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THE INFLUENCE OF VORTICES UPON THE DRAG EXPERIENCED BY AN ELLIPTIC CYLINDER MOVING THROUGH AN INVISCID LIQUID

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SUMMARY

Assuming the existence of a vortex-pair behind an elliptic cylinder moving with a constant velocity through an unlimited mass of an incompressible inviscid fluid in a direction perpendicular to the axis of the cylinder as well as to the major-axis of its cross-section, the drag experienced by the cylinder is discussed.

To this end, the complex velocity potential for the flow around the elliptic cylinder is first obtained in terms of elliptic coordinates, and the drag acting on the cylinder is then computed in two different ways: namely, (i) by directly summing up the fluid pressures acting on the surface of the body, and (ii) by applying the theorem of momentum to an infinite mass of fluid surrounding the cylinder.

As we should have expected, one and the same expression for the drag on the cylinder is obtained, irrespective of the method of computation.

1. Introduction

It is well known that when a cylinder with arbitrary cross-sectional shape is moving, with an appropriate velocity, through a fluid, two or more discrete vortices are created in its wake and in consequence the pressure distribution over the surface of the body differs greatly, especially in the rear part of the body, from that computed on the basis of the theory of continuous flow of an incompressible inviscid fluid.

Assuming two symmetrically disposed vortices of equal strength κ, with circulations in the senses indicated, at two points A and B in the wake of a circular cylinder as shown in Fig. 1, Föppl (**1**) and Bickley (**2**) computed independently the drag acting on a circular cylinder. The former calculated the drag by applying the theorem of momentum to an unlimited mass of fluid surrounding the body, while the latter obtained the drag by directly summing up the fluid pressures over the surface of the cylinder.

According to Bickley, the drag X experienced by a circular cylinder is expressed in the form:

$$X=2\kappa\rho\left\{u_A\frac{a^2}{c^2}\sin 2\gamma+v_A\left(1-\frac{a^2}{c^2}\right)\cos 2\gamma\right\}, \qquad (1.1)$$

where u_A, v_A are the x- and y-components of the velocity of vortex A, a the radius of the cylinder, c the distance of vortices A and B from the centre O of the cylinder, 2γ the angle AOB and ρ the density of the fluid concerned.

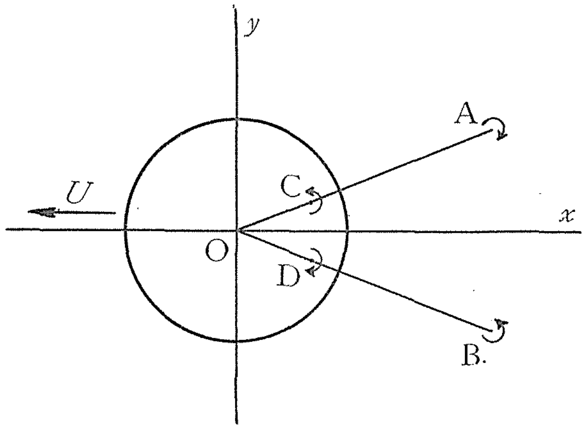

Fig. 1.

If we introduce the rectangular components of the velocities of image vortices C and D at the inverse points of A and B with respect to the circular cylinder respectively (Fig. 1), which are connected with u_A and v_A as:

$$\left.\begin{aligned} u_C&=-\frac{a^2}{c^2}(u_A\cos 2\gamma+v_A\sin 2\gamma),\\ v_C&=-\frac{a^2}{c^2}(u_A\sin 2\gamma-v_A\cos 2\gamma),\\ u_D&=u_C, \qquad v_D=-v_C, \end{aligned}\right\} \qquad (1.2)$$

the expression (1.1) for X becomes

$$X=2\kappa\rho(v_A-v_C), \qquad (1.3)$$

which, on taking account of the obvious relations:

$$v_A=\frac{d}{dt}(c\sin\gamma), \qquad v_C=\frac{d}{dt}\left(\frac{a^2}{c}\sin\gamma\right), \qquad (1.4)$$

can ultimately be written in the form:

$$X=2\kappa\rho\frac{d}{dt}\left\{\left(c-\frac{a^2}{c}\right)\sin\gamma\right\}. \qquad (1.5)$$

It is to be noted here that Föppl's formula for the drag on a circular cylinder differs from the above Bickley's formula (1.5). Namely, using the same notations as above, Föppl's formula takes the form:

$$X = 3\kappa\rho \frac{d}{dt}\left\{\left(c - \frac{a^2}{c}\right)\sin\gamma\right\}. \tag{1.6}$$

The cause of the discrepancy between Föppl's and Bickley's formulae was discussed by Sugawara and the senior writer in 1938 (**3**). In computing the drag experienced by a circular cylinder accompanied by a vortex-pair in its wake, both the direct method as used by Bickley of summing up the fluid pressures acting on the surface of the body and the indirect method as employed by Föppl of applying the theorem of momentum to an infinite mass of fluid surrounding the cylinder were employed.* Thus, it has been shown there that Bickley's formula is correct, but Föppl's one is erroneous.

