Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXVIII, No. 4, Article 3, 1958.

INTRODUCTION OF THE TRANSLATIONAL OPERATOR IN FREQUENCY DOMAIN AND TREATMENT OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS, II.

BY

Tadashi WATANABE

(*Received December 5, 1957*)

ABSTRACT

In this paper, by introducing the translational operator in frequency domain defined in part I, the theory of the linear fixed electrical network is extended to the linear network containing circuit elements which are expressed by E-type functions of time described in part I.

Also the treatment of transmission lines with the above-mentioned network inserted is discussed and especially the extensions of reflection coefficient and transmission coefficient are described.

1. Introduction

As to the linear varying electrical network whose elements might not necessarily be of E-type, only special cases were discussed by several authors, e.g. J. R. Carson (**1**), J. Neufeld (**2**) and L. A. Pipes (**3**).

In this country, M. Akiyama (**4**, **5**, **6**) and Z. Kiyasu (**11**) discussed the theory of the linear periodically varying electrical network.

Recently, L. A. Zadeh (**7**, **8**, **9**, **10**) established the general theory of the linear varying electrical network by introducing the system function. But in order to derive system functions, we generally have to solve higher order linear differential equations with the coefficients which are functions of time. And further it seems generally difficult to know how the impedance and the admittance, i.e. the system functions are constructed by circuit elements.

In the present paper the author deals with principally the linear varying network which includes real circuit elements* varying periodically in time. The consideration can be extended, with a little modification, to the case of the linear varying network whose circuit elements can be expressed by E-type functions**.

2. Linear fixed electrical network

We shall begin with a review of the linear fixed electrical network.

* They are restricted to the functions which can be expanded in Fourier series.

** Reference should be made to part I.

If the degrees of freedom of the network under consideration be N, we can express the circuit equations of the network in the time domain as follows:

$$\left.\begin{aligned} \sum_{k=1}^{N} \bar{Z}_{jk} i_k(t) &= e_j(t)\,, \\ \bar{Z}_{jk} &= DL_{jk} + R_{jk} + S_{jk}\frac{1}{D}\,, \\ D = \frac{d}{dt}\,, \quad \frac{1}{D} &= \int^t\,, \end{aligned}\right\} \tag{2.1}$$

where $i_k(t)$ is the mesh current flowing along the k-th mesh and $e_j(t)$ the algebraic sum of e.m.f. involved in the j-th mesh.

The positive sense of the current $i_k(t)$ is selected uniquely over all meshes, e.g. clockwise, while the positive sense of $e_j(t)$ is taken so as to increase the mesh current $i_j(t)$.

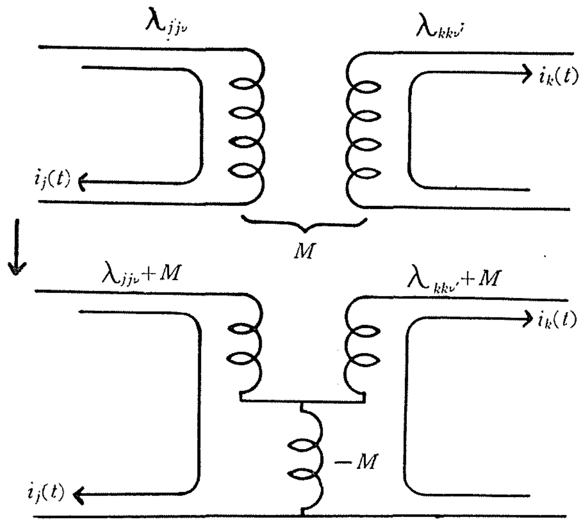


Fig. 1

Next we consider the constructions of L_{jk}, R_{jk} and S_{jk}. When $j \neq k$, $-L_{jk}$ is the sum of all self-inductive coefficients $-\lambda_{jk\kappa}$'s involved in the branch common to the j-th and the k-th meshes. When the j-th and the k-th meshes include coupling by mutual induction, we replace the coupling part by the equivalent circuit as shown in Fig. 1, where M is the mutual inductive coefficient, and then we take the sum described above. Thus:

$$L_{jk} = \sum_{\kappa} \lambda_{jk\kappa}\,. \tag{2.2}$$

Similarly R_{jk} and S_{jk} are obtained by taking sums of all resistances $-\rho_{jk\mu}$'s and all elastic coefficients $-\sigma_{jk\nu}$'s respectively, included in the branch common to the

j-th and the k-th meshes and changing the signs.

Namely:

$$R_{jk} = \sum_{\mu} \rho_{jk\mu} \qquad (j \neq k), \tag{2.3}$$

$$S_{jk} = \sum_{\nu} \sigma_{jk\nu} \qquad (j \neq k). \tag{2.4}$$

When the inductive coefficients, resistances and elastic coefficients which belong to the j-th mesh but not to the other meshes are represented by $\lambda_{jj\kappa}$'s, $\rho_{jj\mu}$'s and $\sigma_{jj\nu}$'s respectively, the expressions L_{jj}, R_{jj} and S_{jj} are given as follows:

$$L_{jj} = \sum_{\kappa} \lambda_{jj\kappa} - \sum_{l}{}' \sum_{\kappa} \lambda_{jl\kappa}, \tag{2.2'}$$

$$R_{jj} = \sum_{\mu} \rho_{jj\mu} - \sum_{l}{}' \sum_{\mu} \rho_{jl\mu}, \tag{2.3'}$$

$$S_{jj} = \sum_{\nu} \sigma_{jj\nu} - \sum_{l}{}' \sum_{\nu} \sigma_{jl\nu}, \tag{2.4'}$$

where the dash in $\sum_{l}{}'$ indicates that terms for which $l=j$ are excluded from the summation. As will easily be seen, we have

$$L_{jk} = L_{kj}, \quad R_{jk} = R_{kj}, \quad S_{jk} = S_{kj},$$

so that we obtain:

$$\bar{Z}_{jk} = \bar{Z}_{kj} \qquad \text{(reciprocity relation)}. \tag{2.5}$$

Also we can express (2.1) in a matrix form:

$$[\bar{Z}][i(t)] = [e(t)], \tag{2.1'}$$

where

$$[e(t)] = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_N(t) \end{bmatrix}, \quad [i(t)] = \begin{bmatrix} i_1(t) \\ i_2(t) \\ \vdots \\ i_N(t) \end{bmatrix}, \quad [\bar{Z}] = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \cdots \bar{Z}_{1N} \\ \bar{Z}_{21} & \bar{Z}_{22} & \cdots \bar{Z}_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \bar{Z}_{N1} & \bar{Z}_{N2} & \cdots \bar{Z}_{NN} \end{bmatrix}.$$

Taking the Laplace transform of (2.1'), we obtain:

$$[Z(p)][I(p)] = [E'(p)], \tag{2.1''}$$

where $[Z(p)]$ is $[\bar{Z}]$ with D replaced by p, $[I(p)]$ the Laplace transform of $[i(t)]$, and $[E'(p)]$ the Laplace transform of $[e(t)]$ added with the terms which include the initial values of currents $[i(t)]$ and charges $[q(t)]$, where $q_j(t) = \int^t i_j(t')\,dt'$.

