MEMOIRS OF THE COLLEGE OF SCIENCE, University of Kyoto, Series A,
Vol. XXVIII, No. 3, Article 1, 1957

DIFFRACTION OF ELECTROMAGNETIC WAVES BY A CIRCULAR CYLINDER OF INFINITE LENGTH*

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(Received May 28, 1957)

ABSTRACT

The diffraction of electromagnetic plane waves by a circular cylinder of infinite length is discussed in case when the wave is incident normally to the axis of the cylinder and the magnetic and the electric fields are parallel and perpendicular to the cylinder-axis respectively.

1. Introduction

The problems on the diffraction of electromagnetic waves by a circular cylinder have been treated by several authors. Recently, C. H. Papas (**1**) has calculated the total scattering cross section of a circular cylinder of infinite length for the plane wave incident normally to the cylinder-axis, in case when the electric field of the incident wave is parallel to the cylinder-axis and the magnetic field perpendicular to it. In this paper, the authors discuss a case in which the magnetic field is parallel to the cylinder-axis and the electric field perpendicular to it. While Papas calculated the cross section at high frequencies only to the zeroth approximation for his case, we shall here perform the calculation to the first approximation for our present case.

2. Fundamental equations

We assume that (1) a circular cylinder of radius a is perfectly conducting, (2) the external medium is vacuum with constants ε_0 and μ_0, and (3) the wave incident normally to the cylinder-axis has its $\boldsymbol{H}$–vector parallel to the axis and its $\boldsymbol{E}$–vector perpendicular to it. Taking the z–axis along the cylinder-axis, the fundamental equations in the cylindrical coordinates ρ, φ, z are given by

$$\left.\begin{aligned} -i\omega\varepsilon_0 E_\rho &= \frac{1}{\rho}\frac{\partial H_z}{\partial \varphi}, \\ -i\omega\varepsilon_0 E_\varphi &= -\frac{\partial H_z}{\partial \rho}, \end{aligned}\right\} \tag{1}$$

* Read at the annual meeting of the Physical Society of Japan, November 2, 1950. Some of the results here given are quoted from our manuscript, which was sent to Dr. Papas in 1951, in F. E. Borgnis and C. H. Papas's book entitled "*Randwertprobleme der Mikrowellenphysik*" (Springer Verlag, 1955), pp. 59–60.

$$i\omega\mu_0 H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\varphi}, \tag{2}$$

where we have assumed $E_z = H_\rho = H_\varphi = 0$ since the state of polarization of the incident wave and the infinite length of the circular cylinder justify this assumption, and the time dependence of $\boldsymbol{E}$ and $\boldsymbol{H}$ is expressed by $e^{-i\omega t}$.

Writing $H_z = \psi$, $k^2 = \varepsilon_0\mu_0\omega^2$, we obtain, from (1) and (2),

$$(\Delta_2 + k^2)\psi = 0, \tag{3}$$

where Δ_2 is the two-dimensional Laplacian.

Further, introducing two-dimensional Green's function G defined by

$$(\Delta_2 + k^2)G(\rho, \varphi; \rho', \varphi') = -\frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi'), \tag{4}$$

we have, by Green's theorem,

$$\int_v \left\{G(\rho, \varphi; \rho', \varphi')(\Delta_2' + k^2)\psi(\rho', \varphi') - \psi(\rho', \varphi')(\Delta_2' + k^2)G(\rho, \varphi; \rho', \varphi')\right\}dv'$$
$$= \int_{S_1+S_2}\left\{G(\rho, \varphi; \rho', \varphi')\frac{\partial}{\partial n'}\psi(\rho', \varphi') - \psi(\rho', \varphi')\frac{\partial}{\partial n'}G(\rho, \varphi; \rho', \varphi')\right\}dS',$$

where v is the volume of the hollow cylinder of unit length formed by the conducting cylinder surface S_1 of radius a and the cylindrical surface S_2 of radius a' ($>a$). A solution of (4) is

$$G(\rho, \varphi; \rho', \varphi') = \frac{i}{4}H_0^{(1)}(kx), \tag{5}$$

with

$$x = \{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')\}^{\frac{1}{2}}.$$