The corresponding problem for the case of an elliptic cylinder was treated by Sanuki and Arakawa in 1931 by applying the theorem of momentum (**4**). However, the expression for the drag on a circular cylinder as deduced, as a limiting form, from their formula for the drag on an elliptic cylinder is neither in accord with Föppl's formula nor with Bickley's. In effect, in the same notations as above, their formula gives the following formula:

$$X = \kappa\rho \frac{d}{dt}\left\{\left(c - \frac{a^2}{c}\right)\sin\gamma\right\} \tag{1.7}$$

for the drag experienced by a circular cylinder.

The object of the present paper is to compute the drag acting on an elliptic cylinder moving, with a constant velocity U, through an unlimited mass of an incompressible inviscid fluid in a direction perpendicular to the axis of the cylinder as well as to the major-axis of its cross-section, by assuming the existence of a vortex-pair behind the cylinder.

For the computation of the drag on the body, two different methods are both employed as in our old paper (**3**) cited before: namely, the drag is calculated (i) by using the direct method of summing up the fluid pressures acting on the surface of the cylinder, and (ii) by applying the theorem of momentum to an infinite mass of fluid surrounding the body. Thus, as we should have expected, one and the same expression for the drag is obtained, which gives, as its limiting case, the correct Bickley's formula for the drag on a dircular cylinder. The sources of error in Sanuki and Arakawa's formula are indicated.

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* Recently it has been found out that in our paper (**3**) cited above, there are some errors in the course of analysis developed on the basis of the theorem of momentum, but fortunately the final result there given needs no alteration. For drawing our attention to these errors, we should like to express our cordial thanks to Dr. K. Tamada. The correct analysis will be given briefly in Appendix of the present paper.

I. Calculation of the Drag by summing up the Fluid Pressures acting on an Elliptic Cylinder

2. The complex velocity potential

In the first place, we shall calculate the drag experienced by an elliptic cylinder by directly summing up the fluid pressures acting on the surface of the body. Adopting the usual artifice, we shall consider a stationary elliptic cylinder placed in a uniform stream of velocity U flowing in the positive direction of the x-axis. We consider the case where the major-axis of the ellipse lies on the y-axis, the origin of the coordinate-axes being taken at the centre of the ellipse.

Now, it is convenient to introduce the elliptic coordinates (ξ, η) connected with the rectangular coordinates (x, y) as:

$$\left.\begin{aligned} x+iy&=c\sinh(\xi+i\eta), \\ \text{or}\qquad z&=c\sinh\zeta. \\ (0<\xi<\infty,\quad &-\pi<\eta<\pi) \end{aligned}\right\} \tag{2.1}$$

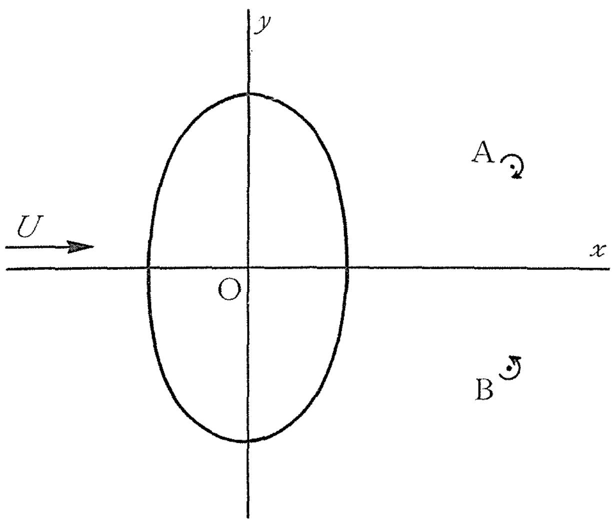

Fig. 2.

We assume that the elliptic cylinder under consideration is defined by $\xi=\xi_1$. Since the two vortices A and B, of strength κ with circulations in the senses as indicated in Fig. 2, are symmetrically situated with respect to the x-axis, we can define their positions (x_A, y_A) and (x_B, y_B) as (ξ_0, η_0) and $(\xi_0, -\eta_0)$ respectively. Thus, we have

$$\left.\begin{aligned} x_A&=x_B=c\sinh\xi_0\cos\eta_0, \\ y_A&=-y_B=c\cosh\xi_0\sin\eta_0. \end{aligned}\right\} \tag{2.2}$$

The complex velocity potential w for the flow around the elliptic cylinder consists of two parts: one part, w_1 say, is the complex velocity potential for the continuous uniform stream past the cylinder, while the other, w_2 say, is the complex velocity potential for the vortex motion. The former can be expressed as:

$$w_1 = Uce^{\xi_1}\cosh(\zeta-\xi_1), \tag{2.3}$$

and, after several calculations, the latter can be determined as (**6**):

$$w_2 = \frac{\kappa}{2\pi i}\log\frac{\cosh(\zeta-\xi_1)-\cosh(\bar{\zeta}_0-\xi_1)}{\cosh(\zeta-\xi_1)-\cosh(\zeta_0-\xi_1)}, \tag{2.4}$$

where $\bar{\zeta}_0$ denotes the conjugate complex of ζ_0.