3. Circuit equations of linear varying network whose elements are expressed by real periodic functions of time*

We assume that $\lambda_{jk\kappa}$, $\rho_{jk\mu}$ and $\sigma_{jk\nu}$ are periodic functions of time with the periods $\omega_{jk\kappa}$, $\omega_{jk\mu}$ and $\omega_{jk\nu}$, respectively:

---

* Read in part before the joint meetings of Kansai branches of three Electrical Societies, Oct., 18, 1952 and Oct., 17, 1953 (**12, 13**).

$$\left.\begin{aligned} \lambda_{jk\kappa} &= \sum_{n=-\infty}^{\infty} \lambda_{jk\kappa}^{(n)} e^{in\omega_{jk\kappa}t}, \\ \rho_{jk\mu} &= \sum_{n=-\infty}^{\infty} \rho_{jk\mu}^{(n)} e^{in\omega_{jk\mu}t}, \\ \sigma_{jk\nu} &= \sum_{n=-\infty}^{\infty} \sigma_{jk\nu}^{(n)} e^{in\omega_{jk\nu}t}. \end{aligned}\right\} \tag{3.1}$$

Since the circuit constants are real, the following relations are obtained:

$$\left.\begin{aligned} \widetilde{\lambda_{jk\kappa}^{(n)}} &= \lambda_{jk\kappa}^{(-n)}, \\ \widetilde{\rho_{jk\mu}^{(n)}} &= \rho_{jk\mu}^{(-n)}, \\ \widetilde{\sigma_{jk\nu}^{(n)}} &= \sigma_{jk\nu}^{(-n)}, \end{aligned}\right\} \tag{3.2}$$

where the quantities with $\sim$ represent the complex conjugates of the corresponding quantities. In this case, the circuit equations in the time domain are the same as (2.1), except that DL_{jk} and $S_{jk}\frac{1}{D}$ should both be kept unchanged in order.

Next, considering that L_{jk}, R_{jk} and S_{jk} are constructed by $\lambda_{jk\kappa}$, $\rho_{jk\mu}$ and $\sigma_{jk\nu}$ as shown in (3.1) respectively*, we take the Laplace transforms of (2.1) in which, however, L_{jk}, R_{jk} and S_{jk} are constructed by linear varying circuit elements given by (3.1).

Then we obtain:

$$\left.\begin{aligned} &\sum_k \sum_n [\sum_\kappa \{p\lambda_{jk\kappa}^{(n)}(I_k(p-in\omega_{jk\kappa})-I_j(p-in\omega_{jk\kappa})\,\delta'_{jk})\} \\ &\quad +\sum_\mu \{\rho_{jk\mu}^{(n)}(I_k(p-in\omega_{jk\mu})-I_j(p-in\omega_{jk\mu})\,\delta'_{jk})\} \\ &\quad +\sum_\nu \{\sigma_{jk\nu}^{(n)}(p-in\omega_{jk\nu})^{-1}(I_k(p-in\omega_{jk\nu})-I_j(p-in\omega_{jk\nu})\,\delta'_{jk})\}] \\ &= E'_j(p), \\ &E'_j(p) = E_j(p)+\sum_k \{L_{jk}(0)\, i_k(0)-p^{-1}S'_{jk}(p)\, q_k(0)\}^{**}, \end{aligned}\right\} \tag{3.3}$$

where

$$\left.\begin{aligned} E_j(p) &= L(e_j(t)), \quad I_j(p) = L(i_j(t)), \\ q_j(t) &= \int^t i_j(t')\,dt', \quad \delta'_{jk} = \begin{cases} 0 & (j=k), \\ 1 & (j\neq k), \end{cases} \\ S'_{jk}(p) &= p\{\sum_\nu \sum_n \sigma_{jk\nu}^{(n)}(p-in\omega_{jk\nu})^{-1}\} \qquad (j\neq k), \\ S'_{jj}(p) &= p\{\sum_\nu \sum_n \sigma_{jj\nu}^{(n)}(p-in\omega_{jj\nu})^{-1}-\sum_l{}' \sum_\nu \sum_n \sigma_{jl\nu}^{(n)}(p-in\omega_{jl\nu})^{-1}\}. \end{aligned}\right\} \tag{3.4}$$

When we can solve $I_k(p)$'s from the fundamental equations (3.3), $i_k(t)$'s can be obtained as inverse Laplace transforms of $I_k(p)$'s.

Further, since it is difficult to make (3.3) in this form correspond to (2.1″), we must introduce the translational operator in frequency domain into (3.3).

---

* The processes of the constructions are the same as in § 2.