In the limit $a' \to \infty$, by (3), (4), the second equation of (1) and the boundary condition on the conducting cylinder surface, the above relation from Green's theorem is rewritten as:

$$\psi(\rho, \varphi) = -\int_{S_1}\psi(\rho', \varphi')\frac{\partial}{\partial n'}G(\rho, \varphi; \rho', \varphi')dS'$$
$$+\lim_{a'\to\infty}\int_{S_2}\left\{G(\rho, \varphi; \rho', \varphi')\frac{\partial}{\partial n'}\psi(\rho', \varphi') - \psi(\rho', \varphi')\frac{\partial}{\partial n'}G(\rho, \varphi; \rho', \varphi')\right\}dS'. \tag{6}$$

Since the induced current on the conducting surface is proportional to H_z, namely to ψ, the first term of (6) can be anticipated to represent the scattered waves and accordingly, if so, the second term must represent the incident wave. In fact, if we assume the incident wave to be

$$\psi = e^{ik\rho\cos(\varphi-\varphi_1)}, \tag{7}$$

and put it for ψ in the second term of (6), we can demonstrate

$$\lim_{a'\to\infty}\int_{S_2}\left\{G(\rho,\ \varphi\ ;\ \rho',\ \varphi')\frac{\partial}{\partial n'}\psi(\rho',\ \varphi')-\psi(\rho',\ \varphi')\frac{\partial}{\partial n'}G(\rho,\ \varphi\ ;\ \rho',\ \varphi')\right\}dS'$$
$$=e^{ik\rho\cos(\varphi-\varphi_1)},$$

when we utilize the relations:

$$G(\rho,\ \varphi\ ;\ \rho',\ \varphi')=\frac{i}{4}\sum_{n=0}^{\infty}\varepsilon_n J_n(k\rho)H_n^{(1)}(k\rho')\cos n(\varphi-\varphi'),\quad(\rho<\rho')$$

and

$$e^{ik\rho'\cos(\varphi'-\varphi_1)}=\sum_{n=0}^{\infty}\varepsilon_n i^n J_n(k\rho')\cos n(\varphi'-\varphi_1),$$

$$\text{with}\quad \varepsilon_n=\begin{cases}1 & (n=0)\\ 2 & (n\geqq 1)\end{cases}$$

Thus we obtain, instead of (6),

$$\psi(\rho,\ \varphi)=-\int_{S_1}\psi(\rho',\ \varphi')\frac{\partial}{\partial n'}G(\rho,\ \varphi\ ;\ \rho',\ \varphi')dS'+e^{ik\rho\cos(\varphi-\varphi_1)}.\tag{8}$$

Considering that the value of $\partial\psi(\rho,\ \varphi)/\partial\rho$ at $\rho=a$ vanishes because of the second equation of (1) and the boundary condition, and that $\partial/\partial n'=-\partial/\partial\rho'$, the following equation can be shown to hold:

$$-ik\cos(\varphi-\varphi_1)\exp[ika\cos(\varphi-\varphi_1)]=a\int_0^{2\pi}\psi(a,\ \varphi')g(\varphi-\varphi')d\varphi',\tag{9}$$

where

$$g(\varphi-\varphi')=\left\{\frac{\partial^2}{\partial\rho\,\partial\rho'}G(\rho,\ \varphi\ ;\ \rho',\ \varphi')\right\}_{\substack{\rho=a\\ \rho'=a}}.\tag{10}$$

If we can solve the integral equation (9) with respect to $\psi(a,\ \varphi')$ i.e. H_z on the conducting surface, we can obtain, by putting $\psi(a,\ \varphi')$ into (8), the general H_z, i.e. $\psi(\rho,\ \varphi)$.

