should be kept constant by means of the feed-back, whose mechanism is as follows. First, the motor speed is detected by a tachometer generator as a speed voltage, which passing through a filter is compared with the set value of the speed voltage and difference voltage, after amplified 1400 times as large, is converted into a shift angle of phase by a phase shifter, and deformed into a peak voltage through the peak-generating device, and this phase shift of peak voltage, impressed on the grids of two thyratrons, controls their ignition angles, and their output current excites the

Table I

Department	Symbol	Element	Rating
Controlled system	PM	3 phase induction motor for prime mover	3 HP, 1430rpm, 110V
	G	Ward-Leonard D.C. generator	2kW, 1500rpm, 110V
	M	Controlled D.C. shunt motor	1kW, 1500rpm, 105V
	LG	D. C. generator for load	1kW, 1500rpm, 105V
	Ra	Total resistance of armature circuit of Leonard {motor, generator}	
	R_L	Resistance load (lamp bank)	
Detecting device	T	Tachometer generator	1000 rpm, 6V, 3 mA
	r_T	Armature resistance of tachometer generator	100 Ω
	R_V	Internal resistance of voltmeter	2 kΩ
	r_1	Resistance of choking coil	1.5 kΩ
	L_1	Inductance of choking coil	30 H
	C_1	Condenser	3.5 μF
	R_1	Resistance	30 kΩ
Amplifier	V_1	First step amplifier tube UZ-6C6	$\mu=1500$, $r_p=1.5$MΩ
	V_6	Glow tube for stabilizing voltage	VRA 135V/60mA
	V_7	Glow tube for stabilizing voltage	VRA 65V/80mA
	R_2	Resistance load of the 1st step amplifier tube	100kΩ
	C_2	Condenser	2.46μF
	V_2	Second step amplifier tube UY-76	$\mu=13.8$, $r_p=9.5$kΩ
	V_8	Glow tube for stabilizing voltage	VRA 150V/30mA
	R_3	Resistance load of the 2nd step amp. tube	30kΩ
Phase shifter	V_3	Vacuum tube for variable resistance of phase shifter	UX-2A3
	C_3	Condenser (phase shifting circuit)	0.76μF
	R_4	Resistance for phase adjusting	3.6 kΩ
Peak generating device	V_4	Vacuum tube for peak generator	UZ-42
	R_3	Resistance load of peak generating tubes	
Manipuator	Th	Grid glow mercury tube TX-920	200V, 2.5A peak load 15A

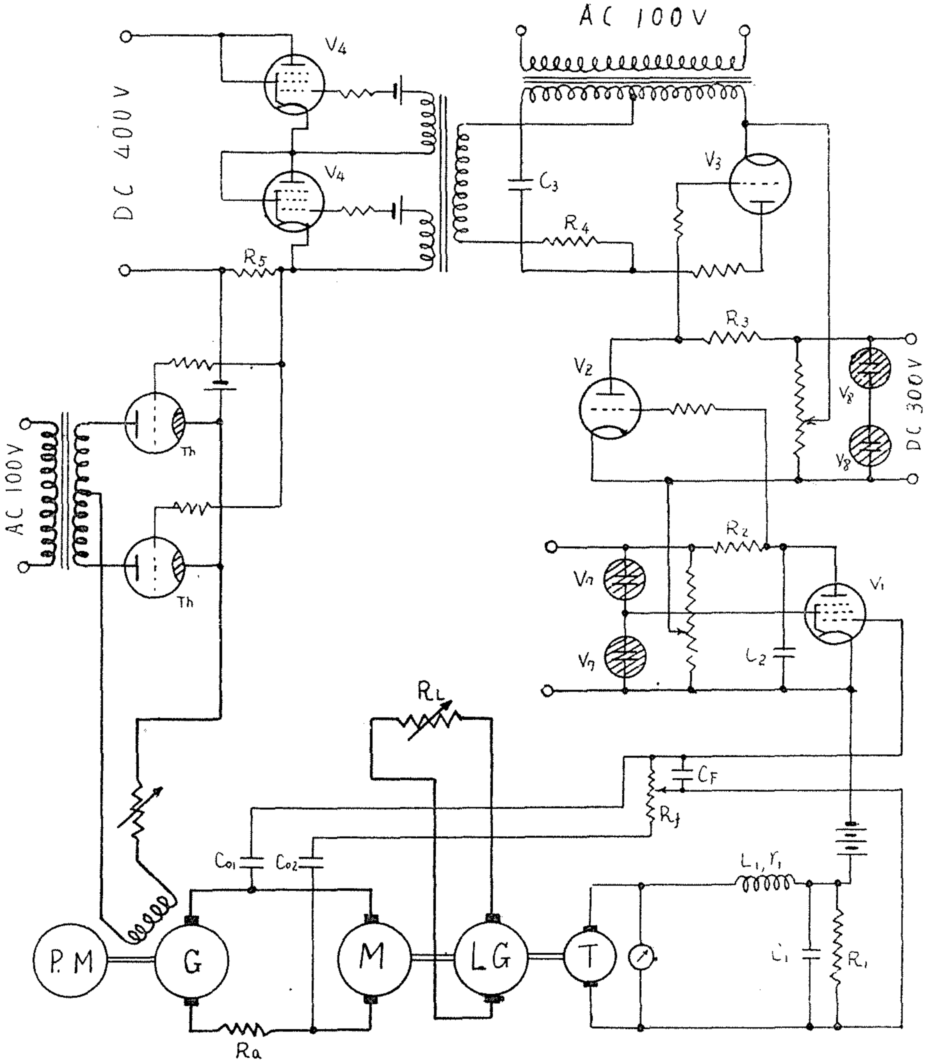
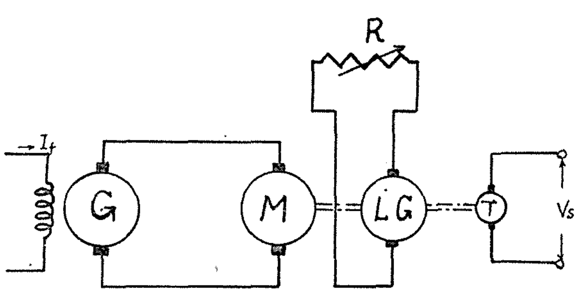


Fig. 14. Diagram of the automatic speed control in Ward-Leonard system



G: Leonard generator
M: Controlled motor
LG: Load-generator
T: Tachometer generator

Fig. 15.

field of the Ward-Leonard generator so that the motor speed is kept constant.

Since the deviation of the motor speed is very small, the system may be assumed to be linear, and on this assumption the calculation has been performed.

ii) Characteristic impedance of each element

a) *Controlled system*

The controlled system consists of a set of Ward-Leonard system with loads and a tachometer generator as shown in Fig. 15, and the input signal is the deviation in the field current of Leonard-generator ΔI_f, while the output signal is the deviation in the generated voltage of tachometer ΔV_s.

If the deviation of field current ΔI_f is small, it may be assumed to be proportional to the change of generated voltage ΔE_g of the generator G, that is,

$$\Delta E_g = K_f \Delta I_f . \qquad \text{(see Fig. 16)}$$

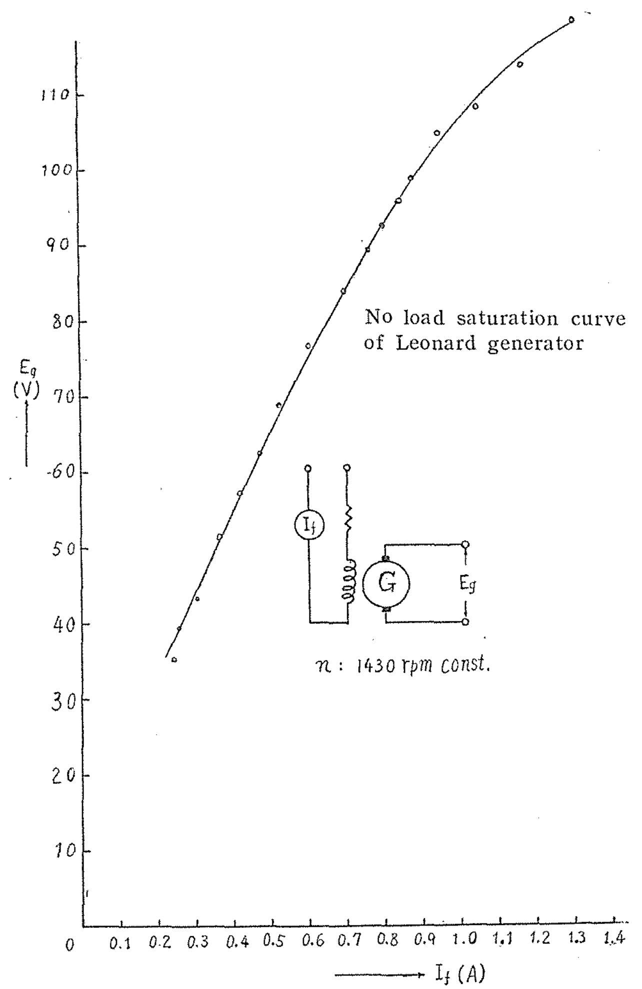


Fig. 16.