** $L_{jk}(0)=L_{jk}(t)|_{t=0}$.

As described in part I, when $f(p)$ is a function of p, the translational operator is defined as follows:

$$T(\omega)\,f(p) = f(p-i\omega)\,. \tag{3.5}$$

Using the operator, we can rewrite (3.3) and (3.4) as follows:

$$\begin{aligned}\sum_k \sum_n [\sum_\kappa \{p\lambda_{jk\kappa}^{(n)} T^n(\omega_{jk\kappa})(I_k(p)-I_j(p)\,\delta'_{jk})\}\\ +\sum_\mu \{\rho_{jk\mu}^{(n)} T^n(\omega_{jk\mu})(I_k(p)-I_j(p)\,\delta'_{jk})\}\\ +\sum_\nu \left\{\sigma_{jk\nu}^{(n)} T^n(\omega_{jk\nu})\frac{1}{p}\,(I_k(p)-I_j(p)\,\delta'_{jk})\right\}\\ = E'_j(p)\,,\end{aligned} \tag{3.6}$$

$$\left.\begin{aligned} S'_{jk}(p) &= p\left\{\sum_\nu \sum_n \sigma_{jk\nu}^{(n)} T^n(\omega_{jk\nu})\frac{1}{p}\right\} \qquad (j \neq k)\,,\\ S'_{jj}(p) &= p\{\sum_\nu \sum_n \sigma_{jj\nu}^{(n)} T^n(\omega_{jj\nu}) - \sum_l{}' \sum_\nu \sum_n \sigma_{jl\nu}^{(n)} T^n(\omega_{jl\nu})\}\,\frac{1}{p}\,.\end{aligned}\right\} \tag{3.7}$$

Now, when $L_{jk}(T, p)$, $R_{jk}(T, p)$ and $S_{jk}(T, p)$ are defined by the following expressions:

$$\left.\begin{aligned} L_{jk}(T,p) &= \{\sum_\kappa \sum_n \lambda_{jk\kappa}^{(n)} T^n(\omega_{jk\kappa})\} - \{\sum_l{}' \sum_\kappa \sum_n \lambda_{jl\kappa}^{(n)} T^n(\omega_{jl\kappa})\}\,\delta_{jk}\,,\\ R_{jk}(T,p) &= \{\sum_\mu \sum_n \rho_{jk\mu}^{(n)} T^n(\omega_{jk\mu})\} - \{\sum_l{}' \sum_\mu \sum_n \rho_{jl\mu}^{(n)} T^n(\omega_{jl\mu})\}\,\delta_{jk}\,,\\ S_{jk}(T,p) &= \{\sum_\nu \sum_n \sigma_{jk\nu}^{(n)} T^n(\omega_{jk\nu})\} - \{\sum_l{}' \sum_\nu \sum_n \sigma_{jl\nu}^{(n)} T^n(\omega_{jl\nu})\}\,\delta_{jk}\,,\end{aligned}\right\} \tag{3.8}$$

the equations (3.6) and (3.7) becomes as follows:

$$\sum_k \left(pL_{jk}(T,p)+R_{jk}(T,p)+S_{jk}(T,p)\,\frac{1}{p}\right) I_k(p) = E'_j(p)\,, \tag{3.6'}$$

$$S'_{jk}(p) = pS_{jk}(T,p)\,\frac{1}{p}\,. \tag{3.7'}$$

Equation (3.6′) is further rewritten:

$$\sum_k Z_{jk}(T,p)\,I_k(p) = E'_j(p)\,, \tag{3.6''}$$

with

$$Z_{jk}(T,p) = pL_{jk}(T,p)+R_{jk}(T,p)+S_{jk}(T,p)\,\frac{1}{p}\,. \tag{3.9}$$

(3.6″) is expressed in a matrix form:

$$[Z(T,p)][I(p)] = [E'(p)]\,, \tag{3.6'''}$$

where

$$[I(p)] = \begin{bmatrix} I_1(p)\\ I_2(p)\\ \vdots\\ I_N(p)\end{bmatrix}, \quad [E'(p)] = \begin{bmatrix} E'_1(p)\\ E'_2(p)\\ \vdots\\ E'_N(p)\end{bmatrix},$$

$$[Z(T, p)] = \begin{bmatrix} Z_{11} & Z_{12} & \cdots\cdots Z_{1N} \\ Z_{21} & Z_{22} & \cdots\cdots Z_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ Z_{N1} & Z_{N2} & \cdots\cdots Z_{NN} \end{bmatrix}^{*}.$$

Representing the inverse matrix of $[Z(T, p)]$ by $[Y(T, p)]$, we obtain:

$$[I(p)] = [Y(T, p)][E'(p)]. \tag{3.10}$$

We will call $[Z(T, p)]$ and $[Y(T, p)]$ generalized impedance operator matrix and generalized admittance operator matrix respectively. From (3.10), we see that it reduces to the discovery of the inverse matrix $[Y(T, p)]$, to obtain $i_k(t)$'s.

We can easily show that Kirchhoff's laws, the principle of superposition, Thévenin-Ho's theorem and the duality of network are valid in the varying network.

Next we derive the nodal equations of the linear varying network by setting the following correspondence:

inductance (L)	$\longrightarrow$	capacitance (C)
resistance (R)	$\longrightarrow$	conductance (G)
elastance (S)	$\longrightarrow$	reciprocal inductance (Γ)
mesh current	$\longrightarrow$	nodal voltage
mesh voltage	$\longrightarrow$	nodal current
mesh	$\longrightarrow$	node.

From the above duality relations, the nodal equations corresponding to (3.6′) are obtained:

$$\sum_k \left(pC_{jk}(T, p) + G_{jk}(T, p) + \Gamma_{jk}(T, p)\frac{1}{p}\right) E_k(p) = I'_j(p), \tag{3.11}$$

where

$$I'_j(p) = I_j(p) + \sum_k \left\{ C_{jk}(0)\, e_k(0) - \Gamma_{jk}(T, p)\, \frac{1}{p}\, \phi_k(0)\right\},^{**}$$

$$\phi_k(t) = \int^t e_k(t')\, dt'.$$

Putting

$$Y_{jk}(T, p) = pC_{jk}(T, p) + G_{jk}(T, p) + \Gamma_{jk}(T, p)\,\frac{1}{p} \tag{3.12}$$

and defining the generalized admittance operator by the following matrix:

$$[Y(T, p)] = \begin{bmatrix} Y_{11} & Y_{12} & \cdots\cdots Y_{1N} \\ Y_{21} & Y_{22} & \cdots\cdots Y_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ Y_{N1} & \cdots & \cdots\cdots Y_{NN} \end{bmatrix}^{***},$$

* The matrix element Z_{lm} is an abbreviation of $Z_{lm}(T, p)$.

** $C_{jk}(0) = C_{jk}(t)|_{t=0}$.

*** Y_{lm} is an abbreviation of $Y_{lm}(T, p)$.

we can rewrite (3.11) as follows:

$$[Y(T, p)][E(p)] = [I'(p)], \tag{3.11$'$}$$

which corresponds to (3.6‴).