Thus, the required complex velocity potential w becomes

$$w = Uce^{\xi_1}\cosh(\zeta-\xi_1)+\frac{\kappa}{2\pi i}\log\frac{\cosh(\zeta-\xi_1)-\cosh(\bar{\zeta}_0-\xi_1)}{\cosh(\zeta-\xi_1)-\cosh(\zeta_0-\xi_1)}, \tag{2.5}$$

or

$$w = \frac{1}{2}\,Uc(e^{\zeta}+e^{2\xi_1-\zeta})+\frac{\kappa}{2\pi i}\log\frac{(e^{\zeta}-e^{\bar{\zeta}_0})(e^{\zeta}-e^{2\xi_1-\bar{\zeta}_0})}{(e^{\zeta}-e^{\zeta_0})(e^{\zeta}-e^{2\xi_1-\zeta_0})}. \tag{2.6}$$

Separating the real and imaginary parts on both sides, we have the velocity potential ϕ and the stream function ψ in the forms:

$$\begin{aligned}\phi = Uce^{\xi_1}\cosh(\xi-\xi_1)\cos\eta+\frac{\kappa}{2\pi}\Big\{&\tan^{-1}\frac{\sinh(\xi_0-\xi_1)\sin(\eta_0+\eta)}{\cosh(\xi-\xi_1)-\cosh(\xi_0-\xi_1)\cos(\eta_0+\eta)}\\ &+\tan^{-1}\frac{\sinh(\xi_0-\xi_1)\sin(\eta_0-\eta)}{\cosh(\xi-\xi_1)-\cosh(\xi_0-\xi_1)\cos(\eta_0-\eta)}\Big\},\end{aligned} \tag{2.7}$$

$$\begin{aligned}\psi =& Uce^{\xi_1}\sinh(\xi-\xi_1)\sin\eta\\ &+\frac{\kappa}{4\pi}\log\frac{\{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}\{\cosh(\xi+\xi_0-2\xi_1)-\cos(\eta+\eta_0)\}}{\{\cosh(\xi-\xi_0)-\cos(\eta+\eta_0)\}\{\cosh(\xi+\xi_0-2\xi_1)-\cos(\eta-\eta_0)\}}.\end{aligned} \tag{2.8}$$

3. The velocities of the vortices

Making use of either of the above expressions for the velocity potential ϕ and the stream function ψ, the fluid velocity at any point not coinciding with either of the vortices can easily be obtained. Thus, if we denote the velocity components perpendicular to and along an ellipse $\xi=$const by v_ξ and v_η respectively, we have

$$v_\xi = \frac{1}{h}\frac{\partial\psi}{\partial\eta}, \qquad v_\eta = -\frac{1}{h}\frac{\partial\psi}{\partial\xi}, \tag{3.1}$$

where $h=c\sqrt{\cosh^2\xi-\sin^2\eta}$.

In order to obtain the velocity of a vortex, e.g. the vortex $(-\kappa)$ at A, itself, however, we must first subtract from ψ the stream function ψ_A due to this vortex alone. Thus, the velocity components $[v_\xi]_A$, $[v_\eta]_A$ of the vortex $(-\kappa)$ at A (ξ_0, η_0) can be obtained by differentiating $\psi-\psi_A$ by ξ and η respectively, and we have

$$
\left.\begin{aligned}
[v_\xi]_A &= \frac{1}{c\sqrt{\cosh^2\xi_0-\sin^2\eta_0}}\Big[Uc\,e^{\xi_1}\sinh(\xi_0-\xi_1)\cos\eta_0 \\
&\qquad +\frac{\kappa}{4\pi}\Big\{\frac{\sin 2\eta_0}{\cosh 2(\xi_0-\xi_1)-\cos 2\eta_0}-\frac{\sin 2\eta_0}{1-\cos 2\eta_0}+\frac{\sin 2\eta_0}{\cosh 2\xi_0+\cos 2\eta_0}\Big\}\Big], \\
[v_\eta]_A &= \frac{1}{c\sqrt{\cosh^2\xi_0-\sin^2\eta_0}}\Big[-Uc\,e^{\xi_1}\cosh(\xi_0-\xi_1)\sin\eta_0 \\
&\qquad +\frac{\kappa}{4\pi}\Big\{\frac{\sinh 2(\xi_0-\xi_1)}{\cosh 2(\xi_0-\xi_1)-1}-\frac{\sinh 2(\xi_0-\xi_1)}{\cosh 2(\xi_0-\xi_1)-\cos 2\eta_0}+\frac{\sinh 2\xi_0}{\cosh 2\xi_0+\cos 2\eta_0}\Big\}\Big].
\end{aligned}\right\} \quad (3.2)
$$