The deviation of motor speed $\varDelta\dot{\theta}$, which is induced from the voltage deviation $\varDelta E_g$, is given by the following equation:

$$J\frac{d\varDelta\dot{\theta}}{dt}+\left(f+\frac{K_1^2}{R_a}+\frac{K_2^2}{R_L}\right)\varDelta\dot{\theta}=\frac{K_1}{R_a}\varDelta E_g,$$

where K_1, K_2 are torque coefficient and induced emf coefficient of motor and generator, because the motor and the generator are of the same structure.

As the speed of motor $\dot{\theta}$ is proportional to the induced emf V_s of the tachometer generator T, that is, $\dot{\theta}=K_T V_S$, hence, if the input signal is the field current diviation $\varDelta I_f$ of Leonard-generator and the output signal is the deviation of speed emf $\varDelta V_S$, the performance equation of controlled system is

$$J'\frac{d\varDelta V_P}{dt}+f'\varDelta V_S=\varDelta I_f,$$

where

$$\begin{cases} J'=\dfrac{K_T}{K_f}\cdot\dfrac{R_a}{K_1}J, \\ f'=\dfrac{K_T}{K_f}\cdot\dfrac{R_a}{K_1}\left(f+\dfrac{K_1^2}{R_a}+\dfrac{K_2^2}{R_L}\right), \end{cases}$$

with

$$\begin{cases} J: \text{ moment of inertia of motor with loads,} \\ f: \text{ friction coefficient of motor with loads.} \end{cases}$$

Thus, the impedance of controlled system is given by

$$Z_1=J'p+f'.$$

The measured values of J' and f' are as follows (see Fig. 17).

Loads	Time constant	J'	f'
no load	0.258	0.0445	0.172
1 load	0.228	0.0438	0.192
2 loads	0.203	0.0440	0.217

b) Detector with filter

The detecting device is a tachometer generator with filter, which converts the speed signal into the voltage signal. However, as the performance equation of detector, that is $\dot{\theta}=K_T V_S$, has been contained in the performance equation of controlled system, it is sufficient here to consider the performance of the filter only. The filter is illustrated in Fig. 18, in which

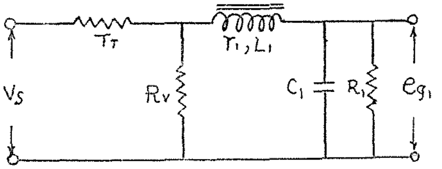
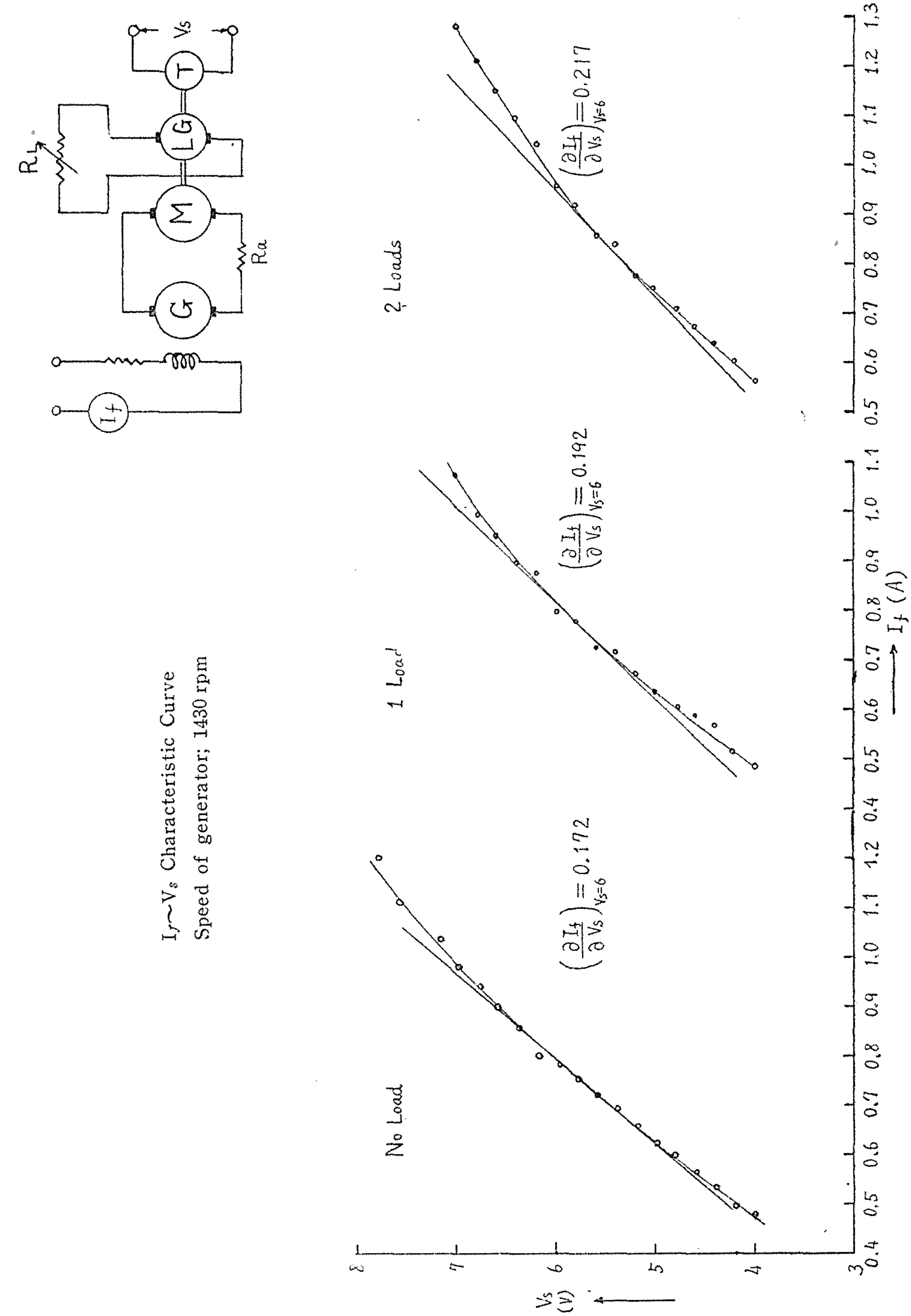


Fig. 18.

r_T: armature resistance of tachometer generator (100 Ω),

R_V: internal resistance of voltmeter (2 KΩ),



$I_f \sim V_s$ Characteristic Curve
Speed of generator; 1430 rpm
No Load
1 Load
2 Loads
$\left(\frac{\partial I_f}{\partial V_s}\right)_{V_s=6} = 0.172$
$\left(\frac{\partial I_f}{\partial V_s}\right)_{V_s=6} = 0.192$
$\left(\frac{\partial I_f}{\partial V_s}\right)_{V_s=6} = 0.217$
Vs (V)
I_f (A)
R_L
Ra
G
M
LG
T
Vs

Fig. 17.

r_1, L_1: resistance and inductance of choking coil (1.5 KΩ, 30 H),
C_1 : condenser (3.5 μF),
R_1 : resistance (30 KΩ).

The performance equation of this filter is

$$k_1 V_S = L_1 C_1 \frac{d^2 e_{g1}}{dt^2} + \left(\frac{L_1}{R_1} + C_1 r_1\right) \frac{de_{g1}}{dt} + \left(\frac{r_1}{R_1} + 1\right) e_{g1},$$

where $k_1 = R_V/(r_T + R_V) = 2/(2+0.1) = 0.95$.

Hence, the performance relation between the input signal ΔV_S and the output signal Δe_{g1} of detector is expressed as follows:

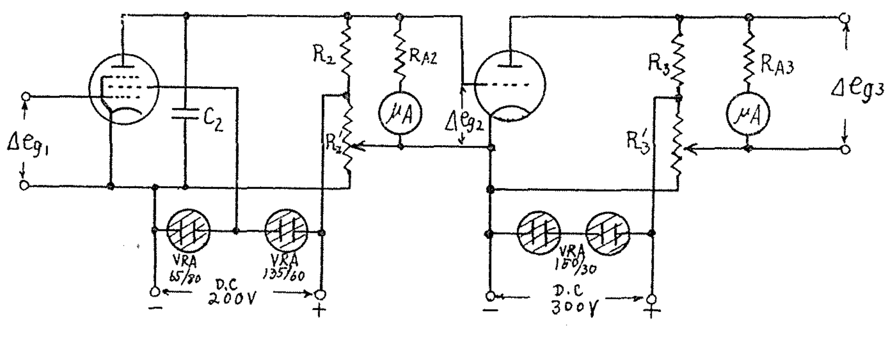
$$k_1 \Delta V_S = \left\{L_1 C_1 p^2 + \left(\frac{L_1}{R_1} + C_1 r_1\right) p + \left(\frac{r_1}{R_1} + 1\right)\right\} \Delta e_{g1},$$

or

$$0.95\, \Delta V_s = (1.05 \times 10^{-4} p^2 + 6.25 \times 10^{-3} p + 1.05)\, \Delta e_{g1}.$$

c) Amplifier

Fig. 19 shows the actual circuit of the two stage amplifier.



R_2: load resistance of UZ-6C6 (100 KΩ),
R_3: load resistance of UY-76 (30 KΩ),
R_2': variable resistance (500 KΩ),
R_3': variable resistance (350 KΩ),
R_{A2}: resistance (500 KΩ),
R_{A3}: resistance (250 KΩ),
C_2: condenser (2.45 μF),
Δe_{g1}: deviation of input signal voltage in the 1st stage,
Δe_{g2}: deviation of input signal voltage in the 2nd stage,
Δe_{g3}: deviation of output signal voltage in the amplifier,
μ: amplification constant of UZ-6C6,
r_{p2}: internal resistance of UZ-6C6,
R_2'': resultant resistance of parallel connection of R_2 and the anode part of R_2'.