3. Scattering cross section

The scattered field term of (8) is:

$$\varphi^{\text{scat}}(\rho,\ \varphi)=a\int_0^{2\pi}\psi(a,\ \varphi')\left\{\frac{\partial}{\partial\rho'}G(\rho,\ \varphi\ ;\ \rho',\ \varphi')\right\}_{\rho'=a}d\varphi'.$$

We shall find its asymptotic representation at $\rho\to\infty$. Since

$$G(\rho,\ \varphi\ ;\ \rho',\ \varphi')\simeq\frac{i}{4}\left(\frac{-2i}{\pi k\rho}\right)^{\frac{1}{2}}\exp[ik\{\rho-\rho'\cos(\varphi-\varphi')\}]$$

holds at $\rho'\ll\rho\to\infty$, we obtain

$$\psi^{\text{scat}}(\rho,\ \varphi)=\frac{k}{4}\left(\frac{-2i}{\pi k}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\rho}}\exp(ik\rho)A(\varphi,\ \varphi_1),\tag{11}$$

where

$$A(\varphi,\ \varphi_1)=a\int_0^{2\pi}\psi_{\varphi_1}(a,\ \varphi')\cos(\varphi-\varphi')\exp\{-ika\cos(\varphi-\varphi')\}d\varphi'\,, \tag{12}$$

and the subscript φ_1 of ψ_{φ_1} means that the corresponding incident wave propagates in the φ_1 direction. Then, from the geometrical symmetry with respect to two incident angles φ_1 and φ_2, there follows

$$A(\pi+\varphi_1,\ \varphi_2)=A(\pi+\varphi_2,\ \varphi_1)\,. \tag{13}$$

From (12) and the integral equation (9), we obtain

$$\frac{ik}{A(\pi+\varphi_2,\ \varphi_1)}=\frac{ik}{A(\pi+\varphi_1,\ \varphi_2)}$$
$$=\frac{\int_0^{2\pi}\int_0^{2\pi}d\varphi'\psi_{\varphi_1}(a,\ \varphi')g(\varphi-\varphi')\psi_{\varphi_2}(a,\ \varphi)d\varphi}{\int_0^{2\pi}\psi_{\varphi_2}\cos(\varphi-\varphi_1)\exp\{ika\cos(\varphi-\varphi_1)\}d\varphi\int_0^{2\pi}\cos(\varphi-\varphi_2)\exp\{ika\cos(\varphi-\varphi_2)\}\psi_{\varphi_2}d\varphi}\,. \tag{14}$$

From (14), (12) and (9) we can derive the stationarity condition of

$$A=A(\pi+\varphi_1,\ \varphi_2)=A(\pi+\varphi_2,\ \varphi_1)\,,$$

against the variation of ψ_{φ_1} and ψ_{φ_2}, that is, the condition for $\delta A=0$.

We shall now concern ourselves with the total scattering cross section defined by

$$\sigma=\frac{\text{The time average scattered power per unit length of cylinder-axis}}{\text{The time average incident power through unit area}}$$
$$=\bar{P}^{\text{scat}}/\bar{S}^{\text{inc}}\,.$$

Since

$$\bar{S}^{\text{inc}}=\frac{1}{2}\text{Re}(\boldsymbol{E}^{\text{inc}}\times\boldsymbol{H}^{\text{inc}*})=\frac{k}{2\omega\varepsilon_0}\,,$$

and

$$\bar{P}^{\text{scat}}=\frac{1}{2}\text{Re}\int_0^{2\pi}E_\varphi^{\text{scat}}(a,\ \varphi)H_z^{\text{scat}*}(a,\ \varphi)ad\varphi\,,$$

we have

$$\sigma=-\text{Re}\,A(\varphi_1,\ \varphi_1)\,. \tag{15}$$

4. Approximation of σ at low frequencies

If, in (14), we put $\varphi_1=0$, $\varphi_2=\pi$ and $\psi_0(a,\ \varphi)=\psi_\pi(a,\ \varphi)=1$ to the zeroth approximation, we can obtain

$$A(\pi,\ \pi)=-\frac{4}{k}\frac{J_0'(ka)}{H_0^{(1)\prime}(ka)}\,,$$

and therefore, from (15), we have

$$\sigma=\frac{4}{k}\frac{\{J_0'(ka)\}^2}{\{J_0'(ka)\}^2+\{N_0'(ka)\}^2}\,. \tag{16}$$

