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TRANSIENT PHENOMENA IN THE WAVE GUIDE*

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ABSTRACT

Transient phenomena in the wave guide are discussed in the present paper. To this end, we deal with the case in which various modes of electromagnetic waves are excited in the wave guide by an electric or a magnetic dipole lying along the guide axis, assuming that for $t<0$, neither an electric nor a magnetic dipole exists and at $t=0$, either an electric or a magnetic dipole appears, giving rise to the excitation of the electromagnetic waves in the guide.

1. Introduction

With the development of microwave technique, the wave guide, one of the main components, has been discussed more in detail. The discussions, however, seem to have been confined so far to the steady-state problems, without being extended to the transient phenomena. In this paper, the authors try to treat the transient phenomena. We consider the case in which various modes of electromagnetic waves are excited in the wave guide by an electric or a magnetic dipole oriented along the guide axis on the following assumptions: For $t<0$, neither an electric nor a magnetic dipole exists, and so no field quantities exist. At $t=0$, either an electric or a magnetic dipole appears and so the electromagnetic waves are excited in the guide. On these assumptions, we derive the Hertzian vectors of excited waves.

The electric and magnetic fields can be derived by differentiating the Hertzian vectors with respect to the time and spatial coordinates.

2. Laplace transforms of the Maxwell equations

The Maxwell equations in the isotropic homogeneous non-dispersive medium are expressed in M. K. S. units as follows:

$$\nabla \times \boldsymbol{E}(\vec{r},\ t)+\mu \frac{\partial \boldsymbol{H}(\vec{r},\ t)}{\partial t}=0,$$

$$\nabla \times \boldsymbol{H}(\vec{r},\ t)-\varepsilon \frac{\partial \boldsymbol{E}(\vec{r},\ t)}{\partial t}=\boldsymbol{J}(\vec{r},\ t), \qquad (2.1)$$

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$$\nabla\cdot\varepsilon\boldsymbol{E}(\vec{r},\ t)=\rho(\vec{r},\ t)\,,$$
$$\nabla\cdot\mu\boldsymbol{H}(\vec{r},\ t)=0\,.$$

Multiplying each of these equations by e^{-pt} and integrating from 0 to ∞ with respect to t, namely, by operating Laplace transform, we obtain, by using the assumptions made in §1:

$$\begin{aligned}
&\nabla\times\boldsymbol{E}(\vec{r},\ p)+\mu p\boldsymbol{H}(\vec{r},\ p)=0\,,\\
&\nabla\times\boldsymbol{H}(\vec{r},\ p)-\varepsilon p\boldsymbol{E}(\vec{r},\ p)=\boldsymbol{J}(\vec{r},\ p)\,,\\
&\nabla\cdot\varepsilon\boldsymbol{E}(\vec{r},\ p)=\rho(\vec{r},\ p)\,,\\
&\nabla\cdot\mu\boldsymbol{H}(\vec{r},\ p)=0\,,
\end{aligned}\tag{2.2}$$

where $\boldsymbol{E}(\vec{r},p)$, $\boldsymbol{H}(\vec{r},p)$, $\boldsymbol{J}(\vec{r},p)$ and $\rho(\vec{r},p)$ are Laplace transforms of $\boldsymbol{E}(\vec{r},t)$, $\boldsymbol{H}(\vec{r},t)$, $\boldsymbol{J}(\vec{r},t)$ and $\rho(\vec{r},t)$ and similar symbols will be used in the following.

When Eqs. (2.1) represent harmonic waves, that is, when the time dependence is expressed by $e^{i\omega t}$, the forms of (2.2) coincide with those which (2.1) will take, when $i\omega$ is replaced by p and the time factor e^{pt} is omitted. Hence, in order to solve Eqs. (2.2), it is sufficient to replace $i\omega$ in the steady-state solution by p.