Fig. 19.

The actual circuit of the first stage of the amplifier is equivalent to the simple circuit as shown in Fig. 20, for small deviation of input signal voltage. Thus, we have

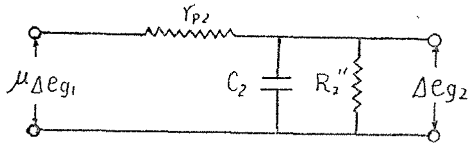


Fig. 20.

$$-\mu\Delta e_{g1}=\left\{C_2 r_{p2} p+\left(1+\frac{r_{p2}}{R_2''}\right)\right\}\Delta e_{g2}\,,$$

or

$$-1260\,\Delta e_{g1}=(3.7p+18.7)\Delta e_{g2}\,. \qquad \text{(see Fig. 21)}$$

The performance of the second stage is expressed by

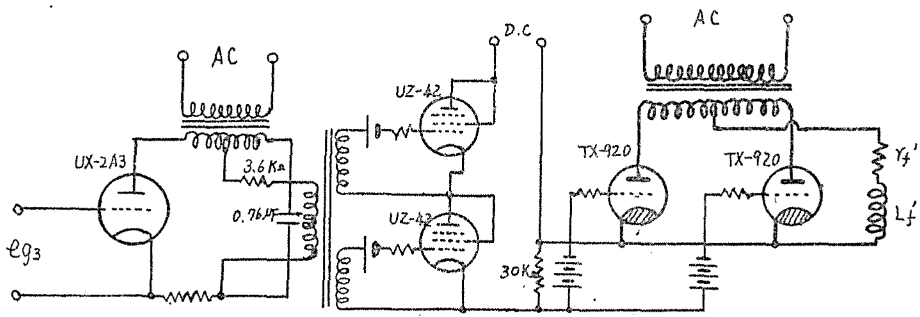
$$K_3'\Delta e_{g2}=\Delta e_{g3}\,. \qquad \text{(see Fig. 22)}$$

As $K_3'=-11$ is obtained by experiment, the performance equation of the whole set of amplifier is obtained as follows:

$$-\mu\Delta e_{g1}=\left[C_2 r_{p2} p+\left(1+\frac{r_{p2}}{R_2''}\right)\right]\frac{1}{K_3'}\Delta e_{g3}\,.$$

d) Manipulator with phase shifter

The manipulator consists of three parts, i. e., phase shifter, impulse generator and thyratron circuit. The input signal, that is, the output voltage deviation of amplifier, is converted into the shift angle of phase by phase shifter, and is deformed by impulse generator into the phase shift of peak voltage, which is impressed on the grids of thyratrons, and the output current of the thyratrons controls the field of Leonard-generator, as shown in Fig. 23.



(Phase shifter) (Impulse generator) (Thyratron circuit)

Fig. 23.

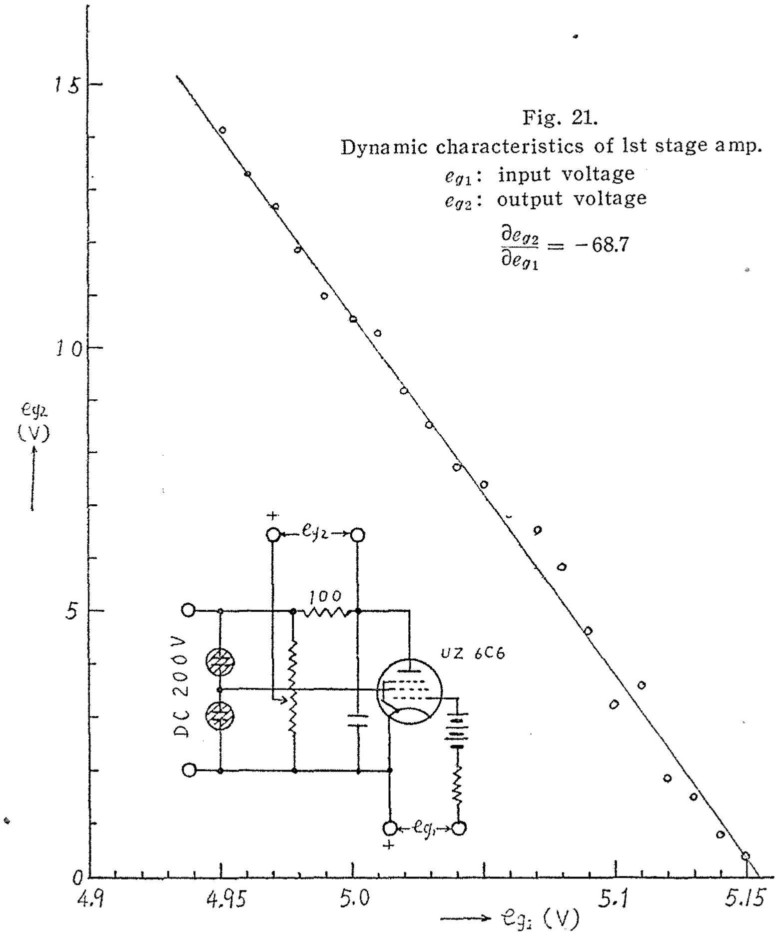


Fig. 21.
Dynamic characteristics of 1st stage amp.
e_{g1}: input voltage
e_{g2}: output voltage

$$\frac{\partial e_{g2}}{\partial e_{g1}} = -68.7$$

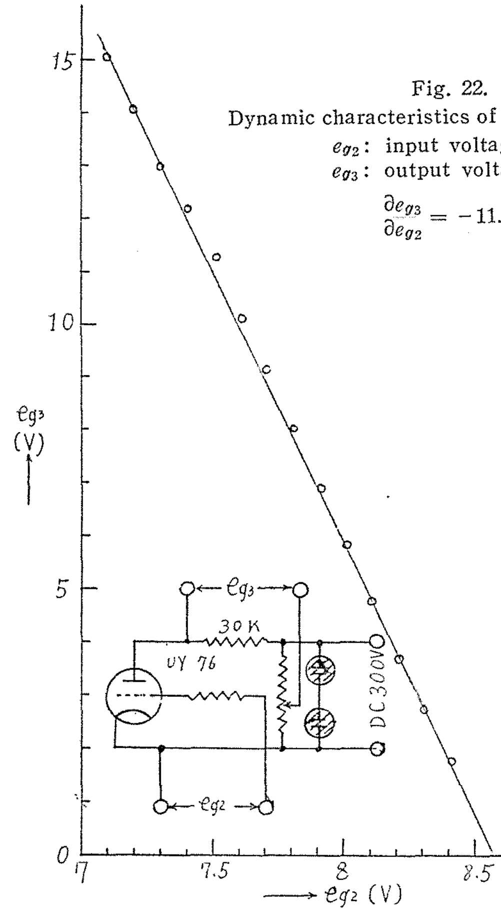


Fig. 22.
Dynamic characteristics of 2nd stage amp.
e_{g2}: input voltage
e_{g3}: output voltage