Now, in order to improve the approximation, we assume

$$\psi_{\varphi_1} = \sum_{n=0}^{\infty} A_n \cos n(\varphi - \varphi_1) , \tag{17}$$

where A_n's are unknown coefficients. From (17) and (14)

$$\frac{ik}{A(\pi, \pi)} = -\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m C_{mn} A_n}{(\sum_{n=0}^{\infty} A_n B_n)^2} , \tag{18}$$

where

$$\left.\begin{aligned} B_n &= 2(-i)^n J_n'(ka) , \\ C_{mn} &= \delta_n^m \frac{(-1)^m i\pi^2 k^2}{m} J_m'(ka) H_m^{(1)\prime}(ka) . \end{aligned}\right\} \tag{19}$$

If we give variations to A_n's in (18) there follows by virtue of the stationarity of $A(\pi, \pi)$,

$$\frac{A(\pi, \pi)}{ik} \sum_{n=0}^{\infty} A_n C_{mn} = -B_m \sum_{n=0}^{\infty} A_n B_n . \tag{20}$$

Further, to calculate after the method of Schwinger (**2**), we introduce D_m defined by

$$\sum_{n=0}^{\infty} C_{mn} D_n = B_m , \qquad (m = 0, 1, 2, \cdots) .$$

By multiplying (20) by D_m and summing up with respect to m, we obtain

$$\frac{A(\pi, \pi)}{ik} \sum_m \{ (\sum_n A_n C_{mn}) D_m \} = -(\sum_m D_m B_m)(\sum_n A_n B_n) ,$$

which gives

$$\frac{A(\pi, \pi)}{ik} = -\sum_m D_m B_m = -\sum_m \frac{B_m^2}{C_{mm}} .$$

Therefore, we obtain finally

$$\sigma = \frac{4}{k} \sum_{n=0}^{\infty} \varepsilon_n \frac{\{J_n'(ka)\}^2}{\{J_n'(ka)\}^2 + \{N_n'(ka)\}^2} . \tag{21}$$

As will be shown in Appendix I, this result can also be derived from the results of Ignatowsky (**3**).

In order to compare, at low frequencies, our result with that of Papas, we denote our σ by $\sigma_\perp$ and Papas's σ by $\sigma_{||}$. Papas has given the results that

$$\sigma_{||} = \frac{4}{k} \frac{\{J_0(ka)\}^2}{\{J_0(ka)\}^2 + \{N_0(ka)\}^2} ,$$

$$\sigma_{||} = \frac{4}{k} \sum_{n=0}^{\infty} \varepsilon_n \frac{\{J_n(ka)\}^2}{\{J_n(ka)\}^2 + \{N_n(ka)\}^2} ,$$

which correspond respectively to (16) and (21) in our present case. Then, if we retain only the first term in each expansion of cylindrical functions for $ka \ll 1$, we have

$$\frac{\sigma_\perp}{\sigma_\parallel} = \left\{\frac{(ka)^2}{2 \ln (ka)}\right\}^2, \tag{22}$$

and

$$\frac{\sigma_\perp}{\sigma_\parallel} \to 0 \quad \text{as} \quad ka \to 0 .$$

5. Approximation of σ at high frequencies

In this case, we can approximate the distribution of $\psi_{\varphi_1}(a, \varphi)$ such that

$$\psi_{\varphi_1}(a, \varphi) = \sum_{n=0}^{\infty} b_n \cos n(\varphi - \varphi_1) \exp\{ika \cos(\varphi - \varphi_1)\} \quad \text{on the illuminated side,}$$

$$\psi_{\varphi_1}(a, \varphi) = 0 \quad \text{on the shadow side.}$$

To the zeroth approximation, we put

$$\begin{aligned} \psi_{\varphi_1}(a, \varphi) &= \exp\{ika \cos(\varphi - \varphi_1)\} && \text{on the illuminated side,} \\ \psi_{\varphi_1}(a, \varphi) &= 0 && \text{on the shadow side,} \end{aligned} \tag{23}$$

that is, we assume Kirchhoff's approximation.