We will call (3.6‴) and (3.11′) generalized mesh equations and generalized nodal equations respectively.

From (3.6‴), (3.9), (3.11′) and (3.12), we can see that in our treatment, the circuit equations of linear fixed network are naturally extended to the linear varying network where the elements are expressed by periodic functions of time.

In Kiyasu's theory, impedances are expressed by infinite dimensional matrices and so it seems to the author that the author's method utilizing the translational operator is more convenient.

The procedure described can evidently be extended to the case when $\lambda_{jk\kappa}$, $\rho_{jk\mu}$ and $\sigma_{jk\nu}$ are varying with the following angular frequencies:

$$\left.\begin{array}{lll} \omega_{jk\kappa}^{(1)},\ \omega_{jk\kappa}^{(2)},\ \cdots\cdots\cdots\cdots,\ \omega_{jk\kappa}^{(l)} & \text{for} & \lambda_{jk\kappa}, \\ \omega_{jk\mu}^{(1)},\ \omega_{jk\mu}^{(2)},\ \cdots\cdots\cdots\cdots,\ \omega_{jk\mu}^{(m)} & \text{for} & \rho_{jk\mu}, \\ \omega_{jk\nu}^{(1)},\ \omega_{jk\nu}^{(2)},\ \cdots\cdots\cdots\cdots,\ \omega_{jk\nu}^{(n)} & \text{for} & \sigma_{jk\nu}. \end{array}\right\} \tag{3.13}$$

And when $\omega_{jk\kappa}^{(i)}$, $\omega_{jk\mu}^{(i)}$ and $\omega_{jk\nu}^{(i)}$ are complex quantities, the above process can be adapted by making the following modification. Namely, in order that the circuit elements may be expressed by real functions, (3.1) are modified as follows*:

$$\left.\begin{aligned} \lambda_{jk\kappa} &= \lambda_{jk\kappa}^{(0)} + \sum_{n=1}^{\infty} \lambda_{jk\kappa}^{(n)}\, e^{in\omega_{jk\kappa}t} + \sum_{n=1}^{\infty} \lambda_{jk\kappa}^{(-n)}\, \widetilde{e^{in\omega_{jk\kappa}t}}, \\ \rho_{jk\mu} &= \rho_{jk\mu}^{(0)} + \sum_{n=1}^{\infty} \rho_{jk\mu}^{(n)}\, e^{in\omega_{jk\mu}t} + \sum_{n=1}^{\infty} \rho_{jk\mu}^{(-n)}\, \widetilde{e^{in\omega_{jk\mu}t}}, \\ \sigma_{jk\nu} &= \sigma_{jk\nu}^{(0)} + \sum_{n=1}^{\infty} \sigma_{jk\nu}^{(n)}\, e^{in\omega_{jk\nu}t} + \sum_{n=1}^{\infty} \sigma_{jk\nu}^{(-n)}\, \widetilde{e^{in\omega_{jk\nu}t}}. \end{aligned}\right\} \tag{3.1$'$}$$

The relations (3.2) remain to be valid. From (3.1′), it is evident that (3.8) and (3.7) have only to be modified as follows. Namely, while the translational operators behind the factors, $\lambda_{jk\kappa}^{(n)}$, $\rho_{jk\mu}^{(n)}$ and $\sigma_{jk\nu}^{(n)}$ $(n>0)$ are left unchanged, the operators behind $\lambda_{jk\kappa}^{(-n)}$, $\rho_{jk\mu}^{(-n)}$ and $\sigma_{jk\nu}^{(-n)}$ $(n>0)$ are replaced by the following operators:

$$T(\widetilde{\omega_{jk\kappa}}),\quad T(\widetilde{\omega_{jk\mu}}),\quad T(\widetilde{\omega_{jk\nu}})$$

respectively, with the power indices of the translational operator remaining unchanged.

Similar extensions are possible for the nodal equations.

---

* Here is considered the case when $l=m=n=1$ in (3.13) and $\omega_{jk\kappa}^{(1)}$, $\omega_{jk\mu}^{(1)}$ and $\omega_{jk\nu}^{(1)}$ are represented by $\omega_{jk\kappa}$, $\omega_{jk\mu}$ and $\omega_{jk\nu}$ respectively. It is easy to extend the discussion to the case when l, m and n are all greater than unity.

4. Linear varying network whose circuit elements vary periodically with one common period

The case shown in the title corresponds to conditions that

$$T(\omega_{jk\kappa}) = T(\omega_{j'k'\mu}) = T(\omega_{j''k''\nu}) \tag{4.1}$$

for all (j, k, κ), (j', k', μ) and (j'', k'', ν).

Now putting

$$\omega_{jk\kappa} = \omega_{j'k'\mu} = \omega_{j''k''\nu} = \omega_0\,,$$

we obtain:

$$[Z(T, p)] = \sum_{n=-\infty}^{\infty} [Z^{(n)}(p)]\, T^n(\omega_0)\,. \tag{4.2}$$

$[Z^{(n)}(p)]$ is given as follows:

$$[Z^{(n)}(p)] = \begin{pmatrix} Z_{11}^{(n)}(p) & Z_{12}^{(n)}(p) \cdots Z_{1N}^{(n)}(p) \\ Z_{21}^{(n)}(p) & Z_{22}^{(n)}(p) \cdots Z_{2N}^{(n)}(p) \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ Z_{N1}^{(n)}(p) & Z_{N2}^{(n)}(p) \cdots Z_{NN}^{(n)}(p) \end{pmatrix}, \tag{4.3}$$

where

$$\left.\begin{aligned} Z_{jk}^{(n)}(p) &= p\lambda_{jk}^{(n)} + \rho_{jk}^{(n)} + p^{-1}\sigma_{jk}^{(n)\prime}\,, \\ p\lambda_{jk}^{(n)} &= p\{\sum_{\kappa} \lambda_{jk\kappa}^{(n)} - (\sum_{l}{}' \sum_{\kappa} \lambda_{jl\kappa})\, \delta_{jk}\}\,, \\ \rho_{jk}^{(n)} &= \sum_{\mu} \rho_{jk\mu}^{(n)} - (\sum_{l}{}' \sum_{\mu} \rho_{jl\mu}^{(n)})\, \delta_{jk}\,, \\ p^{-1}\sigma_{jk}^{(n)\prime} &= (p - in\omega_0)^{-1}\{\sum_{\nu} \sigma_{jk\nu}^{(n)} - (\sum_{l}{}' \sum_{\nu} \sigma_{jl\nu}^{(n)})\, \delta_{jk}\}\,. \end{aligned}\right\} \tag{4.4}$$

We can also write (4.2) as follows:

$$\left.\begin{aligned} [Z(T, p)] &= \sum_{n=-\infty}^{\infty} T^n(\omega_0)[Z^{(n)\prime}(p)]\,, \\ Z_{jk}^{(n)\prime}(p) &= Z_{jk}^{(n)}(p + in\omega_0)\,. \end{aligned}\right\} \tag{4.5}$$

In the treatment described above is included the case of the network of which only one circuit element is varying and the others are fixed. According as the varying element is an inductor or a capacitor, we obtain, using (4.2) or (4.5), the circuit equation which does not include ω_0 explicitly excepting in $T(\omega_0)$.