$$\frac{\partial e_{g3}}{\partial e_{g2}} = -11.0$$

$e_{g3} \sim I_f$ characteristics

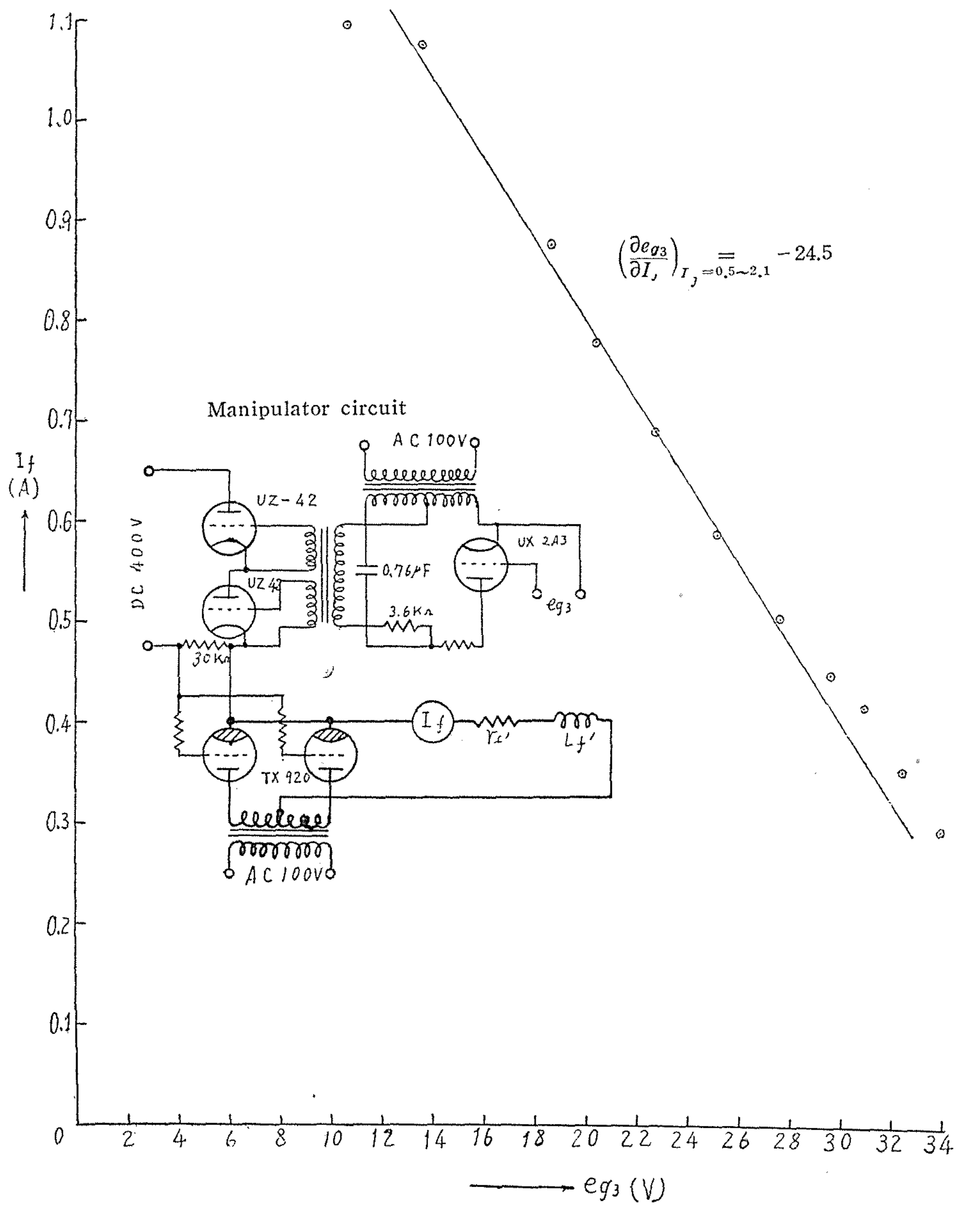


Fig. 24.

The performance characteristics of manipulator have been obtained as follows by oscillographical measurements. The time constants of phase shifter and impulse generator are estimated to be negligibly small both by measurements and by calculation (ca. 0.8/1000 sec) from the internal resistance of UX-2A3 (0.8 KΩ) and the capacity of condenser $C_3(1\,\mu F)$. Thus, the time constant of the whole set depends only upon that of the field winding of Leonard-generator, and is obtained experimentally. Then, the values of constants r_f' and L_f' are calculated from $e_{g3} \sim I_f$ characteristics at operating conditions and the measured time constant $T = 0.155$ sec of the field winding as follows:

$$r_f' = \left(\frac{\partial e_{g3}}{\partial I_f}\right)_{1\cdot 0 > I_f > 0\cdot 5} = 24.5, \quad L_f' = 24.5 \times 0.155 = 3.8\,.$$

Accordingly, the performance equation of the whole set of manipulator is given by

$$-\Delta e_{g3} = K_3 \Delta e_{g2} = (L_f' p + r_f')\Delta I_f\,, \qquad \text{(see Fig. 24)}$$

and $$K_3 = -K_3' = 11\,.$$

Thus, $$Z \equiv L_f' p + r_f' = 3.8\,p + 24.5\,.$$

iii) Characteristic vector

The block diagram of this automatic control system is obtained from the above-mentioned impedances of all elements as follows:

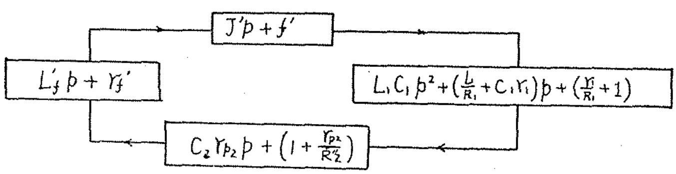


Fig. 25.

Hence, the characteristic matrix of this control system is

$$[\boldsymbol{Z}] = \begin{pmatrix} J'p+f' & & & 1 \\ -K_1 & L_1C_1p^2 + \left(\frac{L_1}{R_1} + C_1 r_1\right)p + \left(\frac{r_1}{R_1} + 1\right) & & \\ & -K_2 & C_2 r_{p2} p + \left(1 + \frac{r_{p2}}{R_2''}\right) & \\ & & -K_3 & L_f' p + r_f' \end{pmatrix}.$$

It gives the characteristic equation of the system as:

$$|[\boldsymbol{Z}]| = 0\,,$$

or

$$\left(J'p + f'\right)\left\{L_1C_1p^2 + \left(\frac{L_1}{R_1} + C_1r_1\right)p + \left(\frac{r_1}{R_1} + 1\right)\right\}\left\{C_2 r_{p2} p + \left(1 + \frac{r_{p2}}{R_2}\right)\right\}\left(L_f' p + r_f'\right) + K_1K_2K_3 = 0.$$

Arranging the equation in descending powers of p, we get

$$a_0p^5+a_1p^4+a_2p^3+a_3p^2+a_4p+a_5=0\,.$$

If the characteristic vector $[a\mu/a_0]$, (μ=1, 2, 3, 4, 5) is transformed by

$$\xi_{2\nu}=\frac{a_{2\nu}}{a_0}\Big/\left(\frac{a_2}{a_0}\right)^{\nu},\qquad \xi_{2\nu+1}=\frac{a_{2\nu+1}}{a_1}\Big/\left(\frac{a_2}{a_0}\right)^{\nu},\qquad (\nu=0,1,2)$$

the following components of the normalized characteristic vector are obtained:

$$\xi_0=1,\quad \xi_1=1,\quad \xi_2=1,$$
$$\xi_3=\frac{a_3}{a_1}\Big/\left(\frac{a_2}{a_0}\right),\ \xi_4=\frac{a_4}{a_0}\Big/\left(\frac{a_2}{a_0}\right)^2,\ \xi_5=\frac{a_5}{a_1}\Big/\left(\frac{a_2}{a_0}\right)^2.$$

Putting numerical values of physical constants at different states of load, we obtain:

a) *No load*:

The characteristic equation is given by

$$6.24\times10^{-5}\,p^5+4.933\times10^{-3}\,p^4+7.09\times10^{-1}\,p^3+10.38\,p^2+51.0\,p+13483=0,$$

and the characteristic vector has the components:

$$\begin{cases}\xi_3=\dfrac{10.38\times6.54}{4.933\times7.09}\times10^{-1}=0.203\,,\\ \xi_4=\dfrac{51.9\times6.54}{7.09\times7.09}\times10^{-3}=0.00663\,,\\ \xi_5=\dfrac{13483\times6.54^2}{4.933\times7.09^2}\times10^{-5}=0.0232\,.\end{cases}$$

b) *One load*:

The characteristic equation is

$$6.44\times10^{-5}\,p^5+4.9\times10^{-3}\,p^4+7.11\times10^{-1}\,p^3+10.58\,p^2+54.05\,p+13492=0\,,$$

and the characteristic vector has the components:

$$\begin{cases}\xi_3=\dfrac{10.58\times6.44}{4.90\times7.11}10^{-1}=0.1955\,,\\ \xi_4=\dfrac{54.05\times6.44}{7.11^2}10^{-3}=0.00683\,,\\ \xi_5=\dfrac{13492\times6.44^2}{4.90\times7.11^2}10^{2-}=0.0223\,.\end{cases}$$

c) Two loads:

The characteristic equation is

$$6.48\times10^{-5}\,p^5+4.95\times10^{-3}\,p^4+7.17\times10^{-1}\,p^3+11.0\,p^2+58.6\,p+13504=0,$$

and the characteristic vector has the components:

$$\begin{cases} \xi_3=\dfrac{11.0\times6.48}{4.95\times7.17}\times10^{-1}=0.2005\,, \\ \xi_4=\dfrac{58.6\times6.48}{7.17^2}\times10^{-3}=0.0074\,, \\ \xi_5=\dfrac{13504\times6.48^2}{4.95\times7.17^2}\times10^{-5}=0.0225\,. \end{cases}$$

From the above-mentioned calculations, we get the following table of the characteristic vector components:

States of load	ξ_3	ξ_4	ξ_5	Stability
no load	0.2030	0.00663	0.0232	unstable
1 load	0.1955	0.00683	0.0223	unstable
2 loads	0.2005	0.00740	0.0225	unstable

Thus, the automatic control system in each of the above-mentioned cases a), b) and c) has been concluded to be unstable as shown with the point P in Fig. 29, and this instability has also been shown in the experiments in which the remarkable hunting has appeared.

iv) Feed-back to detector

In order to stabilize the unstable system, the additional feed-back circuit has been inserted between the output of manipulator Z_4 and the input of detector Z_2. The additional feed-back circuit $Y_{f.b.}$ is shown in Fig. 26.

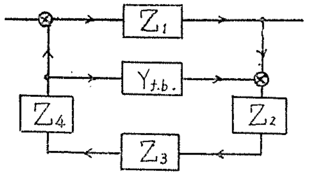


Fig. 26.

$Z_1=J'p+f'$: controlled element (Leonard-motor),

$Z_2=L_1C_1p^2+\left(\frac{L_1}{R_1}+C_1r_1\right)p+\left(\frac{r_1}{R_1}+1\right)$: detector with filter,

$Z_3=C_2r_{p2}p+\left(\frac{r_{p2}}{R_2''}+1\right)$: amplifier,

$Z_4=L_f'p+r_f'$: manipulator,

$Y_{f.b.}=xp$: feed-back element.