For the sake of convenience, we consider $\psi_{\frac{\pi}{2}}$ and $\psi_{\frac{3\pi}{2}}$ as given by

$$\left.\begin{aligned} \psi_{\frac{\pi}{2}}(a, \varphi) &= \exp(ika \sin \varphi), && (\pi \leqq \varphi \leqq 2\pi) \\ &= 0, && (0 \leqq \varphi \leqq \pi) \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \psi_{\frac{3\pi}{2}}(a, \varphi) &= \exp(-ika \sin \varphi), && (0 \leqq \varphi \leqq \pi) \\ &= 0. && (\pi \leqq \varphi \leqq 2\pi) \end{aligned}\right\} \tag{25}$$

Applying these to (14), we have

$$\frac{ik}{A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = \frac{1}{4}\int_0^{\pi} d\varphi \int_{\pi}^{2\pi} \exp\{-ika(\sin \varphi - \sin \varphi')\} g(\varphi - \varphi') d\varphi' . \tag{26}$$

If we change the integration variables from φ, φ' to α, β by the relations

$$\varphi - \varphi' + 2\pi = 2\alpha , \quad \varphi + \varphi' - 2\pi = 2\beta , \tag{27}$$

(26) becomes

$$\frac{ik}{A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = 2\int_0^{\frac{\pi}{2}} g(2\alpha)\, d\alpha \int_0^{\alpha} \exp(-2ika \sin \alpha \sin \beta) d\beta . \tag{28}$$

Carrying out the integration with respect to β for $ka \gg 1$ by the method of stationary phase, we have

$$\frac{ik}{A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = \left(\frac{i\pi}{ka}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} g(2\alpha) \frac{\exp(-2ika \sin \alpha)}{\sqrt{\sin \alpha}} d\alpha , \tag{29}$$

where

$$g(2\alpha)=\frac{-ik^2}{4}\left\{H_1^{(1)\prime}(2ka\sin\alpha)\sin^2\alpha-\frac{1}{ka}H_1^{(1)}(2ka\sin\alpha)\frac{\cos^2\alpha}{2\sin\alpha}\right\}. \tag{30}$$

Now, $\alpha=0$ is a singular point of the integrand of (29). If $2ka\sin\alpha\gg 1$, however, it can be proved that the second term of $g(2\alpha)$ gives only imaginary contribution to A and thus no contribution to σ.

In order to secure the convergency of the integral, we replace the lower limit 0 by α_0 such that $2ka\sin\alpha_0\gg 1$. In this case, we write $A_{\alpha_0}(\pi/2, \pi/2)$ in place of $A(\pi/2, \pi/2)$ and (29) becomes

$$\frac{ik}{A_{\alpha_0}\left(\frac{\pi}{2},\frac{\pi}{2}\right)}=\frac{-ik}{4a}\int_{\alpha_0}^{\frac{\pi}{2}}\sin\alpha d\alpha=-\frac{ik}{4a}\cos\alpha_0, \tag{31}$$

when the second term of the integrand is conveniently neglected. Thus the corresponding cross section is

$$\sigma_{\alpha_0}=\frac{4a}{\cos\alpha_0}. \tag{32}$$

The dependence of σ_{α_0} on ka is roughly considered as follows. For example, if we put $2ka\sin\alpha_0=10$, Hankel's function can be asymptotically expanded to a good approximation. Then, we have

$$\sigma_{\alpha_0}=\frac{4a}{\left\{1-\left(\frac{5}{ka}\right)^2\right\}^{1/2}}\approx 4a\left\{1+\frac{1}{2}\left(\frac{5}{ka}\right)^2\right\}, \tag{33}$$

which enables us to anticipate the high frequency behaviour of σ. Especially it gives

$$\lim_{ka\to\infty}\sigma_{\alpha_0}=4a,$$

which coincides with the cross section in the geometrical optics.