On account of the symmetry, the velocity components $[v_\xi]_B$, $[v_\eta]_B$ of the vortex (κ) at B $(\xi_0, -\eta_0)$ are given by

$$
[v_\xi]_B=[v_\xi]_A, \qquad [v_\eta]_B=-[v_\eta]_A. \quad (3.3)
$$

For later use the rectangular components of velocities of the vortices will also be calculated. It can readily be seen that the rectangular components of velocity u_A, v_A of the vortex $(-\kappa)$ at A are connected with $[v_\xi]_A$, $[v_\eta]_A$ as:

$$
\left.\begin{aligned}
u_A &= \frac{c}{h_0}\cosh\xi_0\cos\eta_0[v_\xi]_A-\frac{c}{h_0}\sinh\xi_0\sin\eta_0[v_\eta]_A, \\
v_A &= \frac{c}{h_0}\sinh\xi_0\sin\eta_0[v_\xi]_A+\frac{c}{h_0}\cosh\xi_0\cos\eta_0[v_\eta]_A,
\end{aligned}\right\} \quad (3.4)
$$

where $h_0=c\sqrt{\cosh^2\xi_0-\sin^2\eta_0}$. Or, remembering the obvious relations that

$$
[v_\xi]_A=h_0\frac{d\xi_0}{dt}, \qquad [v_\eta]_A=h_0\frac{d\eta_0}{dt}, \quad (3.5)
$$

the expressions for u_A, v_A can be put in the forms:

$$
\left.\begin{aligned}
u_A &= \frac{d}{dt}(c\sinh\xi_0\cos\eta_0), \\
v_A &= \frac{d}{dt}(c\cosh\xi_0\sin\eta_0).
\end{aligned}\right\} \quad (3.6)
$$

Substituting the values of $[v_\xi]_A$, $[v_\eta]_A$ as given by (3.2) into the right-hand sides of (3.4), we have, after several calculations,

$$
\left.\begin{aligned}
u_A &= \frac{1}{\cosh 2\xi_0+\cos 2\eta_0}\Big[Ue^{\xi_1}\{\sinh(2\xi_0-\xi_1)-\sinh\xi_1\cos 2\eta_0\} \\
&\qquad +\frac{\kappa}{\pi c}\Big\{\frac{\cosh(2\xi_0-\xi_1)\cosh(\xi_0-\xi_1)}{\cosh 2(\xi_0-\xi_1)-\cos 2\eta_0}\sin\eta_0-\frac{\cosh\xi_0}{2\sin\eta_0} \\
&\qquad -\frac{\sin\eta_0}{\sinh(\xi_0-\xi_1)}\,\frac{\sinh(2\xi_0-\xi_1)\sinh^2\xi_0+\sinh\xi_1\cos^2\eta_0}{\cosh 2\xi_0+\cos 2\eta_0}\Big\}\Big], \\
v_A &= \frac{\cos\eta_0}{\cosh 2\xi_0+\cos 2\eta_0}\Big[-2Ue^{\xi_1}\cosh\xi_1\sin\eta_0 \\
&\qquad +\frac{2\kappa}{\pi c}\sin^2\eta_0\Big\{\frac{1}{\cosh 2(\xi_0-\xi_1)-\cos 2\eta_0}\,\frac{\cosh(2\xi_0-\xi_1)}{2\sinh(\xi_0-\xi_1)}+\frac{\sinh\xi_0}{\cosh 2\xi_0+\cos 2\eta_0}\Big\}\Big].
\end{aligned}\right\} \quad (3.7)
$$

On account of the symmetry, the rectangular components of velocity u_B, v_B of the vortex (κ) at B are given by

$$u_B = u_A, \qquad v_B = -v_A. \tag{3.8}$$

4. Calculation of the force acting on the cylinder

We next proceed to the calculation of the drag experienced by the elliptic cylinder $(\xi=\xi_1)$ under consideration by directly summing up the fluid pressures acting on the cylinder. The pressure distribution over the surface of the cylinder can be calculated by the well-known pressure equation, namely:

$$p = F(t) - \frac{1}{2}\rho q^2 - \rho \frac{\partial \phi}{\partial t}. \tag{4.1}$$

If we denote the x- and y-components of the force acting on the cylinder by X and Y respectively, we have (7)

$$\begin{aligned} X - iY &= -i\oint p\, d\bar{z} \\ &= \frac{1}{2} i\rho \oint \left(\frac{dw}{dz}\right)^2 dz + i\rho \frac{d}{dt}\oint \bar{w}\, d\bar{z}, \end{aligned} \tag{4.2}$$

where $\bar{w}$ denotes the conjugate complex of w, and both the two integrals are taken, in the counter-clockwise sense, once round the circumference of the cylinder.

After some calculations, the first term on the right-hand side of (4.2), which we denote simply by F_q, can be evaluated as:

$$\begin{aligned} F_q &= \frac{1}{2} i\rho \oint \left(\frac{dw}{dz}\right)^2 dz \\ &= 2\kappa\rho \frac{\cos\eta_0}{\cosh 2\xi_0 + \cos 2\eta_0}\Big[-2Ue^{\xi_1}\cosh\xi_1 \sin\eta_0 \\ &\quad + \frac{2\kappa}{\pi c}\sin^2\eta_0 \Big\{\frac{1}{\cosh 2(\xi_0-\xi_1) - \cos 2\eta_0}\frac{\cosh(2\xi_0-\xi_1)}{2\sinh(\xi_0-\xi_1)} + \frac{\sinh\xi_0}{\cosh 2\xi_0 + \cos 2\eta_0}\Big\}\Big]. \end{aligned} \tag{4.3}$$