Now, as we wish to consider the electromagnetic fields in the wave guide, the field quantities $\boldsymbol{E}(\vec{r},t)$, $\boldsymbol{H}(\vec{r},t)$ are derived from the combination of the z-components of the electric Hertzian vector and the magnetic Hertzian vector: $\Phi(\vec{r},t)$ and $\Psi(\vec{r},t)$. They are:

$$\begin{aligned}
E_x(\vec{r},\ t)&=\frac{\partial^2\Phi(\vec{r},\ t)}{\partial x\partial z}-\mu\frac{\partial^2\Psi(\vec{r},\ t)}{\partial t\partial y}\,,\\
E_y(\vec{r},\ t)&=\frac{\partial^2\Phi(\vec{r},\ t)}{\partial y\partial z}+\mu\frac{\partial^2\Psi(r,\ t)}{\partial t\partial x}\,,\\
E_z(\vec{r},\ t)&=\frac{\partial^2\Phi(\vec{r},\ t)}{\partial z^2}-\varepsilon\mu\frac{\partial^2\Phi(\vec{r},\ t)}{\partial t^2}\,,\\
H_x(\vec{r},\ t)&=\frac{\partial^2\Psi(\vec{r},\ t)}{\partial x\partial z}+\varepsilon\frac{\partial^2\Phi(\vec{r},\ t)}{\partial t\partial y}\,,\\
H_y(\vec{r},\ t)&=\frac{\partial^2\Psi(\vec{r},\ t)}{\partial y\partial z}-\varepsilon\frac{\partial^2\Phi(\vec{r},\ t)}{\partial t\partial x}\,,\\
H_z(\vec{r},\ t)&=\frac{\partial^2\Psi(\vec{r},\ t)}{\partial z^2}-\varepsilon\mu\frac{\partial^2\Psi(\vec{r},\ t)}{\partial t^2}\,.
\end{aligned}\tag{2.3}$$

If we take their Laplace transforms, we obtain, using the assumptions in §1,

$$\begin{aligned}
E_x(\vec{r},\ p)&=\frac{\partial^2\Phi(\vec{r},\ p)}{\partial x\partial z}-p\mu\frac{\partial\Psi(\vec{r},\ p)}{\partial y}\,,\\
E_y(\vec{r},\ p)&=\frac{\partial^2\Phi(\vec{r},\ p)}{\partial y\partial z}+p\mu\frac{\partial\Psi(\vec{r},\ p)}{\partial x}\,,\\
E_z(\vec{r},\ p)&=\frac{\partial^2\Phi(\vec{r},\ p)}{\partial z^2}-p^2\varepsilon\mu\Phi(\vec{r},\ p)\,,
\end{aligned}\tag{2.4}$$

$$H_x(\vec{r}, p)=\frac{\partial^2\Psi(\vec{r}, p)}{\partial x\partial z}+p\varepsilon\frac{\partial\Phi(\vec{r}, p)}{\partial y},$$
$$H_y(\vec{r}, p)=\frac{\partial^2\Psi(\vec{r}, p)}{\partial y\partial z}-p\varepsilon\frac{\partial\Phi(r, p)}{\partial x},$$
$$H_z(\vec{r}, p)=\frac{\partial^2\Psi(\vec{r}, p)}{\partial z^2}-p^2\varepsilon\mu\Psi(\vec{r}, p),$$

and also

$$\left(\Delta-\varepsilon\mu\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\Psi(\vec{r}, t)\\ \Phi(\vec{r}, t)\end{Bmatrix}=\begin{cases}-\dfrac{1}{\mu}M_z(t)\delta(\vec{r}-\vec{r}_0)\\ -\dfrac{1}{\varepsilon}P_z(t)\delta(\vec{r}-\vec{r}_0),\end{cases} \tag{2.5}$$

where P_z and M_z are respectively the z-components of the electric and magnetic diople moments of the exciting system located at $\vec{r}=\vec{r}_0$.