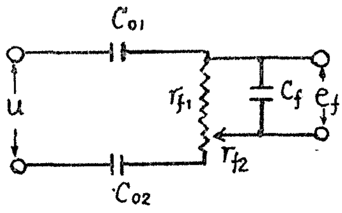


Fig. 27.

u = input voltage,
e_f = output voltage,
$C_{01} = 2.11\ \mu F$,
$C_{02} = 7.0\ \mu F$,
$C_f = 2.46\ \mu F$,
r_{f1}, r_{f2} = variable resistances ($r_{f1}+r_{f2} = 2000\ \Omega$),
$C_0 = C_{01}C_{02}/(C_{01}+C_{02}) = 1.62\ \mu F$.

With R_g = armature resistance of Leonard-generator and R_m = armature resistance of Leonard-motor (controlled motor), we obtain:

$$Y_{f.b.} = C_0 r_{f1} K_f \frac{R_m}{R_g+R_m} p = xp = 0.0562\, r_{f1} p\,,$$

and

$$|[Z]| = \begin{vmatrix} Z_1 & & & 1 \\ -K_1 & Z_2 & & Y_{f.b.} \\ & -K_2 & Z_3 & \\ & & -K_3 & Z_4 \end{vmatrix} = |[Z]_0| + Y_{f.b.}|[Z]_{\mathrm{I}}| = 0\,,$$

where $[Z]_0$: characteristic matrix without any additional feed-back,

$$|[Z]_{\mathrm{I}}| = \begin{vmatrix} Z_1 & & \\ & -K_2 & Z_3 \\ & & -K_3 \end{vmatrix} = K_2K_3Z_1 = K_2K_3(J'p+f')\,.$$

Hence, the feed-back vector is given by

$$\begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix}_{\mathrm{I}} = K_2K_3x \begin{pmatrix} J' \\ f' \end{pmatrix} = 0.0562\, K_2K_3r_{f1} \begin{pmatrix} J' \\ f' \end{pmatrix}.$$

Then, the characteristic vector of the system, with the additional feed-back element $Y_{f.b.}$ ($r_{f1} = 1.0\ \mathrm{K}\Omega$), has the components:

Load	ξ_0	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
no load	6.54×10^{-5}	4.933×10^{-3}	7.09×10^{-1}	45.48	187.0	13483
1 load	6.44×10^{-5}	4.90×10^{-3}	7.11×10^{-1}	45.05	205.5	13492
2 loads	6.48×10^{-5}	4.95×10^{-3}	7.17×10^{-1}	45.80	229.6	13504

and is normalized as follows:

Load	ξ_0''	ξ_1''	ξ_2''	ξ_3''	ξ_4''	ξ_5''	Stability
no load	1	1	1	0.890	0.0244	0.0232	unstable
1 load	1	1	1	0.836	0.0257	0.0223	unstable
2 loads	1	1	1	0.835	0.0290	0.0225	stable

The above-calculated vector-components are illustrated in Fig. 29, and the experiments have quite agreed with these theoretical calculations, showing that the system is completely stabilized only in the case of two loads. And the measured values of resistance r_{f1} for the system being in critical stability are quite consistent with the author's theoretical calculations, as follows.

Load	Theoretical values	Measured values
no load	1.04 KΩ	1.12 KΩ
1 load	1.01 KΩ	1.10 KΩ
2 loads	0.96 KΩ	0.94 KΩ

v) Feed-back to amplifier

When the additional feed-back circuit $Y_{f\cdot b}$, which is shown in Fig. 28, is inserted between the output of manipulator and the input of amplifier, in order to stabilize the hunting system, the block diagram of the whole system is given as follows:

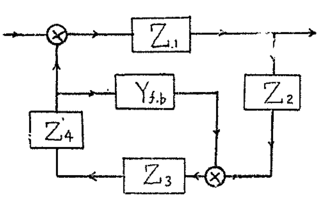


Fig. 28.

$Z_1 = J'p+f'$: controlled element (Leonard-motor),

$Z_2 = L_1C_1p^2+\left(\frac{L_1}{R_1}+C_1r_1\right)p+\left(\frac{r_1}{R_1}+1\right)$: detector with filter,

$Z_3 = C_2r_{p2}p+\left(\frac{r_{p2}}{R_2''}+1\right)$: amplifier,

$Z_4 = L_f'p+r_f'$: manipulator,

$Z_{f\cdot b\cdot} = xp$: feed-back element.

The characteristic equation is

$$\begin{vmatrix} J'p+f' & & & 1 \\ -K_1 & L_1C_1p_2+\left(\frac{L_1}{R_1}+C_1r_1\right)p+\left(\frac{r_1}{R_1}+1\right) & & \\ & -K_2 & C_2r_{p2}p+\left(\frac{r_{p2}}{R_2''}+1\right) & Y_{f\cdot b\cdot} \\ & & -K_3 & L_f'p+r_f' \end{vmatrix} = 0\,.$$

This is expanded as follows:

$$|[Z]_0| + Y_{f\cdot b\cdot}|[Z]_{\mathrm{I}}| = 0\,,$$

where $[Z]_0$: characteristic matrix without any additional feed-back,

$$|[Z]_{\mathrm{I}}| = -\begin{vmatrix} Z_1 & & \\ -K_1 & Z_2 & \\ & & -K_3 \end{vmatrix} = K_3Z_1Z_2 = K_3(J'p+f')\left\{L_1C_1p^2+\left(\frac{L_1}{R_1}+C_1r_1\right)p+\left(\frac{r_1}{R_1}+1\right)\right\}$$

$$= J'L_1C_1K_3p^3+K_3\left\{J'\left(\frac{L_1}{R_1}+C_1r_1\right)+f'L_1C_1\right\}p^2$$

$$+K_3\left\{J'\left(\frac{r_1}{R_1}+1\right)+f'\left(\frac{L_1}{R_1}+C_1r_1\right)\right\}p+K_3f'\left(1+\frac{r_1}{R_1}\right).$$

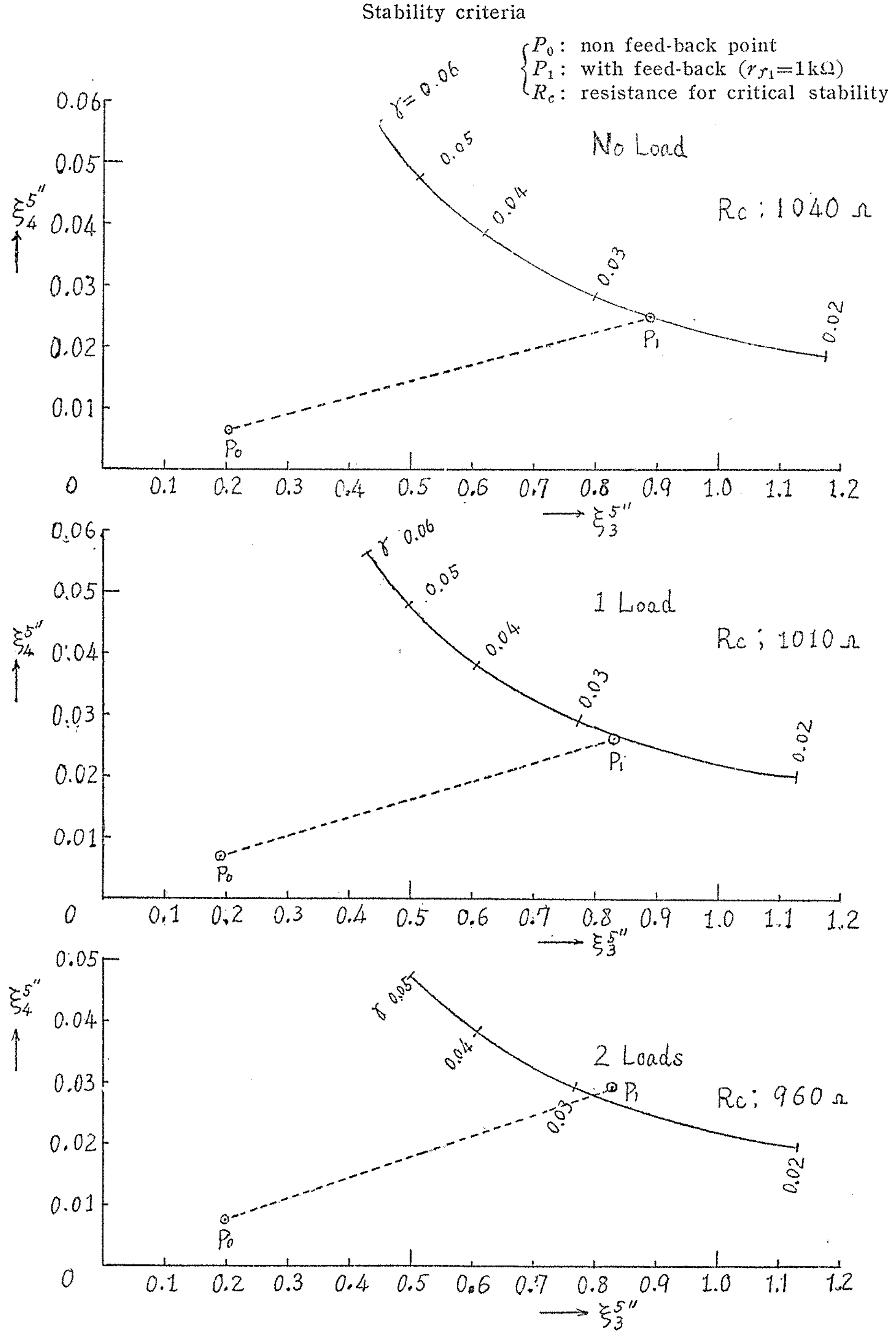

Stability criteria
P_0 : non feed-back point
P_1 : with feed-back (r_{f1}=1kΩ)
R_c : resistance for critical stability
No Load
Rc ; 1040 Ω
1 Load
Rc ; 1010 Ω
2 Loads
Rc ; 960 Ω
γ = 0.06
0.05
0.04
0.03
0.02
P_0
P_1
$\xi_4^{5''}$
$\xi_3^{5''}$
0
0.01
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2

Fig. 29.