This is often convenient for application, e.g. when the method of series expansion is used. If the only varying element is a resistor, the same circuit equation is obtained by use of either expression.

5. Transmission lines with linear periodically varying network inserted*

The following discussion still holds when the linear periodically varying network is replaced by the E-type circuit element.

* Read in part before the joint meeting of Kansai branches of three Electrical Societies, Oct. 17, 1954 (**14**).

We consider how the reflection coefficient and the transmission coefficient are generalized, in case when the linear periodically varying network is inserted in the transmission lines such as Lecher lines, coaxial cable, wave guide, etc.

The transmission lines are assumed to be fixed. Since neither charge nor current exists in the inserted network before the arrival of the incident waves, the following are substituted into (3. 6‴):

$$\left.\begin{aligned} E_j'(p) &= E_j(p) \qquad (j=1,2)\,, \\ E_j'(p) &= 0 \qquad\quad\;\; (j \neq 1,2)\,, \end{aligned}\right\}$$

and eliminating $I_3(p)$, $I_4(p)$,$\cdots I_N(p)$ from the resulting equations, we can easily obtain:

$$\begin{bmatrix} E_1(p) \\ E_2(p) \end{bmatrix} = \begin{bmatrix} \bar{Z}_{11}(T,p) & \bar{Z}_{12}(T,p) \\ \bar{Z}_{21}(T,p) & \bar{Z}_{22}(T,p) \end{bmatrix} \begin{bmatrix} I_1(p) \\ I_2(p) \end{bmatrix}. \tag{5. 1}$$

When, as shown in Fig. 2, the semi-infinite transmission lines (I) are connected to the terminals (1, 1′) of the inserted network and the load $Z_L(p)$ to the terminals (2, 2′), we will derive the reflection coefficient in the transmission lines (I).

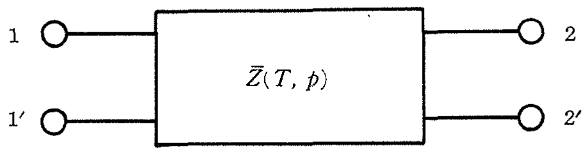


Fig. 2

Now we take the x_1-axis along the transmission lines (I), with the positive sense directed toward terminals (1, 1′). We represent the current and the voltage at the point x_1 (in the frequency domain) by $I(x_1, p)$ and $E(x_1, p)$ respectively and assume that the incident waves have the time factor $e^{i\omega t}$ in the time-domain representation. Then, in the case of constant current source the incident current and voltage are given by:

$$\left.\begin{aligned} I^{(i)}(x_1, p) &= e^{-\gamma_1(p)x_1}\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}\,, \\ E^{(i)}(x_1, p) &= Z_0^{(1)}(p)\, e^{-\gamma_1(p)x_1}\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}\,, \end{aligned}\right\} \tag{5. 2}$$

where $\gamma_1(p)$ and $Z_0^{(1)}(p)$ are the propagation constant and the surge impedance of the transmission lines (I) in the frequency domain respectively. These expressions mean that the incident waves arrive at the inserted network at $t=t_0$, if $t_0>0$.

Next, the reflected current and voltage can be given by:

$$\left.\begin{aligned} I^{(r)}(x_1, p) &= e^{\gamma_1(p)x_1}A(T,p)\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}\,, \\ E^{(r)}(x_1, p) &= -Z_0^{(1)}(p)\, e^{\gamma_1(p)x_1}A(T,p)\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}\,. \end{aligned}\right\} \tag{5. 3}$$

Applying the continuity conditions of the current and voltage at $x_1=0$, i.e. at the terminals $(1, 1')$, to (5. 2) and (5. 3), we obtain:

$$\left.\begin{aligned} A(T, p) &= \{Z_0^{(1)}(p)+\bar{Z}(T, p)\}^{-1}\{Z_0^{(1)}(p)-\bar{Z}(T, p)\}, \\ \text{where} \quad & \\ \bar{Z}(T, p) &= \bar{Z}_{11}(T, p)-\bar{Z}_{12}(T, p)\{Z_L(p)+\bar{Z}_{22}(T, p)\}^{-1}\bar{Z}_{21}(T, p). \end{aligned}\right\} \tag{5. 4}$$

As $Z_L(p)$, the semi-infinite transmission lines (II) are connected to the terminals $(2, 2')$ and we take the x_2-axis along these lines, with the positive sense away from the terminals $(2, 2')$ and the origin set at the terminals $(2, 2')$.

Representing the propagation constant and the surge impedance of the transmission lines (II) in the frequency domain by $\gamma_2(p)$ and $Z_0^{(2)}(p)$ respectively, we consider the transmission coefficient in the transmission lines (II). Now the transmitted current and voltage are assumed to be given by:

$$\left.\begin{aligned} I^{(t)}(x_2, p) &= e^{-\gamma_2(p)x_2}I_2(p), \\ E^{(t)}(x_2, p) &= Z_0^{(2)}(p)\, e^{-\gamma_2(p)x_2}I_2(p). \end{aligned}\right\} \tag{5. 5}$$

The following expressions are substituted into (5. 1):

$$\begin{aligned} I_1(p) &= \{1+A(T, p)\}\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}, \\ E_1(p) &= Z_0^{(1)}(p)\{1-A(T, p)\}\{I_0\, e^{-pt_0}(p-i\omega)^{-1}\}, \\ E_2(p) &= -Z_0^{(2)}(p)\, I_2(p), \end{aligned}$$

where $A(T, p)$ means $A(T, p)$ which is obtained by putting

$$Z_L(p) = Z_0^{(2)}(p)$$

in (5. 4).