Taking the Laplace transforms of (2.5) and using the assumptions in §1, we derive

$$(\Delta-\varepsilon\mu p^2)\begin{Bmatrix}\Psi(\vec{r}, p)\\ \Phi(\vec{r}, p)\end{Bmatrix}=\begin{cases}-\dfrac{1}{\mu}M_z(p)\delta(\vec{r}-\vec{r}_0)\\ -\dfrac{1}{\varepsilon}P_z(p)\delta(\vec{r}-\vec{r}_0).\end{cases} \tag{2.6}$$

According to (2.4) and (2.6), the Laplace transforms of electromagnetic fields in the guide are identified with the steady-state solutions in which $i\omega$ is replaced by p and the time dependent factor e^{pt} is omitted. This is statement made at the beginning of this section.

3. Laplace transforms of Hertzian vectors in the wave guide

As is well known, the electromagnetic waves in the guide that are excited by an electric or a magnetic dipole oriented in an arbitrary direction, can be expressed by a linear combination of z-components of electric and magnetic Hertzian vectors $\Phi(\vec{r}, t)$ and $\Psi(\vec{r}, t)$, where the z-axis is the guide axis. But we consider only the case in which an electric or a magnetic dipole is oriented along the z-axis. Other cases can be treated in the same manner. The symbols to be used in the following are summarized below.

$I(t)$: strength of electric current forming an electric or a magnetic dipole, where the latter is considered as the elementary circulating current,
$I(p)$: Laplace transform of $I(t)$,
$\delta\zeta$: length of an electric dipole,
δF : area of the elementary circulating current,
$\vec{r}_0(x_0, y_0, 0)$: position of dipole,

$\psi_n(x, y)$: a function defined by $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_n^2\right)\psi_n(x, y)=0$ subject to the boundary condition $\partial\psi_n/\partial n=0$ at the wall of the wave guide, where k_n^2 is the nth eigenvalue of $\psi_n(x, y)$ and $k_n^2 \leqq k_{n+1}^2$,

$\varphi_n(x, y)$: a function defined by $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_n'^2\right)\varphi_n(x, y)=0$ subject to the boundary condition $\varphi_n(x, y)=0$ at the wall, where $k_n'^2$ is the nth eigenvalue of $\varphi_n(x, y)$ and $k_n'^2 \leqq k_{n+1}'^2$,

$$\gamma_n = (k_n^2+p^2\varepsilon\mu)^{1/2} \qquad \left(0 \leqq \arg\gamma_n \leqq \frac{\pi}{2}\right),$$

$$\gamma_n' = (k_n'^2+p^2\varepsilon\mu)^{1/2} \qquad \left(0 \leqq \arg\gamma_n' \leqq \frac{\pi}{2}\right),$$

$$J(\psi_n) = \iint \psi_n^2(x, y)dxdy\,,$$

$$J(\varphi_n) = \iint \varphi_n^2(x, y)dxdy\,.$$

According to the statement made at the beginning of §2, the Laplace transform of an electric or a magnetic dipole moment can be expressed as follows:

The Laplace transform of a magnetic dipole moment is $\mu I(p)\delta F$ and that of an electric dipole moment $I(p)\delta\zeta/p$. Now, when an electric or a magnetic dipole is oriented along the z-axis, we have the following expressions which are obtained by putting p in place of $i\omega$ in the usual steady-state solutions (**1**):

(i) For a magnetic dipole parallel to the z-axis,

$$\Psi^{mz}(\vec{r}, p)=\frac{I(p)\delta F}{2}\sum_n\frac{\psi_n(x_0, y_0)\psi_n(x, y)}{\gamma_n J(\psi_n)}e^{\mp\gamma_n z}. \tag{3.1}$$

The electromagnetic field can be derived from the Laplace transform of the z-component of this magnetic Hertzian vector.