Comparing this with the expression for v_A as given by (3.7), we have

$$F_q = 2\kappa\rho\, v_A, \tag{4.4}$$

which, with the help of (3.6), may also be written as:

$$F_q = 2\kappa\rho c \frac{d}{dt}(\cosh\xi_0 \sin\eta_0). \tag{4.5}$$

Next, the second term on the right-hand side of (4.2), which we denote simply by F_ϕ, can be evaluated as:

$$\begin{aligned} F_\phi &= i\rho \frac{d}{dt}\oint \bar{w} d\bar{z} = i\rho \frac{d}{dt}\oint \bar{w}_2 d\bar{z} \\ &= -2\kappa\rho c \frac{d}{dt}(e^{\xi_1-\xi_0}\cosh\xi_1 \sin\eta_0), \end{aligned} \tag{4.6}$$

where $\bar{w}_2$ is the conjugate complex of w_2 which is given by (2.4) and denotes the complex velocity potential for the vortex motion. In terms of the velocity components u_A, v_A of the vortex $(-\kappa)$ at A, the value of F_ϕ can also be expressed in the form:

$$F_\phi=\frac{2\kappa\rho e^{\xi_1}\cosh\xi_1}{\cosh 2\xi_0+\cos 2\eta_0}\{(u_A\sin 2\eta_0-v_A\cos 2\eta_0)-v_A e^{-2\xi_0}\}\,. \tag{4.7}$$

Thus, making use of the values of F_q and F_ϕ as given by (4.5) and (4.6) respectively, we have, by (4.2),

$$\left.\begin{aligned} X&=2\kappa\rho c\, e^{\xi_1}\frac{d}{dt}\{\sinh(\xi_0-\xi_1)\sin\eta_0\}\,,\\ Y&=0\,. \end{aligned}\right\} \tag{4.8}$$

Further, if we use the values of F_q and F_ϕ as given by (4.4) and (4.7) respectively, the drag X can also be expressed in the form:

$$X=\frac{2\kappa\rho e^{\xi_1}}{\cosh 2\xi_0+\cos 2\eta_0}[u_A\cosh\xi_1\sin 2\eta_0+v_A\{\sinh(2\xi_0-\xi_1)-\sinh\xi_1\cos 2\eta_0\}]\,, \tag{4.9}$$

and when the values of u_A and v_A as given by (3.7) are substituted, this becomes ultimately

$$X=\frac{4\kappa^2\rho}{\pi c}e^{\xi_1}\cosh\xi_1\cos\eta_0\frac{\cosh(3\xi_0-2\xi_1)\sin^2\eta_0-\cosh\xi_0\sinh^2(\xi_0-\xi_1)}{\{\cosh 2(\xi_0-\xi_1)-\cos 2\eta_0\}(\cosh 2\xi_0+\cos 2\eta_0)^2}\,. \tag{4.10}$$

It will be seen that in case when the vortices lie on a curve* defined by

$$\sin\eta_0=\sqrt{\frac{\cosh\xi_0\sinh^2(\xi_0-\xi_1)}{\cosh(3\xi_0-2\xi_1)}}\,, \tag{4.11}$$

the cylinder does not experience any drag.

II. Application of the Theorem of Momentum to the Calculation of the Drag

5. The forces acting on the cylinder

We next proceed to the calculation of the drag experienced by the elliptic cylinder $(\xi=\xi_1)$ under consideration, by applying this time the theorem of momentum to an infinite mass of fluid surrounding the cylinder as in the papers by Föppl (**1**) and by Sanuki and Arakawa (**4**).

We take the coordinate-axes (x, y) as shown in Fig. 2 and we assume that the cylinder is moving with a constant velocity U in an unlimited perfect fluid in the

* This is the equation for the locus of equilibrium positions of the vortices (**5**), and is easily obtained by eliminating κ from the following two equations:

$$[v_\xi]_A=0\,,\quad [v_\eta]_A=0\,.$$

negative direction of the x-axis. Then, with the same notation as before, the complex velocity potential w for the flow around the elliptic cylinder ($\xi=\xi_1$) under consideration is given by

$$w=w_1+w_2\,, \tag{5.1}$$

where

$$w_1=\frac{1}{2}\,Uc(e^{2\xi_1-\zeta}+e^{-\zeta})\,, \tag{5.2}$$

$$w_2=\frac{\kappa}{2\pi i}\log\frac{(e^{\zeta}-e^{\bar{\zeta}_0})(e^{\zeta}-e^{2\xi_1-\bar{\zeta}_0})}{(e^{\zeta}-e^{\zeta_0})(e^{\zeta}-e^{2\xi_1-\zeta_0})}\,. \tag{5.3}$$

Now, as shown in Fig. 3, we take an elliptic contour C_1 just round the elliptic cylinder under consideration, a contour C_2 round the cut connecting the two points A, B and a large circular contour C_3 enclosing the whole system, and we assume that these three contours are connected by some straight lines as shown in the figure.

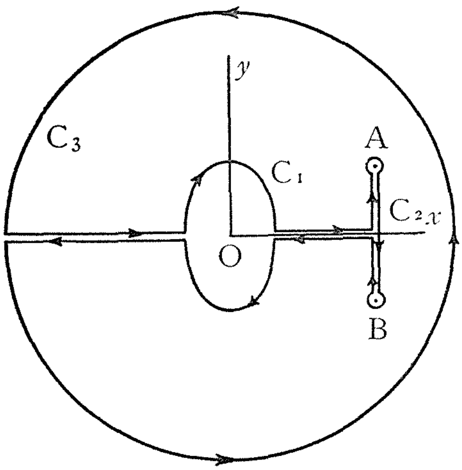

Fig. 3.