Then

$$I_2(p) = B(T, p)\, I_0\{e^{-pt_0}(p-i\omega)^{-1}\},$$

where

$$B(T, p) = -\{Z_0^{(2)}(p)+\bar{Z}_{22}(T, p)\}^{-1}\bar{Z}_{21}(T, p)\{1+A(T, p)\}. \tag{5. 6}$$

Substituting (5. 6) into (5. 5), we obtain:

$$\left.\begin{aligned} I^{(t)}(x_2, p) &= e^{-\gamma_2(p)x_2}B(T, p)\, I_0\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ E^{(t)}(x_2, p) &= Z_0^{(2)}(p)\, e^{-\gamma_2(p)x_2}B(T, p)\, I_0\{e^{-pt_0}(p-i\omega)^{-1}\}. \end{aligned}\right\} \tag{5. 7}$$

We will call $A(T, p)$ and $B(T, p)$ defined by (5. 4) and (5. 6) respectively the generalized reflection coefficient operator and transmission coefficient operator for the constant current source.

For the constant voltage source, we can also define the generalized reflection coefficient operator and transmission coefficient operator which are represented by $A'(T, p)$ and $B'(T, p)$ respectively, related with $A(T, p)$ and $B(T, p)$ such that:

$$\left.\begin{aligned} A'(T, p) &= Z_0^{(1)}(p)\, A(T, p)\, Z_0^{(1)}(p)^{-1}, \\ B'(T, p) &= Z_0^{(2)}(p)\, B(T, p)\, Z_0^{(2)}(p)^{-1}. \end{aligned}\right\} \tag{5.8}$$

When $A'(T, p)$ and $B'(T, p)$ are used, the expressions of the current and voltage in the frequency domain for the constant voltage source are given by the following: On the side of the terminals $(1, 1')$;

$$\left.\begin{aligned} I_1(x_1, p) &= Z_0^{(1)}(p)^{-1}\{e^{-\gamma_1(p)x_1}+e^{\gamma_1(p)x_1}A'(T, p)\}\, E_0\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ E_1(x_1, p) &= \{e^{-\gamma_1(p)x_1}-e^{\gamma_1(p)x_1}A'(T, p)\}E_0\{e^{-pt_0}(p-i\omega)^{-1}\}, \end{aligned}\right\} \tag{5.9}$$

and on the side of the terminals $(2, 2')$;

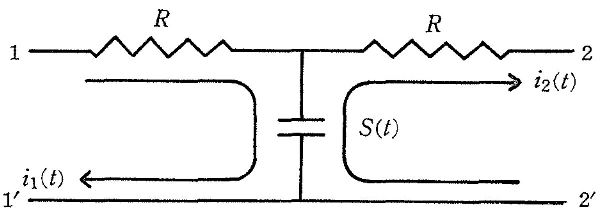
$$\left.\begin{aligned} I_2(x_2, p) &= Z_0^{(2)}(p)^{-1}\, e^{-\gamma_2(p)x_2}B'(T, p)\, E_0\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ E_2(x_2, p) &= e^{-\gamma_2(p)x_2}B'(T, p)\, E_0\{e^{-pt_0}(p-i\omega)^{-1}\}. \end{aligned}\right\} \tag{5.10}$$

As seen from (5.8), when the transmission lines (I) and (II) are non-dispersive i.e. $Z^{(1)}(p)$ and $Z^{(2)}(p)$ do not include p, $A'(T, p)$ and $B'(T, p)$ coincide with $A(T, p)$ and $B(T, p)$ respectively.

The author is sure that such a problem as has been described in this section, cannot be easily treated by other methods.

6. Some examples

(i) As an example of the linear varying network described in §4, we will consider the circuit shown in Fig. 3, which acts as a varying low pass filter.



$S(t)=\sigma^{(0)}+2\sigma^{(1)}\cos(\omega_0 t)$

Fig. 3

The circuit equations are given as follows:

$$\begin{bmatrix} E_1(p) \\ E_2(p) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1(p) \\ I_2(p) \end{bmatrix}, \tag{6.1}$$

where

$$q_1(0) = q_2(0) = 0,$$

and

$$\left.\begin{aligned} Z_{11}(T, p) &= Z_{22}(T, p) = R+\{\sigma^{(0)}+\sigma^{(1)}(T(\omega_0)+T^{-1}(\omega_0))\}\frac{1}{p}, \\ Z_{12}(T, p) &= Z_{21}(T, p) = -\{\sigma^{(0)}+\sigma^{(1)}(T(\omega_0)+T^{-1}(\omega_0))\}\frac{1}{p}. \end{aligned}\right\} \tag{6.2}$$

From (6.1) we obtain:

$$\frac{1}{2}(E_1(p)+E_2(p)) = R\frac{1}{2}(I_1(p)+I_2(p)),$$
$$\frac{1}{2}(E_1(p)-E_2(p)) = \left\{R+2\{\sigma^{(0)}+\sigma^{(1)}(T(\omega_0)+T^{-1}(\omega_0))\}\frac{1}{p}\right\}\frac{1}{2}(I_1(p)-I_2(p)),$$

and further by these equations,

$$\left.\begin{aligned}\frac{1}{2}(I_1(p)+I_2(p)) &= \frac{1}{R}\frac{1}{2}(E_1(p)+E_2(p)),\\ \frac{1}{2}(I_1(p)-I_2(p)) &= \frac{1}{R}p\,e^{-\beta(T(\omega_0)-T^{-1}(\omega_0))}\\ &\quad\times\frac{1}{p+\tau^{-1}}e^{\beta(T(\omega_0)-T^{-1}(\omega_0))}\frac{1}{2}(E_1(p)-E_2(p)),\end{aligned}\right\} \tag{6.3}$$

where $\beta=\frac{2\sigma^{(1)}}{iR\omega_0}$ and $\tau=\frac{R}{2\sigma^{(0)}}$.