(ii) For an electric dipole parallel to the z-axis,

$$\Phi^{ez}(\vec{r}, p)=\frac{I(p)\delta\zeta}{2\varepsilon}\sum_n\frac{\varphi_n(x_0, y_0)\varphi_n(x, y)}{p\gamma_n' J(\varphi_n)}e^{\mp\gamma_n' z}. \tag{3.2}$$

The double signs in the above two expressions are to be taken in the same order. The minus sign in the exponential factor corresponds to the waves travelling in the direction of z increasing, while the plus sign corresponds to the waves travelling in the direction of z decreasing, when p is replaced by $i\omega$. In the following, we consider only the case in which the exponential factors are $\exp(-\gamma_n z)$ and $\exp(-\gamma_n' z)$, $(z>0)$. The case in which the

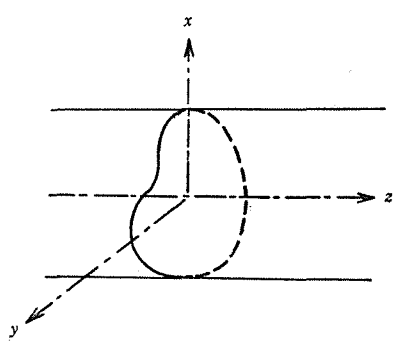

Fig. 1.

exponential factors are $\exp\begin{Bmatrix}\gamma_n z\\ \gamma_n' z\end{Bmatrix}$ $(z<0)$ can be treated analogously. Operating inverse transform to these expressions, we can obtain the corresponding $\Psi(\vec{r}, t)$ and $\Phi(\vec{r}, t)$.

4. Case 1 when the impressed current $I_0 e^{i\omega t}$ begins to be impressed at $t=0$.

In this case, we take

$$I(t) = \begin{cases} 0 & \text{when } t<0\,, \\ I_0 e^{i\omega t} & \text{when } t\geqq 0\,, \end{cases} \tag{4.1}$$

where I_0 represents a complex amplitude of current. By taking the Laplace transform of (4.1), we obtain

$$I(p) = \frac{I_0}{p-i\omega}\,.$$

Substituting this into (3.1) and (3.2) and taking their inverse transforms, we obtain $\Psi(\vec{r}, t)$ and $\Phi(\vec{r}, t)$ for the present case.

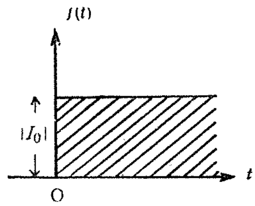

Fig. 2.

In the case (ii) in §3 we have (2), for $z>0$,

$$\begin{aligned}\frac{e^{-\gamma_n z}}{\gamma_n} &= \frac{e^{-z(k_n^2+p^2\varepsilon\mu)^{1/2}}}{(k_n^2+p^2\varepsilon\mu)^{1/2}} = \int_0^\infty J_0(k_n\beta)\frac{\exp\left(-p\sqrt{\varepsilon\mu}\sqrt{\beta^2+z^2}\right)}{(\beta^2+z^2)^{1/2}}\beta d\beta \\ &= a\int_{z/a}^\infty J_0(k_n\sqrt{a^2\tau^2-z^2})e^{-p\tau}d\tau \\ &= a\phi_n(z,p)\,,\end{aligned} \tag{4.3}$$

where we have put

$$a = 1/\sqrt{\varepsilon\mu}\,, \qquad \sqrt{\beta^2+z^2} = \tau/\sqrt{\varepsilon\mu} = a\tau\,,$$

and $\phi_n(z, p)$ is the Laplace transform of $\phi_n(z, t)$ defined by:

$$\phi_n(z, t) = \begin{cases} 0 & \text{when } t<z/a\,, \\ J_0(k_n\sqrt{a^2t^2-z^2}) & \text{when } t\geqq z/a\,. \end{cases} \tag{4.4}$$

Thus, using this formula, we obtain the expression for $\Psi^{mz}(\vec{r}, t)$ in the form:

$$\Psi^{mz}(\vec{r}, t) = \frac{I_0\delta F}{2}\sum_n\frac{\psi_n(x_0, y_0)\psi_n(x, y)}{J(\psi_n)}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{-\gamma_n z}}{\gamma_n}\frac{e^{pt}}{p-i\omega}dp\,, \tag{4.5}$$

where α is the real part of p.