For a better approximation of σ calculation, we take, instead of (23), the first two terms in the expansion of ψ_{φ_1} and thus we put

$$\begin{aligned}\psi_{\varphi_1}&=\{1+b\cos(\varphi-\varphi_1)\}\exp\{ika\cos(\varphi-\varphi_1)\} &&\text{on the illuminated side,}\\ &=0 &&\text{on the shadow side.}\end{aligned} \tag{34}$$

Then,

$$\begin{aligned}\psi_{\frac{\pi}{2}}&=(1+b\sin\varphi)\exp(ika\sin\varphi), &&(\pi\leqq\varphi\leqq 2\pi)\\ &=0, &&(0\leqq\varphi\leqq\pi)\\ \psi_{\frac{3\pi}{2}}&=(1-b\sin\varphi)\exp(-ika\sin\varphi), &&(0\leqq\varphi\leqq\pi)\\ &=0, &&(\pi\leqq\varphi\leqq 2\pi)\end{aligned}$$

which correspond respectively to (24) and (25). If we introduce these into (14), we obtain

$$\frac{ik}{A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = \frac{P+2bQ+b^2R}{P'+2bQ'+b^2R'}, \tag{35}$$

where

$$\left.\begin{aligned}
P &= \int_{\pi}^{2\pi} \exp(ika\sin\varphi')d\varphi' \int_0^{\pi} g(\varphi-\varphi') \exp(-ika\sin\varphi)d\varphi\,,\\
Q &= \frac{1}{2}\int_{\pi}^{2\pi} \sin\varphi' \exp(ika\sin\varphi')d\varphi' \int_0^{\pi} g(\varphi-\varphi') \exp(-ika\sin\varphi)d\varphi\\
&\quad -\frac{1}{2}\int_{\pi}^{2\pi} \exp(ika\sin\varphi')d\varphi' \int_0^{\pi} g(\varphi-\varphi') \sin\varphi \exp(-ika\sin\varphi)d\varphi\,,\\
R &= -\int_{\pi}^{2\pi} \sin\varphi' \exp(ika\sin\varphi')d\varphi' \int_0^{\pi} g(\varphi-\varphi') \sin\varphi \exp(-ika\sin\varphi)d\varphi\,,\\
P' &= -\int_0^{\pi} \sin\varphi d\varphi \int_{\pi}^{2\pi} \sin\varphi d\varphi\,,\\
Q' &= \frac{1}{2}\int_0^{\pi} \sin^2\varphi d\varphi \int_{\pi}^{2\pi} \sin\varphi d\varphi - \frac{1}{2}\int_0^{\pi} \sin\varphi d\varphi \int_{\pi}^{2\pi} \sin^2\varphi d\varphi\,,\\
R' &= \int_0^{\pi} \sin^2\varphi d\varphi \int_{\pi}^{2\pi} \sin^2\varphi d\varphi\,.
\end{aligned}\right\} \tag{36}$$

Further, (35) can be transformed into

$$\frac{ik}{A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)} = \frac{b_0^2P+2b_0b_1Q+b_1^2R}{b_0^2P'+2b_0b_1Q'+b_1^2R'} = \frac{\sum_{m=0}^{1}\sum_{n=0}^{1} b_m M_{mn} b_n}{\sum_{m=0}^{1}\sum_{n=0}^{1} b_m N_{mn} b_n}, \tag{37}$$

where

$$\left.\begin{aligned}
&b_0 = 1\,, \quad && M_{00} = P\,, && N_{00} = P'\,,\\
&b_1 = b\,, && M_{01} = M_{10} = Q\,, \quad && N_{01} = N_{10} = Q'\,,\\
& && M_{11} = R\,, && N_{11} = R'\,.
\end{aligned}\right\} \tag{38}$$

When we give variation to b_m in (37) corresponding to the variation of ψ_{φ_1}, we have from the stationarity of A,

$$A\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \sum_{n=0}^{1} M_{mn} b_n = ik \sum_{n=0}^{1} N_{mn} b_n\,. \tag{39}$$