Solving (6.3) for $I_1(p)$ and $I_2(p)$, we obtain:

$$\begin{bmatrix}I_1(p)\\I_2(p)\end{bmatrix}=\begin{bmatrix}Y_{11} & Y_{12}\\Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}E_1(p)\\E_2(p)\end{bmatrix}, \tag{6.4}$$

where

$$\left.\begin{aligned}Y_{11}(T,p) = Y_{22}(T,p) &= \frac{1}{2R}\left\{1+pU^{-1}(T(\omega_0))\frac{1}{p+\tau^{-1}}U(T(\omega_0))\right\},\\ Y_{12}(T,p) = Y_{21}(T,p) &= \frac{1}{2R}\left\{1-pU^{-1}(T(\omega_0))\frac{1}{p+\tau^{-1}}U(T(\omega_0))\right\},\\ \text{and}\qquad U(T(\omega_0)) &= \exp\left\{\frac{2\sigma^{(1)}}{i\omega_0R}(T(\omega_0)-T^{-1}(\omega_0))\right\}.\end{aligned}\right\} \tag{6.5}$$

Now, assuming that

$$e_1(t) = A_1e^{i\omega t},$$
$$e_2(t) = A_2e^{i\omega t},$$

and taking into account the considerations in §4, part I, we obtain $i_1(t)$ and $i_2(t)$ by converting (6.4) to the representation in the time domain as follows:

$$\begin{bmatrix}i_1(t)\\i_2(t)\end{bmatrix}=\begin{bmatrix}y_{11}(t) & y_{12}(t)\\y_{21}(t) & y_{22}(t)\end{bmatrix}\begin{bmatrix}e_1(t)\\e_2(t)\end{bmatrix}, \tag{6.6}$$

where

$$\left.\begin{aligned}y_{11}(t) = y_{22}(t) &= \frac{1}{2R}(1+\eta(t)),\\ y_{12}(t) = y_{21}(t) &= \frac{1}{2R}(1-\eta(t)),\end{aligned}\right\} \tag{6.7}$$

and

$$\eta(t) = e^{-i\omega t}\frac{d}{dt}\left\{U^{-1}(e^{i\omega_0t})\,e^{-\frac{t}{\tau}}\int_0^t e^{(\frac{1}{\tau}+i\omega)t'}U(e^{i\omega_0t'})\,dt'\right\}.$$

It seems worth noticing that the steady state solution can be obtained if, for the lower limit of integration in $\eta(t)$, zero is replaced by $-\infty$ corresponding to $\tau > 0$, while $+\infty$ is taken when $\tau < 0$.

(ii) As an example of §5, we consider the non-dispersive transmission lines in which the varying condenser is inserted.

The elastic coefficient of the condenser is assumed to vary sinusoidally with angular frequency ω_0, so that it can be expressed as follows:

$$S(t) = S_0 + 2\Delta S \cos(\omega_0 t + \varphi),$$

or

$$S(t) = \sigma^{(0)} + \sigma^{(1)} e^{i\omega_0 t} + \sigma^{(-1)} e^{-i\omega_0 t},$$

where

$$S_0 = \sigma^{(0)}, \quad \Delta S e^{i\varphi} = \sigma^{(1)}, \quad \Delta S e^{-i\varphi} = \sigma^{(-1)}.$$

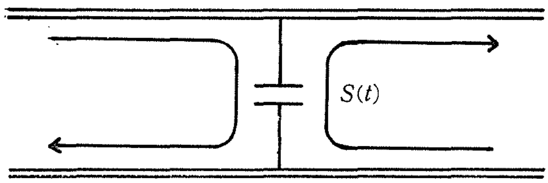


Fig. 4

Now, the x-axis is taken along the transmission lines and the origin set at the position of the inserting terminals, the right-hand side of the terminals being positive and the left-hand side negative.

Next the surge impedance and the propagation constant in the frequency domain are represented by Z_0 and $\gamma(p)$ respectively.

We assume that for the constant voltage source the waves are incident from the negative side and they are given by the following expressions:

$$\left.\begin{aligned} E^{(i)}(x, p) &= E_0 e^{-\gamma(p)x}\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ I^{(i)}(x, p) &= Z_0^{-1} E_0 e^{-\gamma(p)x}\{e^{-pt_0}(p-i\omega)^{-1}\}. \end{aligned}\right\}$$

Then the currents and the voltages for $x < 0$ and $x > 0$ are given as follows:

for $x < 0$,

$$\left.\begin{aligned} E(x, p) &= E_0\{e^{-\gamma(p)x} - e^{\gamma(p)x} A'(T, p)\}\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ I(x, p) &= Z_0^{-1} E_0\{e^{-\gamma(p)x} + e^{\gamma(p)x} A'(T, p)\}\{e^{-pt_0}(p-i\omega)^{-1}\}, \end{aligned}\right\} \tag{6.8}$$

and for $x > 0$,

$$\left.\begin{aligned} E(x, p) &= E_0 e^{-\gamma(p)x} B'(T, p)\{e^{-pt_0}(p-i\omega)^{-1}\}, \\ I(x, p) &= Z_0^{-1} E_0 e^{-\gamma(p)x} B'(T, p)\{e^{-pt_0}(p-i\omega)^{-1}\}, \end{aligned}\right\} \tag{6.9}$$

where $A'(T, p)$ and $B'(T, p)$ are given by the following expressions:

$$\left.\begin{aligned} A'(T, p) &= \{2Z(T, p) Z_0^{-1} + 1\}^{-1}, \\ B'(T, p) &= 1 - A'(T, p) = \{1 + 2Z(T, p) Z_0^{-1}\}^{-1}\{2Z(T, p) Z_0^{-1}\}, \\ Z(T, p) &= \left\{\sigma^{(0)} + (\sigma^{(1)} T(\omega_0) + \sigma^{(-1)} T^{-1}(\omega_0))\right\} \frac{1}{p}. \end{aligned}\right\} \tag{6.10}$$

Now, if we assume that these transmission lines are lossless Lecher lines or coaxial cables with induction coefficient per unit length l and capacity per unit length c, the expressions for $A'(T, p)$ and $B'(T, p)$ become as follows:

$$\left.\begin{aligned} A'(T, p) &= pU^{-1}(T(\omega_0))\frac{1}{p+\tau^{-1}}U(T(\omega_0)), \\ B'(T, p) &= 1-pU^{-1}(T(\omega_0))\frac{1}{p+\tau^{-1}}U(T(\omega_0)), \end{aligned}\right\} \tag{6.11}$$

where

$$U(T(\omega_0)) = \exp\left\{\frac{2}{iZ_0\omega_0}(\sigma^{(1)}T(\omega_0)-\sigma^{(-1)}T^{-1}(\omega_0))\right\},$$

$$\tau = \frac{2Z_0}{\sigma^{(0)}}.$$

Substituting (6.11) in (6.8) and taking the inverse Laplace transforms of the resulting expressions, we obtain the following expressions for $i(x, t)$ and $e(x, t)$ which denote the inverse Laplace transforms of $I(x, p)$ and $E(x, p)$ respectively.