As usual, the above integral can be modified to a contour integral on the complex p plane, the contour being taken to be the combination of a straight line $\alpha-i\infty \to \alpha+i\infty$ and a semicircle of radius ∞ on the right-hand or left-hand side, according as $t<z/a$ or $t>z/a$, as shown in Fig. 3. Let the Laplace transform and its inverse transform be designated by the operators L and L^{-1} respectively. Then we have

$$
\begin{aligned}
I_n &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{-\gamma_n z}}{\gamma_n}\,\frac{I_0 e^{pt}}{p-i\omega}\,dp \\
&= aL^{-1}\{\phi_n(z,\,p)I(p)\} \\
&= a\int_0^t I(t-\tau)\phi_n(z,\,\tau)d\tau \\
&= \begin{cases} 0 & \text{when } t<z/a\,, \\ I_0\displaystyle\int_{z/a}^t e^{i\omega(t-\tau)}J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau & \text{when } t\geqq z/a\,. \end{cases} \qquad (4.6)
\end{aligned}
$$

Thus, we have finally

$$
\Psi^{mz}(\vec{r},\,t)=\frac{I_0\delta F}{2}a\sum_n \frac{\psi_n(x_0,\,y_0)\psi_n(x,\,y)}{J(\psi_n)} \times \begin{cases} 0 & \text{when } t<z/a, \\ \displaystyle\int_{z/a}^t e^{i\omega(t-\tau)}J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau & \text{when } t\geqq z/a. \end{cases} \qquad (4.7)
$$

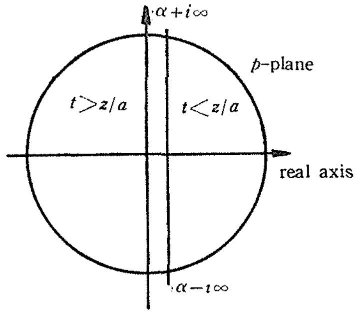

Fig. 3.

The above expression shows clearly that the electromagnetic waves propagate with speed $a=1/\sqrt{\varepsilon\mu}$ and that the phase velocity in the guide which appears in the steady-state solutions, is not the true propagation velocity.

The case (ii) in §3 can be treated in a similar manner. Only the final result is given here.

$$
\begin{aligned}
\Phi^{ez}(\vec{r},\,t) &= \frac{I_0\delta\zeta}{2\varepsilon}\sum_n \frac{\varphi_n(x_0,\,y_0)\varphi_n(x,\,y)}{J(\varphi_n)}\,\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{-\gamma_n' z}}{\gamma_n'}\,\frac{e^{pt}}{p(p-i\omega)}\,dp \\
&= \frac{I_0\delta\zeta}{2\varepsilon}\,a\sum_n \frac{\varphi_n(x_0,\,y_0)\varphi_n(x,\,y)}{J(\varphi_n)} \\
&\quad\times\begin{cases} 0 & \text{when } t<z/a\,, \\ \displaystyle\int_{z/a}^t \frac{2}{\omega}\sin\frac{\omega}{2}(t-\tau)e^{i\omega(t-\tau)/2}J_0(k_n'\sqrt{a^2\tau^2-z^2})d\tau & \text{when } t\geqq z/a\,. \end{cases} \qquad (4.8)
\end{aligned}
$$