Then, multiplying (39) with an indeterminate quantity E_m and summing up with respect to m, we obtain

$$A\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = ik \frac{\sum_m \sum_n E_m N_{mn} b_n}{\sum_m \sum_n E_m M_{mn} b_n}\,. \tag{40}$$

The above E_m can be defined by

$$\sum_{m=0}^{1} E_m N_{mn} = \lambda \sum_{m=0}^{1} E_m M_{mn} \qquad (n = 0, 1)\,, \tag{41}$$

that is,

$$\left.\begin{aligned}(N_{00}-\lambda M_{00})E_0+(N_{10}-\lambda M_{10})E_1=0\,,\\(N_{01}-\lambda M_{01})E_0+(N_{11}-\lambda M_{11})E_1=0\,,\end{aligned}\right\}\tag{41'}$$

where λ is an undetermined constant.

In order that a nontrivial solution of E_m may exist, the following equation must be satisfied:

$$\begin{vmatrix}N_{00}-\lambda M_{00} & N_{10}-\lambda M_{10}\\ N_{01}-\lambda M_{01} & N_{11}-\lambda M_{11}\end{vmatrix}=0\,,$$

that is,

$$\begin{vmatrix}P'-\lambda P & Q'-\lambda Q\\ Q'-\lambda Q & R'-\lambda R\end{vmatrix}=0\,,\tag{42}$$

which determines the value of λ. Accordingly, E_m ($m=0, 1$) can also be determined, but for the determination of A which in turn gives σ by (15), it suffices only to know the value of λ, because from (40) and (41), we have

$$A\left(\frac{\pi}{2},\ \frac{\pi}{2}\right)=ik\lambda\,.\tag{43}$$

For $ka\gg1$ and $2ka\sin\alpha_0\gg1$, the elements of the determinantal equation (42) become:

$$\left.\begin{aligned}P&=\frac{-ik}{a}\cos\alpha_0\,,\\Q&=\frac{ik}{2a}\left(\frac{\pi}{2}-\alpha_0+\frac{\sin 2\alpha_0}{2}\right),\\R&=-\frac{ik}{a}\left(\cos\alpha_0-\frac{\cos^3\alpha_0}{3}\right),\\P'&=4\,,\\Q'&=-\pi\,,\\R'&=\frac{\pi^2}{4}\,,\end{aligned}\right\}\tag{36'}$$

whose derivation is shown in Appendix II. With these values, the solution of (42) is given by

$$\lambda=-\frac{a}{ik}\,\frac{4\left(\cos\alpha_0-\frac{1}{3}\cos^3\alpha_0\right)+\frac{\pi^2}{4}\cos\alpha_0-\frac{\pi^2}{2}+\pi\alpha_0-\frac{\pi}{2}\sin 2\alpha_0}{\cos\alpha_0\left(\cos\alpha_0-\frac{1}{3}\cos^3\alpha_0\right)-\frac{1}{4}\left(\frac{\pi}{2}-\alpha_0+\frac{1}{2}\sin 2\alpha_0\right)^2}\,.\tag{44}$$

To get a rough estimation, we put for example

$$2ka\sin\alpha_0=10\,,$$

then (44) becomes

$$\lambda=-\frac{4a}{ik}\left\{1+\frac{1}{2}\left(\frac{5}{ka}\right)^2\right\},\tag{45}$$

and we obtain by (15)

$$\sigma = 4a\left\{1+\frac{1}{2}\left(\frac{5}{ka}\right)^2\right\}, \tag{46}$$

which tends to

$$\sigma = 4a \qquad \text{as} \quad ka \to \infty . \tag{47}$$

Acknowledgement

The authors' thanks are due to Professor S. Tomotika for his helpful discussion. Especially the junior author (T. W.) wishes to thank Professor Tomotika for his powerful guidance, in mathematical physics, since the time when he was an undergraduate student under the direction of Professor Tomotika.