If, as before, we denote the x- and y-components of the resultant force acting on the cylinder by X and Y respectively, then by applying the theorem of momentum to the fluid in the region S enclosed by the three contours C_1, C_2, C_3, we have

$$X=-\frac{d}{dt}\iint_S \rho u\,dx\,dy-\oint_{C3}\rho u v_n\,ds-\oint_{C3}p\cos(n,\,x)ds\,, \tag{5.4}$$

$$Y=-\frac{d}{dt}\iint_S \rho v\,dx\,dy-\oint_{C3}\rho v v_n\,ds-\oint_{C3}p\cos(n,\,y)ds\,, \tag{5.5}$$

where n denotes the outward normal to the contour C_3, ds is the line-element along the contour C_3 reckoned positive in the counter-clockwise sense, and v_n is the velocity component normal to the element ds.

Since, however, it is obvious on account of the symmetry that $Y=0$, it suffices to consider the drag X only.

Remembering that $u=\partial\phi/\partial x$ and applying Green's theorem, we can transform the surface integral $\iint_S \rho u\,dx\,dy$ into a line integral taken, in the counter-clockwise sense, along the closed contour enclosing the region S, which consists of the three contours C_1, C_2, C_3 and some straight lines connecting them (Fig. 3). We then have

$$\iint_S \rho u\,dx\,dy=\iint_S \rho\frac{\partial\phi}{\partial x}dx\,dy=\left(\oint_{C1}+\oint_{C2}+\oint_{C3}\right)\rho\phi\cos(n,\,x)ds. \tag{5.6}$$

Thus, making use of the pressure equation (4.1), the drag X is expressed as:

$$X=-\frac{d}{dt}\left\{\left(\oint_{C1}+\oint_{C2}\right)\rho\phi\cos(n,\,x)ds\right\}$$
$$-F(t)\oint_{C3}\cos(n,\,x)ds+\frac{1}{2}\rho\oint_{C3}q^2\cos(n,\,x)\,ds-\rho\oint_{C3}uv_n\,ds. \tag{5.7}$$

However, the integral $\oint_{C3}\cos(n,\,x)ds$ evidently vanishes, and it can easily be shown that both the last two integrals tend to zero as the large circular contour C_3 recedes away to infinity. Also, it is found that the velocity potential $\phi_1=\Re(w_1)$ contributes nothing to the first two integrals in the brackets. Thus, we finally obtain

$$X=\frac{dI}{dt}, \tag{5.8}$$

where

$$I=-\left\{\left(\oint_{C1}+\oint_{C2}\right)\rho\phi_2\cos(n,\,x)ds\right\}, \tag{5.9}$$

where ϕ_2 denotes the velocity potential for the vortex motion alone.

6. Evaluation of the drag

Our next problem is to evaluate the two line integrals in (5.9). After simple calculations the first integral becomes

$$\oint_{C1}\rho\phi_2\cos(n,\,x)ds=-\rho c\cosh\xi_1\int_{-\pi}^{\pi}[\phi_2]_{\xi_1}\cos\eta\,d\eta, \tag{6.1}$$

where $[\phi_2]_{\xi_1}$ denotes the value of the function ϕ_2 on the surface of the elliptic cylinder $\xi=\xi_1$.

After various calculations we find however that

$$\int_{-\pi}^{\pi}[\phi_2]_{\xi_1}\cos\eta\,d\eta=\Re\int_{-\pi}^{\pi}[w_2]_{\xi_1}\cos\eta\,d\eta=-2\kappa e^{\xi_1-\xi_0}\sin\eta_0. \tag{6.2}$$

Therefore we have

$$\oint_{C1}\rho\phi_2\cos(n,\,x)ds=2\kappa\rho c e^{\xi_1-\xi_0}\cosh\xi_1\sin\eta_0. \tag{6.3}$$

On the other hand, the remaining second integral can easily be evaluated by noticing that the value of ϕ_2 on the right-hand side of the cut AB exceeds the value of ϕ_2 on the left-hand side by the cyclic constant κ. Thus, denoting the y-coordinate of the point A by y_A, we have

$$\oint_{C_2} \rho\phi_2 \cos(n, x) ds = -2\kappa\rho y_A$$
$$= -2\kappa\rho c \cosh \xi_0 \sin \eta_0. \qquad (6.4)$$

Hence, we get finally

$$I = 2\kappa\rho c e^{\xi_1} \sinh(\xi_0 - \xi_1) \sin \eta_0, \qquad (6.5)$$

and the expression for the drag X experienced by the elliptic cylinder ($\xi = \xi_1$) under consideration becomes

$$X = 2\kappa\rho c e^{\xi_1} \frac{d}{dt} \{\sinh(\xi_0 - \xi_1) \sin \eta_0\}. \qquad (6.6)$$

It will be seen that, as we should have expected, this formula is in perfect agreement with the previous one which has been obtained by directly summing up the fluid pressures acting on the elliptic cylinder ($\xi = \xi_1$).