Next, we shall discuss the behaviours of these expressions at $t\to\infty$, fixing z at a finite value. We here consider only the case (i) in §3, since the other cases can be discussed similarly. Thus, by (4.7) we obtain

$$
\Psi^{mz}(\vec{r},\,t)=\frac{I_0\delta F}{2}\,a\sum_n \frac{\psi_n(x_0,\,y_0)\,\psi_n(x,\,y)}{J(\psi_n)}\,e^{i\omega t}\int_{z/a}^t e^{-i\omega\tau}J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau\,,
$$

where obviously $t>z/a$, since $t\to\infty$. Hence, using (4.3), we obtain

$$
\begin{aligned}
\Psi^{mz}(\vec{r},\,t) &= \frac{I_0\delta F}{2}\,a\sum_n \frac{\psi_n(x_0,\,y_0)\psi_n(x,\,y)}{J(\psi_n)} \\
&\quad\times\lim_{t\to\infty} e^{i\omega t}\int_{z/a}^{\infty} e^{-i\omega\tau}J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau \\
&= \frac{I_0\delta F}{2}\sum_n \frac{\psi_n(x_0,\,y_0)\psi_n(x,\,y)}{J(\psi_n)}\,\frac{e^{-\gamma_n z}}{\gamma_n}\lim_{t\to\infty} e^{i\omega t}\,.
\end{aligned}
$$

This is the steady-state solution, showing that (4. 7) approaches the steady state as time increases.

5. Case 2 when the current I_0 begins to be impressed at $t=0$.

In case when $I(t)$ is a unit function, the expressions (4. 7) and (4. 8) in the preceding section are simplified. Thus, taking into account that when

$$\begin{aligned} I(t) &= 0 \qquad t<0 \\ &= 1 \qquad t\geqq 0\,, \end{aligned} \tag{5. 1}$$

its Laplace transform is

$$I(p)=\frac{1}{p}\,, \tag{5. 2}$$

we obtain the expression for this case by putting $\omega=0$, $I_0=1$ in the expressions in §4. Thus, for the case (i) in §3,

$$\begin{aligned} \Psi^{mz'}(\vec{r},t) &= \frac{\delta F}{2}a\sum_n \frac{\psi_n(x_0,y_0)\psi_n(x,y)}{J(\psi_n)} \\ &\times \begin{cases} 0 & \text{when } t<z/a\,, \\ \displaystyle\int_{z/a}^{t} J_0(k_n\sqrt{a^2\tau^2-z^2})\,d\tau & \text{when } t\geqq z/a\,, \end{cases} \end{aligned} \tag{5. 3}$$

and for the case (ii) in §3,

$$\begin{aligned} \Phi^{ez'}(\vec{r},t) &= \frac{\delta\zeta}{2\varepsilon}\sum_n \frac{\varphi_n(x_0,y_0)\varphi_n(x,y)}{J(\varphi_n)} \\ &\times \begin{cases} 0 & \text{when } t<z/a\,, \\ \displaystyle\int_{z/a}^{t} (t-\tau)J_0(k_n'\sqrt{a^2\tau^2-z^2})\,d\tau & \text{when } t\geqq z/a\,. \end{cases} \end{aligned} \tag{5. 4}$$

6. Case 3 when the impressed currents in cases 1 and 2 end at $t=T$.

In case when the space is excited by a rectangular pulse, the fields can be easily obtained from the above expressions. For example, in the case (i) in §3, the expression (5. 3) is used to obtain

$$\Psi^{mz''}(\vec{r},t)=\Psi^{mz'}(\vec{r},t)-\Psi^{mz'}(\vec{r},t-T)\,,$$

where T is the duration of the pulse.

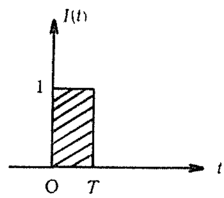

Fig. 4.