The feed-back vector or stablizing vector is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = K_3 x \begin{pmatrix} J'L_1C_1 \\ J'\left(\frac{L_1}{R_1}+C_1r_1\right)+f'L_1C_1 \\ J'\left(\frac{r_1}{R_1}+1\right)+f'\left(\frac{L_1}{R_1}+C_1r_1\right) \\ f'(r_1/R_1+1) \end{pmatrix} = r_{f1} \begin{pmatrix} J'\times 8.3\times 10^{-2} \\ J'\times 4.95+f'\times 8.3\times 10^{-3} \\ J'\times 830+f'\times 4.95 \\ f'\times 830 \end{pmatrix},$$

and it has the following numerical components for $r_{f1}=1.0\,\mathrm{K}\Omega$.

Load	ξ_1	ξ_2	ξ_3	ξ_4
no load	3.7×10^{-4}	0.230	37.85	143.0
1 load	3.66×10^{-4}	0.234	37.55	159.0
2 loads	3.64×10^{-4}	0.236	37.48	180.0

So the characteristic vector, with additional feed-back, has the components:

Load	ξ_0	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
no load	6.54×10^{-5}	5.303×10^{-3}	0.939	48.23	194.0	13483
1 load	6.44×10^{-5}	5.266×10^{-3}	0.945	48.13	213.0	13492
2 loads	6.48×10^{-5}	5.314×10^{-3}	0.953	48.48	238.6	13504

and by normalization, these become:

Load	ξ_0''	ξ_1''	ξ_2''	ξ_3''	ξ_4''	ξ_5''
no load	1	1	1	0.603	0.0144	0.0122
1 load	1	1	1	0.626	0.0154	0.0119
2 loads	1	1	1	0.621	0.0170	0.0118

Next, the feed-back vector for $r_{f1}=1.5\,\mathrm{K}\Omega$ has the following components:

Load	ξ_1	ξ_2	ξ_3	ξ_4
no load	0.555×10^{-3}	0.345	56.78	214.5
1 load	0.549×10^{-3}	0.351	56.33	238.5
2 loads	0.546×10^{-3}	0.354	56.22	270.0

So the characteristic vector with additional feed-back is given by the following components:

Load	ξ_0	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
no load	6.54×10^{-5}	5.488×10^{-3}	1.054	67.16	265.5	13492
1 load	6.44×10^{-5}	5.449×10^{-3}	1.062	66.885	292.5	13492
2 loads	6.48×10^{-5}	5.496×10^{-3}	1.017	67.22	328.6	13504

By normalization, these components become:

Load	ξ_0''	ξ_1''	ξ_2''	ξ_3''	ξ_4''	ξ_5''
no load	1	1	1	0.762	0.0157	0.00945
1 load	1	1	1	0.740	0.0167	0.00905
2 loads	1	1	1	·0738	0.0185	0.00897

By the author's stability criterion, the characteristic vectors for all cases without any additional feed-back and for a case with the additional feed-back of $r_{f1}=1.0\,\mathrm{K}\Omega$, whose components have been calculated above, are theoretically concluded to be unstable, while the characteristic vectors for cases with the additional feed-back of $r_{f1}=1.5\,\mathrm{K}\Omega$ are all theoretically concluded to be stable (see Fig. 30). The experiments have quite agreed with the author's theoretical conclusions.

vi) Period of hunting

For example, we repeat here the case of no load and feed-back to amplifier. The characteristic vector in the state of critical stability has the components:

Load	ξ_0	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
no load	6.54×10^{-5}	5.488×10^{-3}	1.054	67.16	265.5	13492

which are normalized as follows:

Load	ξ_0''	ξ_1''	ξ_2''	ξ_3''	ξ_4''	ξ_5''
no load	1	1	1	0.762	0.0157	0.00945

This normalized characteristic vector is represented by a point, scaled $\gamma''=0.016$ on the normalized line of critical stability of the 5-dimensional vector in Fig. 30.

Accordingly, the hunting frequency of this unstable system is given as follows:

$$f=\frac{\omega}{2\pi}=\frac{\sqrt{\gamma}}{2\pi}=\frac{1}{2\pi}\sqrt{\gamma''\left(\frac{\xi_2}{\xi_0}\right)}=\frac{1}{2\pi}\sqrt{0.016\times\frac{1.054}{6.54\times10^{-5}}}=0.58\ \ (1/\mathrm{sec}),$$

or

$$T=1/f=0.387\ \ (\mathrm{sec}).$$

This theoretical value of hunting frequency may be considered to be quite consistent with the measured value 2.8 (cycles per second), or $T=0.357$ (seconds), if we take into account the non-linearity of elements and the experimental errors.

vii) Straight-line stability criterion

So far we have considered the section of the 5-dimensional vector by (ξ_4, ξ_5)-plane. As described in § 7 of the preceding paper, if (ξ_3, ξ_5)-plane is considered

Stability criteria

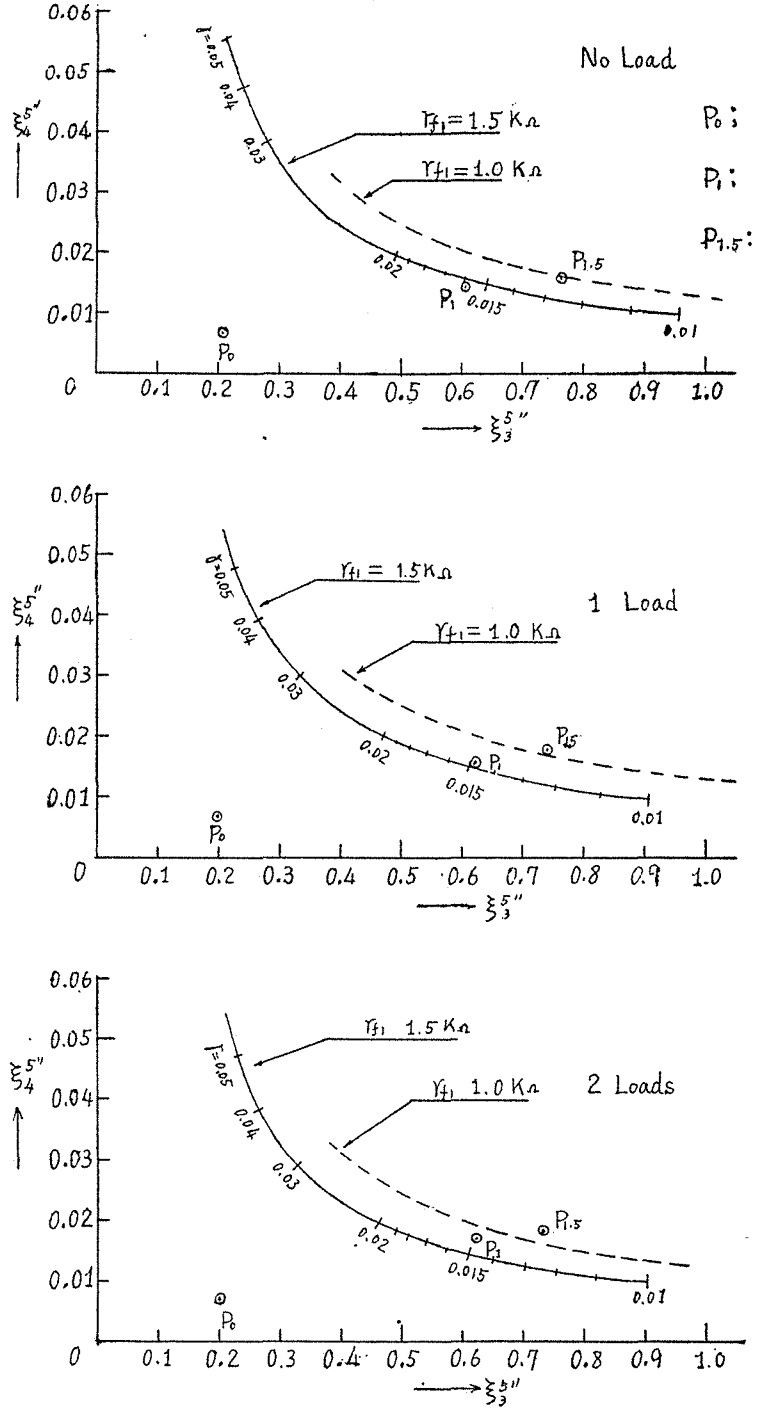


Fig. 30.

instead of (ξ_4, ξ_5)-plane, we obtain a very simple stability criterion which consists of some straight lines.