Putting $v^{-1}=\sqrt{lc}$, we get, for $x<0$,

$$\left.\begin{aligned} e(x, t) &= E_0\{P(x, t)-R(x, t)\}, \\ i(x, t) &= Z_0^{-1}E_0\{P(x, t)+R(x, t)\}, \end{aligned}\right\} \tag{6.12}$$

where

$$\left.\begin{aligned} &\left.\begin{aligned} P(x, t) &= 0 && \left(t-t_0-\frac{x}{v}<0\right), \\ P(x, t) &= e^{i\omega\{(t-t_0)-\frac{x}{v}\}} && \left(t-t_0-\frac{x}{v}>0\right), \end{aligned}\right\} \\ &R(x, t) = 0 \qquad \left(t-t_0+\frac{x}{v}<0\right), \\ &R(x, t) = \frac{d}{dt}\left\{U^{-1}(e^{i\omega_0(t^*+t_0)})\,e^{-\frac{t^*}{\tau}}\int_0^{t^*}e^{(\frac{1}{\tau}+i\omega)t'}U(e^{i\omega_0(t'+t_0)})\,dt'\right\} \\ &\qquad\qquad \left(t-t_0+\frac{x}{v}>0\right) \end{aligned}\right\}$$

with $t^*=t-t_0+\frac{x}{v}$, and for $x>0$,

$$\left.\begin{aligned} &e(x, t) = 0 \qquad \left(t-t_0-\frac{x}{v}<0\right), \\ &i(x, t) = 0 \qquad \left(t-t_0-\frac{x}{v}<0\right), \\ &\left.\begin{aligned} e(x, t) &= E_0 \\ i(x, t) &= Z_0^{-1}E_0 \end{aligned}\right\}\times\Big[e^{i\omega t'^*}-\frac{d}{dt}\Big\{U^{-1}(e^{i\omega_0(t'^*+t_0)}) \\ &\qquad\qquad \times e^{-\frac{t'^*}{\tau}}\int_0^{t'^*}e^{(\frac{1}{\tau}+i\omega)t'}U(e^{i\omega_0(t'+t_0)})\,dt'\Big\}\Big] \\ &\left(t-t_0-\frac{x}{v}>0\right), \end{aligned}\right\} \tag{6.13}$$

where $t'^{*} = t - t_0 - \frac{x}{v}$.

The steady state solution can be obtained by the same procedure as stated in (i).

7. Conclusion

By introducing the translational operator, the circuit equations of the linear fixed network can be more naturally extended to the cases of the linear varying network than by other authors' theories.

Further, the ideas such as impedance, admittance, reflection coefficient and transmission coefficient can also be generalized by the present author's method. The present theory can explicitly give the impedance and the admittance in terms of the circuit elements. Particularly, as described in §5, we can easily perform the treatment of the cases when the linear varying network is inserted in the transmission lines. Such cases have never been treated by other authors.

As described in part I, in order to obtain the solution of the circuit equations, we must generally rely on the method of series expansion.

However, according to this method, the transient parts of the solutions sometimes take complex forms and so it seems necessary to obtain the transient parts in tractable forms by means of a more ingeneous method. Since it is important to consider in detail the transient parts which are associated with the stability, we should make researches on this aspect of the subject. The author hopes to carry out the researches in future.

Finally the author wishes to thank Professor S. Tomotika and Professor I. Takahashi for their powerful guidance and persistent encouragement.

REFERENCES

1. J. R. CARSON, Theory and calculation of variable electrical systems. Phys. Rev. **17** (1921), 116.
2. J. NEUFELD, Extension of the methods of Heaviside's calculus in calculation of circuits containing parameters varying with time. Phil. Mag. [7] **15** (1933), 170.
3. L. A. PIPES, Operational and matrix methods in linear variable networks. Phil. Mag. [7] **25** (1938), 585.
4. M. AKIYAMA, The fundamental theory of the linearly modulating network. J. Inst. Electr. Comm. Japan **119** (1939), 524 (in Japanese).
5. M. AKIYAMA, The theoretical considerations of the linearly modulating four-terminals network. J. Inst. Electr. Comm. Japan **120** (1939), 591 (in Japanese).
6. M. AKIYAMA, The theoretical considerations of the linearly modulating four-terminals network. J. Inst. Electr. Comm. Japan **121** (1939), 699 (in Japanese).
7. L. A. ZADEH, Frequency analysis of variable networks. Proc. Inst. Radio Eng. **38** (1950), 291.
8. L. A. ZADEH, The determination of the impulsive response of variable networks. J. Appl. Phys. **21** (1950), 642.
9. L. A. ZADEH, Circuit analysis of linear varying-parameter networks. J. Appl. Phys. **21** (1950), 1171.

10. L. A. ZADEH, Initial conditions in linear varying-parameter system. J. Appl. Phys. 22 (1950), 782.
11. Z. KIYASU, The theory of the ring modulator. Report Res. Electr. Comm. Lab. No. 1, March (1951) (in Japanese).
12. I. TAKAHASHI and T. WATANABE, On the linear network containing circuit elements periodically varying with time (I). Preparing draft at the joint meeting of Kansai branches of three Electrical Societies, Oct. 18, 1951 (in Japanese).
13. I. TAKAHASHI and T. WATANABE, On the linear network containing circuit elements periodically varying with time (II). Preparing draft at the joint meeting of Kansai branches of three Electrical Societies, Oct. 17, 1953 (in Japanese).
14. I. TAKAHASHI and T. WATANABE, On the linear network containing circuit elements periodically varying with time (III). Preparing draft at the joint meeting of Kansai branches of three Electrical Societies, Oct. 17, 1954 (in Japanese).