Hence,

$$\begin{aligned} \Psi^{mz''}(\vec{r},t) &= \frac{\delta F}{2}a\sum_n \frac{\psi_n(x_0,y_0)\psi_n(x,y)}{J(\psi_n)} \\ &\times \begin{cases} 0 & \text{when } t-T<t<z/a\,, \\ \displaystyle\int_{z/a}^{t} J_0(k_n\sqrt{a^2\tau^2-z^2})\,d\tau & \text{when } t-T<z/a<t\,, \\ \displaystyle\int_{t-T}^{t} J_0(k_n\sqrt{a^2\tau^2-z^2})\,d\tau & \text{when } z/a<t-T<t\,. \end{cases} \end{aligned} \tag{6. 1}$$

The results for the other case can be obtained in a similar procedure.

Next, when the excitation is given by a harmonic wave train of finite duration, the formulae in §4 are available. For example, in the case (i) in §3, the expression (4.7) is used. Thus, if we put $I_0=1$ (say) in (4.7), we obtain

$$\begin{aligned}\Psi^{mz''}(\vec{r}, t) &= \Psi^{mz}(\vec{r}, t) - e^{i\omega T}\Psi^{mz}(\vec{r}, t-T)\\ &= \frac{\delta F}{2} a \sum_n \frac{\psi_n(x_0, y_0)\psi_n(x, y)}{J(\psi_n)}\\ &\times \begin{cases} 0 & \text{when } t-T<t<z/a, \\ \int_{z/a}^{t} e^{i\omega(t-\tau)} J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau & \text{when } t-T<z/a<t, \\ \int_{t-T}^{t} e^{i\omega(t-\tau)} J_0(k_n\sqrt{a^2\tau^2-z^2})d\tau & \text{when } z/a<t-T<t. \end{cases}\end{aligned} \tag{6.2}$$

7. The approximate formulae for the first precursors

While in the above sections we have obtained, by the use of Laplace transforms, the general solutions including the transient solutions, we shall discuss in this section the first precursors of the wave by Sommerfeld's method (**3**). We shall discuss the behaviours of $\Psi(\vec{r}, t)$ and $\Phi(\vec{r}, t)$. In order to obtain the electromagnetic field intensities, it is sufficient to differentiate them with respect to time and spatial coordinates. In this section, only the case considered in §4, namely, the case where the impressed current $I_0 e^{i\omega t}$ begins to be impressed at $t=0$ will be discussed.

(a) For the case (i) in §3, we obtain, starting from (4.5),

$$\Psi^{mz}(\vec{r}, t) = \frac{I_0\delta F}{2}\sum_n \frac{\psi_n(x_0, y_0)\psi_n(x, y)}{J(\psi_n)} \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{-\gamma_n z}}{\gamma_n} \frac{e^{pt}}{p-i\omega} dp, \tag{7.1}$$

and

$$I_n^{mz} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{-\gamma_n z}}{\gamma_n} \frac{e^{pt}}{p-i\omega} dp, \tag{7.2}$$

where, as before, $\gamma_n = (k_n^2 + p^2\varepsilon\mu)^{1/2}$.

Now, if we put $p = i\omega'$, (7.2) becomes

$$I_n^{mz} = \frac{1}{2\pi i}\oint \frac{\exp\{i(\omega' t - \sqrt{\omega'^2\varepsilon\mu - k_n^2}z)\}}{i(\omega'-\omega)(\omega'^2\varepsilon\mu - k_n^2)^{1/2}} d\omega'.$$

The path of integration of the ω'-plane is replaced by a circle of an infinitely large radius with its centre at the origin. Following Sommerfeld this circle is designated by U. As the measure of magnitude of the radius of U, the following may be adopted:

$$|\omega'| \gg \text{the larger of } \omega \text{ and } k_n/\sqrt{\varepsilon\mu}. \tag{A}$$

Hence, when I_n^{mz} is computed, the following approximations are made:

$$\omega' - \omega \approx \omega',$$

$$(\omega'^2\varepsilon\mu-k_n^2)^{1/2}\approx\omega'\sqrt{\varepsilon\mu}\left\{1-\frac{1}{2}\frac{k_n^2}{(\omega'\sqrt{\varepsilon\mu})^2}\right\}.$$