For example, in the case of feed-back to detector, the normalized characteristic vector has the components:

States of load	ξ_0''	ξ_1''	ξ_2''	ξ_3''	ξ_4''	ξ_5''
no load	1	1	1	0.890	0.0244	0.232
1 load	1	1	1	0.836	0.0257	0.223
2 loads	1	1	1	0.835	0.0290	0.225

Now, cutting the hypersurface of critical stability of the 5-dimensional vector,

$$\left.\begin{aligned} \xi_4'' &= \gamma''(1-\gamma''), \\ \xi_5'' &= \gamma''(\xi_3''-\gamma''), \end{aligned}\right\}$$

with a plane $\xi_4'' =$ constant (any value of ξ_4'' in the above table), we obtain (ξ_3'', ξ_5'')-stability criterion which consists of two straight lines as:

$$\xi_5'' = \gamma_1''(\xi_3''-\gamma_1''), \qquad (\gamma_1\text{-line})$$

$$\xi_5'' = \gamma_2''(\xi_3''-\gamma_2''), \qquad (\gamma_2\text{-line})$$

where γ_1'' and γ_2'' are the roots of the equation:

$$\xi_4'' = \gamma''(1-\gamma'')$$

for given value of ξ_4''-component, as shown in the following table:

State of load	ξ_4''	γ_1''	γ_2''
no load	0.0244	0.024	0.976
1 load	0.0257	0.027	0.973
2 loads	0.0290	0.030	0.970

The stability criteria by these straight lines are illustrated in Fig. 32, and the results are completely in accordance with those of (ξ_4'', ξ_5'')-stability criterion.

12. Conclusion

In this paper, the stability of the feed-back physical system, the relations between the stability regions of the n-dimensional characteristic vector and those of the $(n-1)$-dimensional vector and also the relations between those of the n-dimensional vector and the $(n-2)$-dimensional vector have first been discussed in detail, and the methods of constructing the stability criteria of higher dimensions from

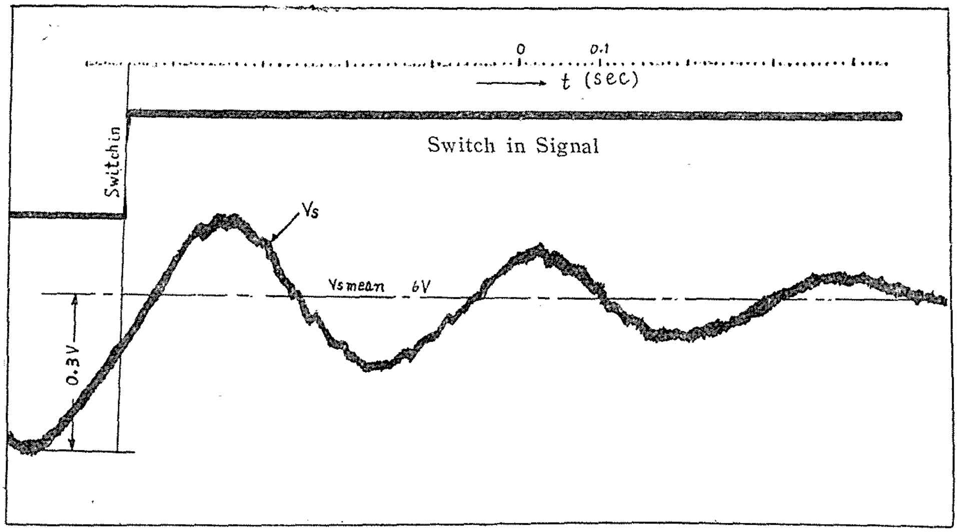


Fig. 31 a. Transient performance with anti-hunting feed-back to amplifier (r_{f1}=1.5 KΩ). No load.

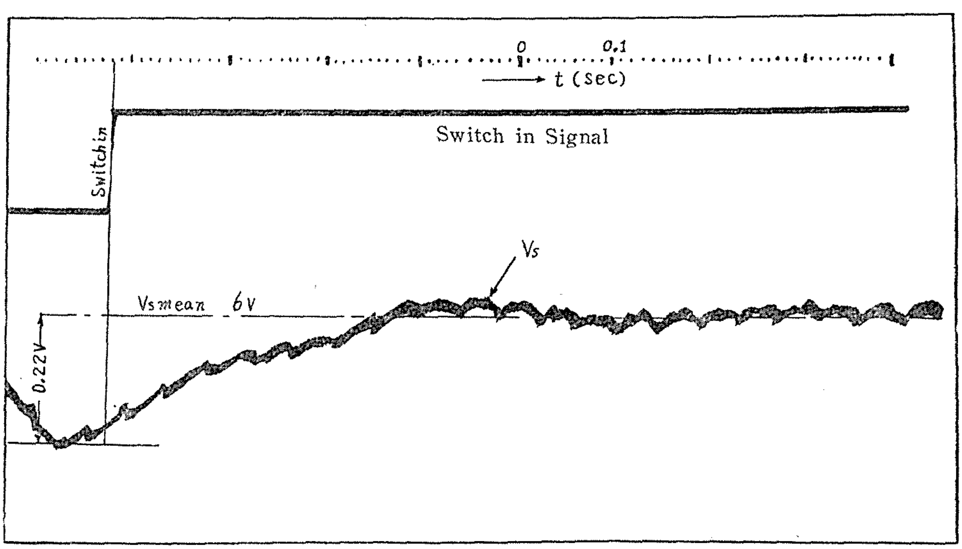


Fig. 31 b. Transient performance with anti-hunting feed-back to amplifier (r_{f1}=2 KΩ). 1 load.

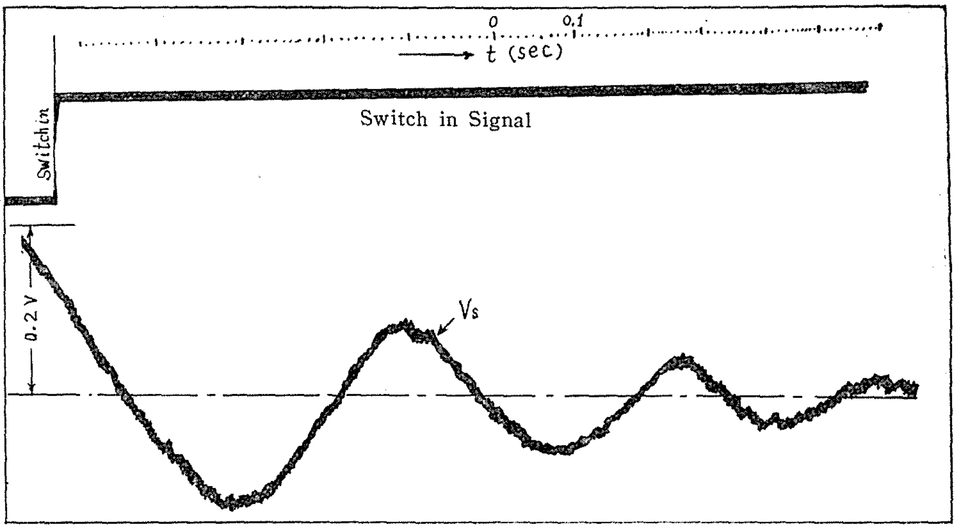


Fig. 31 c. Transient performance with anti-hunting feed-back to amplifier (r_{f1}=1.5 KΩ). 2 loads.

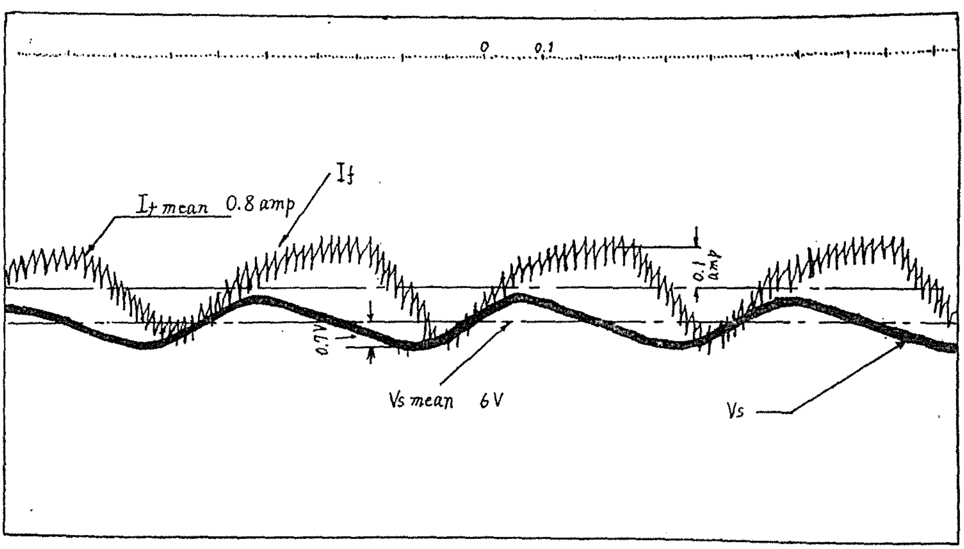


Fig. 31 d. Stationary hunting without feed-back.