III. A Limiting Case

7. The drag on a circular cylinder

As an addendum, we shall lastly derive from the preceding result the expression for the drag on a circular cylinder accompanied by two symmetrically disposed vortices in its wake. We denote the radius of the circular cylinder by a and the distance of the vortices from the axis of the cylinder by r_0. Then, making, in (4.8) or (6.6), c to tend to zero and both ξ_0, ξ_1 to infinity at the same time in such a manner that ce^{ξ_0} and ce^{ξ_1} become equal to $2r_0$ and $2a$ respectively, we obtain the expression for the drag on the circular cylinder. Thus, we have

$$X = 2\kappa\rho \frac{d}{dt}\left\{\left(r_0 - \frac{a^2}{r_0}\right)\sin \eta_0\right\}.$$

It is readily found that except for the difference in notation, this is in perfect agreement with the correct formula (1.5) for the drag on a circular cylinder.

Appendix

Note on the Influence of Vortices upon the Drag of a Circular Cylinder moving through a Fluid

In a paper (**3**) published as early as 1938, Sugawara and the senior writer calculated the drag experienced by a circular cylinder moving, with a constant velocity, through an incompressible perfect fluid, assuming the existence of two symmetrically disposed vortices in its wake. Two different methods of calculating the drag were employed. In one method the drag was obtained by summing up the pressures exerted by the fluid upon the surface of the circular cylinder, while in the other it was calculated by applying the theorem of momentum to an infinite mass of fluid surrounding the cylinder.

Recently Dr. K. Tamada has kindly drawn our attention to some errors in our paper cited above which are found in the course of analysis developed on the basis of the theorem of momentum, though fortunately the final result there given needs no alteration. The correct analysis will now be given.

1. We assume that a circular cylinder of radius a is moving with constant velocity U through an incompressible inviscid fluid in the negative direction of the x-axis, and we also assume that the cylinder is accompanied by two symmetrically disposed vortices with strength κ in its wake.

Referring to Fig. 1 in our 1938 paper, the complex velocity potential w for the flow outside the cylinder is given by*

$$w = U\frac{a^2}{z} + \frac{\kappa}{2\pi i}\log\frac{(z-z_{\rm C})(z-z_{\rm B})}{(z-z_{\rm A})(z-z_{\rm D})}, \qquad (1)$$

where $z_{\rm A}(=ce^{i\gamma})$ and $z_{\rm B}(=ce^{-i\gamma})$ are the complex coordinates of the vortices A and B in the fluid, while $z_{\rm C}$ and $z_{\rm D}$ are the complex coordinates of the corresponding image vortices C and D in the circular cylinder, i. e. $z_{\rm C}=(a^2/c)e^{i\gamma}$ and $z_{\rm D}=(a^2/c)e^{-i\gamma}$.

We take a closed contour similar to that as shown in Fig. 3 in the preceding pages, with only one difference that an elliptic contour C_1 there is now replaced by a circular contour C_1 of radius a. Thus, we take a closed contour Σ which consists of a circular contour C_1 of radius a coinciding with the circular cylinder under consideration, a contour C_2 round the cut connecting the two points A, B and a large circular contour C_3 of radius R enclosing the whole system, and some straight lines connecting these three contours.

Then, if we denote the drag experienced by the circular cylinder by X, we

* It will be noted that the velocity potential ϕ is here defined as $\mathbf{v}=\text{grad}\ \phi$, while in our 1938 paper we defined it as $\mathbf{v}=-\text{grad}\ \phi$.

have, by applying the theorem of momentum to the fluid in the region S enclosed by the above-mentioned closed contour Σ,

$$X=-\frac{d}{dt}\iint_{S}\rho u\,dx\,dy-\oint_{C3}\rho u v_n\,ds-\oint_{C3}p\cos(n,\,x)ds\,, \tag{2}$$

where n denotes the outward normal to the contour C_3, ds is the line-element along the contour C_3 reckoned positive in the counter-clockwise sense, and v_n is the velocity component normal to the element ds.

Remembering that $u=\partial\phi/\partial x$ and applying Green's theorem, we can transform the surface integral $\iint_S \rho u\,dx\,dy$ into a line integral taken, in the counter-clockwise sense, along the closed contour Σ. Thus, we have

$$\iint_{S}\rho u\,dx\,dy=\iint_{S}\rho\frac{\partial\phi}{\partial x}\,dx\,dy=\left(\oint_{C1}+\oint_{C2}+\oint_{C3}\right)\rho\phi\cos(n,\,x)ds\,. \tag{3}$$

Therefore

$$\begin{aligned}X=&-\frac{d}{dt}\left\{\left(\oint_{C1}+\oint_{C2}+\oint_{C3}\right)\rho\phi\cos(n,\,x)ds\right\}\\&-\oint_{C3}\rho u v_n\,ds-\oint_{C3}p\cos(n,\,x)ds\,.\end{aligned} \tag{4}$$