Then we have

$$I_n^{mz}\approx-\frac{1}{2\pi\sqrt{\varepsilon\mu}}\oint\frac{e^{i(\omega't'+\xi_n/\omega')}}{\omega'^2}\,d\omega', \tag{7.3}$$

where

$$t'=t-z\sqrt{\varepsilon\mu},\quad \xi_n=k_n^2z/2\sqrt{\varepsilon\mu}.$$

When we introduce a new integration variable u such that

$$e^{iu}=\omega'\sqrt{\frac{t'}{\xi_n}}, \tag{7.4}$$

we have

$$i(\omega't'+\xi_n/\omega')=i\sqrt{t'\xi_n}\left(\omega'\sqrt{\frac{t'}{\xi_n}}+\frac{1}{\omega'}\sqrt{\frac{\xi_n}{t'}}\right)$$
$$=2i\sqrt{t'\xi_n}\cos u,$$

and therefore

$$I_n^{mz}\approx\frac{1}{2\pi i}\frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{t'}{\xi_n}}\int e^{2i\sqrt{t'\xi_n}\cos u-iu}\,du, \tag{7.5}$$

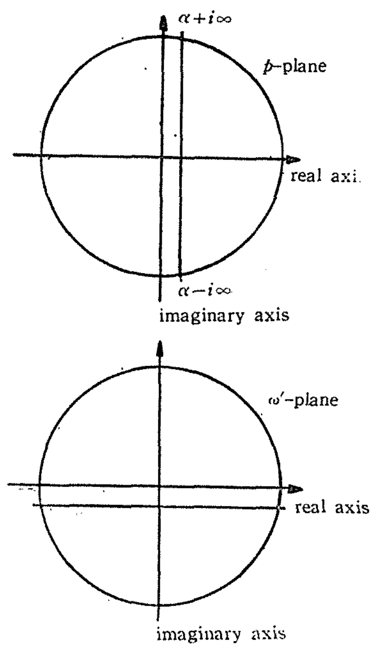

Fig. 5.

the integral being taken from $u=0$ to $u=2\pi$, since, as will be seen from (7.4), when u varies from 0 to 2π, a circle with its centre at the origin is described in the ω'-plane. In case when $t'=t-z\sqrt{\varepsilon\mu}$ is sufficiently small, the radius of this circle satisfies the above-mentioned condition and so the approximation (7.5) is adequate. Therefore, when we take an adequately small t', the approximate expression for I_n^{mz} is

$$I_n^{mz}=\frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{t'}{\xi_n}}J_1(2\sqrt{t'\xi_n}). \tag{7.6}$$

It is to be noted that since there are the condition (4) and the condition $k_n^2\leq k_{n+1}^2$, the accuracy of the approximate formula (7.6) is different for various n with the same t'. So this formula seems to be useful in order to discuss each mode individually.

(b) For the case (ii) in § 3, we have

$$\Phi^{ez}(\vec{r},t)=\frac{I_0\delta\zeta}{2\varepsilon}\sum_n\frac{\varphi_n(x_0,y_0)\varphi_n(x,y)}{J(\varphi_n)}I_n^{ez},$$
$$I_n^{ez}=\frac{1}{2\pi i}\int\frac{e^{-\gamma_n'z}}{\gamma_n'}\frac{e^{pt}}{p(p-i\omega)}\,dp \tag{7.7}$$
$$\approx\frac{1}{\sqrt{\varepsilon\mu}}\frac{t'}{\xi_n'}J_2(2\sqrt{t'\xi_n'}),$$

where

$$t' = t - z\sqrt{\varepsilon\mu}, \qquad \xi'_n = \frac{1}{2}\frac{k_n'^2 z}{\sqrt{\varepsilon\mu}}.$$

From these formulae the behaviours of the first precursors of electromagnetic waves can be investigated by differential operations.

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