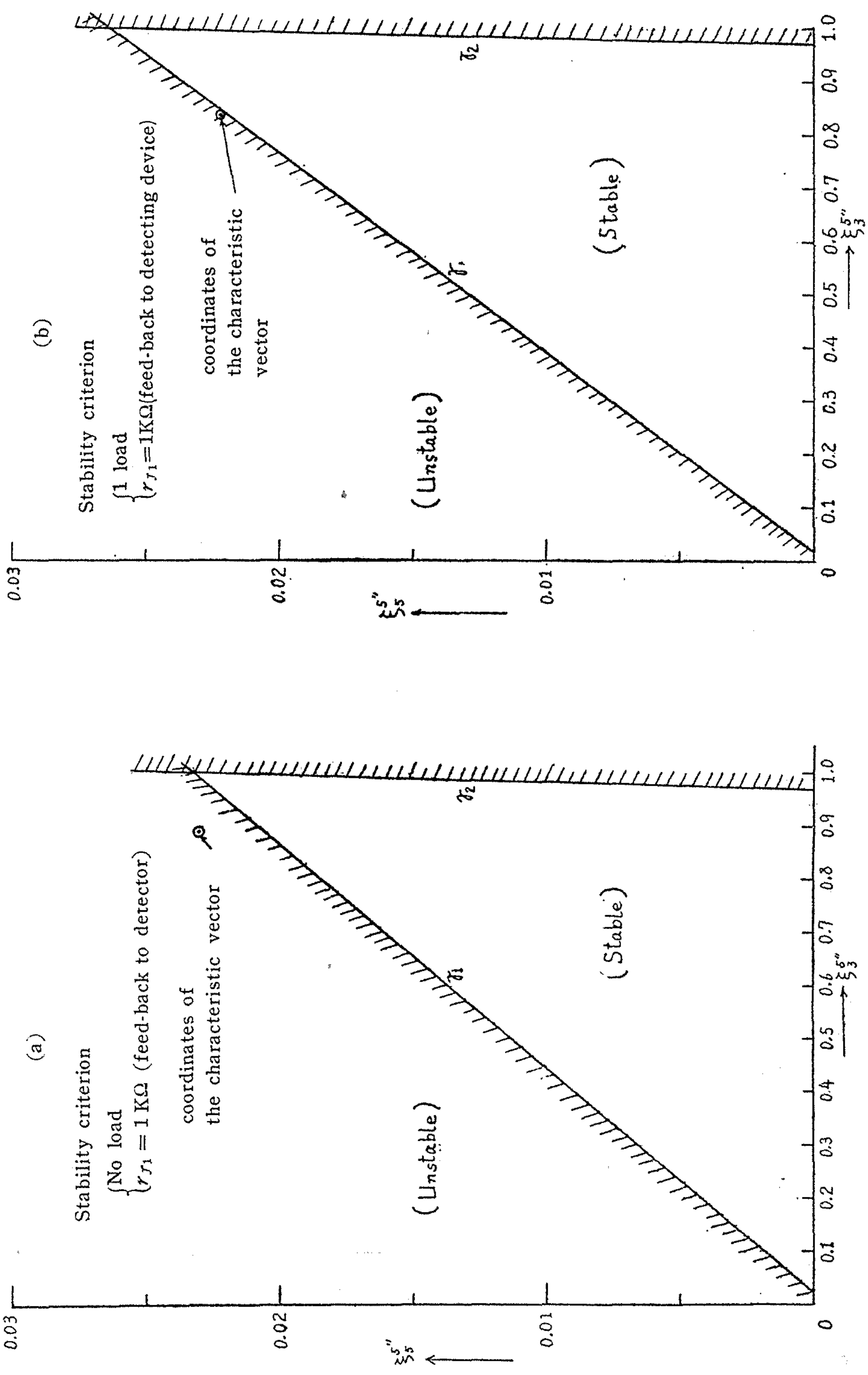

(a)
Stability criterion
No load
$r_{f1} = 1$ KΩ (feed-back to detector)
coordinates of the characteristic vector
(Unstable)
(Stable)
γ_1
γ_2
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\xi_5^{5''}$
$\xi_3^{6''}$
(b)
Stability criterion
1 load
r_{f1}=1KΩ(feed-back to detecting device)
coordinates of the characteristic vector
(Unstable)
(Stable)
γ_1
γ_2
$\xi_5^{5''}$
$\xi_3^{5''}$

Fig. 32.

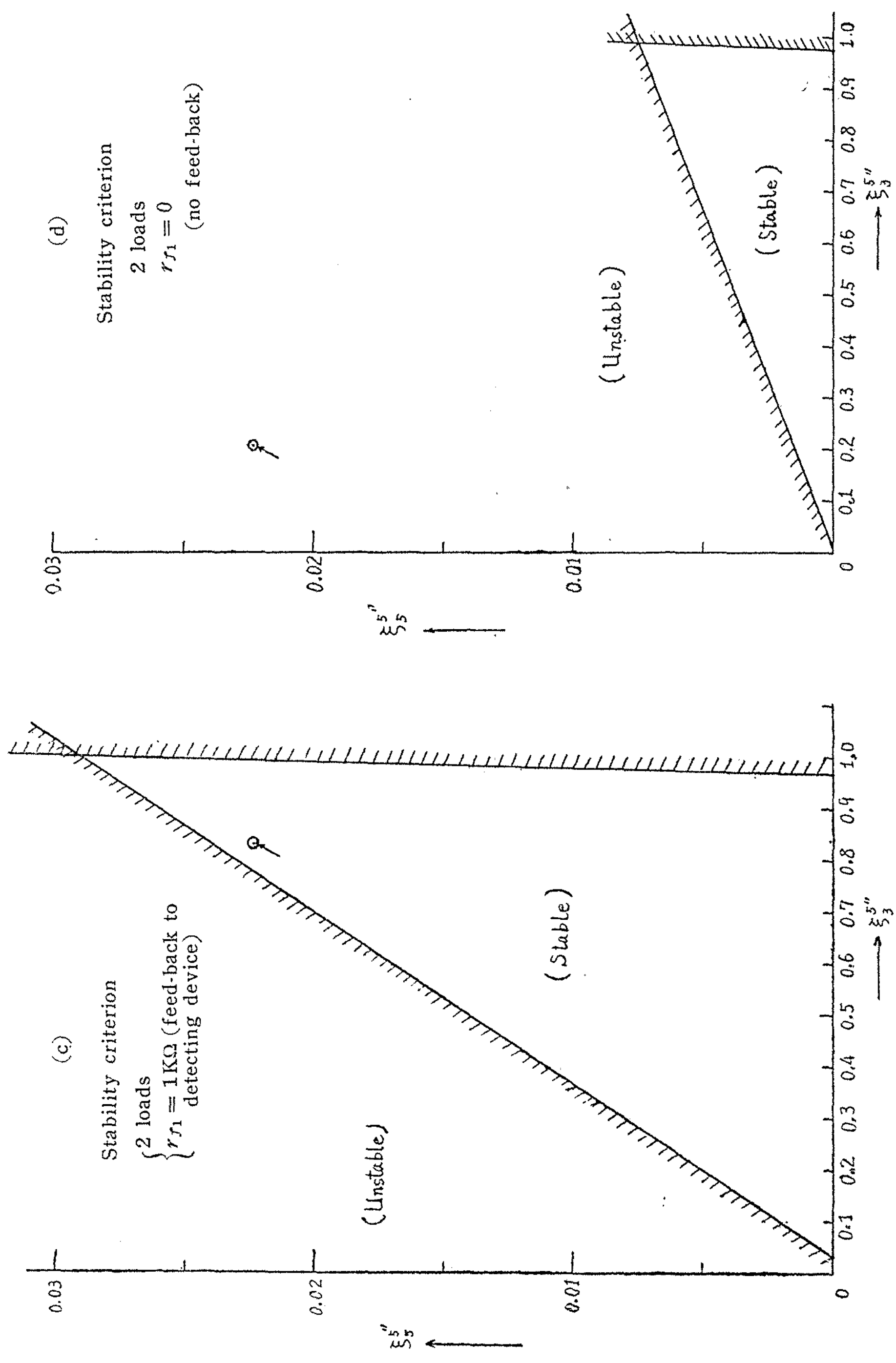

(d)
Stability criterion
2 loads
$r_{f1} = 0$
(no feed-back)
(Unstable)
(Stable)
(c)
Stability criterion
2 loads
$r_{f1} = 1K\Omega$ (feed-back to detecting device)
(Unstable)
(Stable)
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\xi_5^{5''}$
$\xi_3^{5''}$

Fig. 32.

those of lower dimensions have been described. Accordingly, from the simple stability criteria of the 2- or 3-dimensional vectors, the stability criteria of the 4- and 5- and higher dimensional vectors are readily obtained successively.

The stability criteria on any (ξ_r, ξ_{r+1})-plane, besides those on (ξ_{n-1}, ξ_n)-plane, have also been discussed. Moreover, the stability criteria on (ξ_{n-2}, ξ_n)-plane, which consists of some straight lines, have been shown to be very simple and convenient.

It has been shown in § 7 that the stability criteria of higher dimensional vectors are drawn by adding two criteria of lower dimensional vectors. This method makes also the stability criteria very simple and convenient.

We have also shown that from the γ-scale on the line of critical stability, the hunting frequency in the case of automatic control system as well as the oscillating frequency in the case of oscillator is calculated.

It has also been discussed how simply the additional feed-back and the additional feed-forward, which are commonly used for stabilizing the automatic control system, can be treated by the author's vectorial considerations.

Lastly, the experiments on the stabilization of the speed control system have shown the complete accordance with the theoretical results and thus the confirmation of the author's methods has been achieved.

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