It can easily be shown that the integral $\oint_{C3}\rho u v_n\,ds$ tends to zero as the large circular contour C_3 recedes away to infinity. Also, since U is assumed to be constant, it is found that the velocity potential $\phi_1=\Re(Ua^2/z)$ contributes nothing to the three integrals in the brackets as well as to the last integral. Thus, making use of the pressure equation (4.1) and remembering that $\oint_{C3}\cos(n,\,x)ds=0$ and that the integral $\oint_{C3}q^2\cos(n,\,x)ds$ vanishes as C_3 recedes away to infinity, we obtain

$$X=-\frac{d}{dt}\left\{\left(\oint_{C1}+\oint_{C2}+\oint_{C3}\right)\rho\phi_2\cos(n,\,x)ds\right\}+\rho\int_{-\pi}^{\pi}\frac{\partial\phi_2}{\partial t}\cos\theta\cdot R d\theta\,, \tag{5}$$

where

$$\begin{aligned}\phi_2&=\Re\left[\frac{\kappa}{2\pi i}\log\frac{(z-z_C)(z-z_B)}{(z-z_A)(z-z_D)}\right]\\&=\frac{\kappa}{2\pi}\Im\left[\log\frac{(z-z_C)(z-z_B)}{(z-z_A)(z-z_D)}\right].\end{aligned} \tag{6}$$

We shall now evaluate the four integrals in (5). First, from our 1938 paper, we find that

$$\left(\oint_{C1}+\oint_{C2}\right)\rho\phi_2\cos(n,\,x)ds=-2\kappa\rho\left(c-\frac{a^2}{c}\right)\sin\gamma\,. \tag{7}$$

The remaining two integrals can be evaluated as follows. On the circular contour C_3 we have $z=Re^{i\theta}$ and therefore remembering that $z_A=ce^{i\gamma}$, $z_B=ce^{-i\gamma}$, $z_C=(a^2)e^{i\gamma}$, and $z_D=(a^2/c)e^{-i\gamma}$ we have

$$\log(z-z_A)=\log R+i\theta-\frac{c}{R}e^{i(\gamma-\theta)}+O\left(\frac{1}{R^2}\right),$$
$$\log(z-z_B)=\log R+i\theta-\frac{c}{R}e^{-i(\gamma+\theta)}+O\left(\frac{1}{R^2}\right),$$
$$\log(z-z_C)=\log R+i\theta-\frac{a^2}{cR}e^{i(\gamma-\theta)}+O\left(\frac{1}{R^2}\right),$$
$$\log(z-z_D)=\log R+i\theta-\frac{a^2}{cR}e^{-i(\gamma+\theta)}+O\left(\frac{1}{R^2}\right).$$

Thus, we have

$$\phi_2=\frac{\kappa}{2\pi}\left(c-\frac{a^2}{c}\right)\frac{1}{R}\{\sin(\gamma+\theta)+\sin(\gamma-\theta)\}+O\left(\frac{1}{R^2}\right)$$
$$=\frac{\kappa}{\pi}\left(c-\frac{a^2}{c}\right)\frac{1}{R}\sin\gamma\cos\theta+O\left(\frac{1}{R^2}\right), \tag{8}$$

and hence we have,* when $R\to\infty$,

$$\int_{-\pi}^{\pi}\phi_2\cos\theta\cdot Rd\theta=\kappa\left(c-\frac{a^2}{c}\right)\sin\gamma. \tag{9}$$

This gives immediately

$$\rho\int_{-\pi}^{\pi}\frac{\partial\phi_2}{\partial t}\cos\theta\cdot R\,d\theta=\kappa\rho\frac{d}{dt}\left\{\left(c-\frac{a^2}{c}\right)\sin\gamma\right\}. \tag{10}$$

Also, we have

$$\oint_{C3}\rho\phi_2\cos(n,x)ds=\int_{-\pi}^{\pi}\rho\phi_2\cos\theta\cdot R\,d\theta$$
$$=\kappa\rho\left(c-\frac{a^2}{c}\right)\sin\gamma. \tag{11}$$

Thus, we get finally

$$X=2\kappa\rho\frac{d}{dt}\left\{\left(c-\frac{a^2}{c}\right)\sin\gamma\right\}. \tag{12}$$

2. The drag X can also be evaluated in an alternative manner as follows. If use is made of the pressure equation (4.1) in (4), we have

$$X=-\frac{d}{dt}\left\{\left(\oint_{C1}+\oint_{C2}\right)\rho\phi\cos(n,x)ds\right\}$$
$$-F(t)\oint_{C3}\cos(n,x)ds+\frac{1}{2}\rho\oint_{C3}q^2\cos(n,x)ds-\rho\oint_{C3}uv_n ds. \tag{13}$$

The integral $\oint_{C3}\cos(n,x)ds$ evidently vanishes, and it can easily be shown that both the last two integrals tend to zero as the large circular contour C_3 recedes away to infinity. Thus, since, as mentioned before, the velocity potential ϕ_1 contributes nothing to the first two integrals in the brackets, we obtain

$$X=-\frac{d}{dt}\left\{\left(\oint_{C1}+\oint_{C2}\right)\rho\phi_2\cos(n,x)ds\right\}, \tag{14}$$

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* In our 1938 paper we have erroneously mentioned that $\int_{-\pi}^{\pi}\phi_2\cos\theta\cdot Rd\theta$ evidently vanishes when $R\to\infty$.

and therefore, by (7), we get ultimately

$$X=2\kappa\rho\frac{d}{dt}\left\{\left(c-\frac{a^2}{c}\right)\sin\gamma\right\}. \tag{15}$$

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