Appendix I

Using the results derived by Ignatowsky (**3**), σ can be evaluated as follows. Corresponding to

$$H_z^{\text{inc}} = e^{ik\rho\cos(\varphi-\varphi_1)} = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k\rho)\cos n(\varphi-\varphi_1)\,, \tag{A1}$$

we put

$$H_z^{\text{scat}} = \sum_{n=0}^{\infty} \varepsilon_n i^n P_n H_n^{(1)}(k\rho)\cos n(\varphi-\varphi_1)\,, \tag{A2}$$

where $\varepsilon_0=1$, $\varepsilon_n=2$ $(n\geqq 1)$, and P_n is an unknown factor, but it can be determined by the boundary condition to be

$$P_n = -\frac{J_n'(ka)}{H_n^{(1)\prime}(ka)}\,. \tag{A3}$$

Inserting (A3) into (A2), we obtain

$$\begin{aligned}\sigma &= -\frac{1}{k}\operatorname{Im}\left\{\int_0^{2\pi}\left(\frac{\partial H_z^{\text{inc}}}{\partial\rho}\right)_{\rho=a} H_z^*(a,\varphi)\,ad\varphi\right\}\\ &= \frac{4}{k}\sum_{n=0}^{\infty}\varepsilon_n\frac{\{J_n'(ka)\}^2}{\{J_n'(ka)\}^2+\{N_n'(ka)\}^2}\,,\end{aligned} \tag{A4}$$

which coincides with (21).

Appendix II

The evaluation of P', Q' and R' is easy and P is equal to the right-hand side of (26) except for a numerical factor. Therefore, we perform the evaluation of Q and R only. Now,

$$Q = Q_1 + Q_2\,,$$

with

$$Q_1 = \frac{1}{2}\int_{\pi}^{2\pi}\sin\varphi'\exp(ika\sin\varphi')\,d\varphi'\int_0^{\pi} g(\varphi-\varphi')\exp(-ika\sin\varphi)\,d\varphi\,,$$

$$Q_2 = -\frac{1}{2}\int_\pi^{2\pi} \exp\,(ika \sin \varphi')d\varphi' \int_0^\pi g(\varphi-\varphi') \sin \varphi \exp\,(-ika \sin \varphi)d\varphi\,.$$

Changing the integration variables from φ, φ' to α, β by the relations:

$$\varphi-\varphi'+2\pi = 2\alpha\,, \quad \varphi+\varphi'-2\pi = 2\beta\,,$$

we have

$$Q_1 = -4\int_0^{\frac{\pi}{2}} d\alpha\, g(2\alpha) \int_0^\alpha d\beta \sin\,(\alpha-\beta) \exp\,(-2ika \sin \alpha \cos \beta)\,.$$

Carrying out the integration with respect to β for $ka \gg 1$, we obtain

$$Q_1 = -2\left(\frac{i\pi}{ka}\right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} d\alpha\, g(2\alpha)(\sin \alpha)^{\frac{1}{2}} \exp\,(-2ika \sin \alpha)\,.$$

If we replace the lower limit of the above integral by the constant α_0 which statisfies $2ka \sin \alpha_0 \gg 1$, then

$$Q_1 = \frac{ik}{4a}\left(\frac{\pi}{2}-\alpha_0+\frac{1}{2}\sin 2\alpha_0\right)\,.$$

Further, we can easily show that $Q_2 = Q_1$ and thus we obtain finally the following expression for Q:

$$Q = \frac{ik}{2a}\left(\frac{\pi}{2}-\alpha_0+\frac{1}{2}\sin 2\alpha_0\right)\,.$$

Similarly we can evaluate R.

REFERENCES

1. C. H. PAPAS, Diffraction by a cylindrical obstacle. J. Appl. Phys. **21** (1950), 314.
2. H. LEVINE and J. SCHWINGER, On the theory of diffraction by an aperture in an infinite plane screen, I. Phys. Rev. **74** (1943), 953.
3. W. v. IGNATOWSKY, Reflexion elektromagnetischer Wellen an einem Draht. Ann. d. Phys. **18** (1905), 495.