On Generalized Radix Representations

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1. Radix representation and two generalizations

Positive base: Let \( g \geq 2 \) be an integer. Then every \( n \in \mathbb{Z} \) can be represented in the form
\[
n = \pm \sum_{j=0}^{\ell} n_ig^i, \quad 0 \leq n_i < g.
\]

Negative base: V. Grünwald (1885): Let \( g \leq -2 \) be an integer. Then every \( n \in \mathbb{Z} \) can be represented in the form
\[
n = \sum_{j=0}^{\ell} n_ig^i, \quad 0 \leq n_i < g.
\]

You can find details and more material about the here studied kind of questions in the following papers:


1.1. \( \beta \) representation

A. Rényi (1957): Let \( \beta > 1 \) be a real number and \( A = \{0, 1, \ldots, \lfloor \beta \rfloor \} \) be the set of digits. Then each \( \gamma \in [0, \infty) \) can be represented by
\[
\gamma = a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots
\]
with \( a_i \in A \). This \( \beta \)-representation is usually not unique.

Assuming however that
\[
0 \leq \gamma - \sum_{i=n}^{m} a_i\beta^i < \beta^n
\]
hold for all \( n \leq m \) the \( \beta \)-representation become unique. For \( \gamma \in [0, 1) \) this greedy expansion can be given by the \( \beta \)-transformation
\[
T_{\beta}(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor
\]
This concept is the topics of a lot of research:
1.2. CNS representation

Let $\mathbb{Z}_K$ be the ring of integers of the algebraic number field $K$.

\[ \{\alpha, N\}; \quad \alpha \in \mathbb{Z}_K, \quad N = \{0, \ldots, |\text{Norm}(\alpha)| - 1\} \]

is called a canonical number system if every $\nu \in \mathbb{Z}_K$ can be represented in the form

\[ \nu = \sum_{j=0}^{r} n_j \alpha^j, \quad n_j \in N. \]

1.3. CNS polynomials

Observation: If $\mathbb{Z}_K$ is monogenic then $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{Z}_K$. This means $\mathbb{Z}_K \equiv \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$, where $P(x)$ is the minimal polynomial of $\alpha$.

Moreover, \( \{\alpha, N\} \) is a CNS in $\mathbb{Z}_Q(\alpha)$ means nothing else that every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element (a representative) such that its coefficients is bounded by $|p_0| - 1$.

A monic polynomial $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ is called CNS polynomial if every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element

\[ a_0 + a_1x + \cdots + a_kx^k \] (3)

such that $0 \leq a_i < |p_0|$.

2. Comparison of the properties of greedy expansions and of CNS-polynomials

Let $\beta$ be the root of $B(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$.

Let $\text{Fin}(\beta)$ be the set of positive real numbers having finite greedy expansion with respect to $\beta$. We say that $\beta > 1$ has property (F) if

\[ \text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty). \]

Property (F)

\begin{align*}
\beta &\text{ is a Pisot number:} & \beta &> 1, \text{ but its conjugates are } < 1 \\
\text{If } b_1 &\geq \cdots \geq b_d \geq 1, & \text{Frougny and Solomyak (1992)} & \\
\text{If } p_{d-1} &\leq \cdots \leq p_0, & \text{B. Kovács (1981)} & \\
\text{Characterization results if} & & \\
b_1 &> |b_2| + \cdots + |b_d|, & & \text{Akiyama, Petö, (2002),} \\
b_d &\neq 0, \text{ Hollander (1996)} & & \text{Scheicher, Thuswaldner} \\
\end{align*}
3. Shift Radix Systems

Let \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d \). To \( r \) we associate the mapping
\[
\tau_r : \mathbb{Z}^d \rightarrow \mathbb{Z}^d ; \text{if } a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \text{ then let }\
\tau_r(a) = (a_2, \ldots, a_d, -\left\lfloor ra \right\rfloor),
\]
where \( ra = r_1a_1 + \cdots + r_da_d \), i.e. the inner product of \( r \) and \( a \).

Let \( r \) be fixed. We will show: \( r \) gives rise to a Pisot number \( \beta \) with property (F) as well as to a CNS-polynomial if

for all \( a \in \mathbb{Z}^d \exists k > 0 \) with \( \tau_r^k(a) = 0 \). \( \text{(4)} \)

If (4) holds, we will call \( \tau_r \) a *shift radix system* (SRS for short). Hence SRS is a common generalization of greedy expansions with property (F) and CNS-polynomials.

It is clear that \( D_1 = [-1, 1] \) and \( D_2^0 = [0, 1] \).
To illustrate the difficulty of the characterization problem of \( D_2^0 \), we show an approximation of \( D_2^0 \).

4. Relation between SRS and CNS-polynomials

This is a more delicate question.

Let \( F(x) = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x] \)
with \( p_d = 1 \). Every coset of \( \mathbb{Z}[x]/F(x)\mathbb{Z}[x] \) has an element of form
\[
A_0 + A_1 x + \cdots + A_{d-1} x^{d-1}, \quad A_i \in \mathbb{Z}. \quad \text{(5)}
\]

3.1. Relation between SRS and \( \beta \)-expansions

Two basic definitions. Let
\[
D_2^0 := \{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0 \} \quad \text{and}
\]
\[
D_d := \{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d : \{\tau_r^k(a)\}_{k \geq 0} \text{ is ultimately periodic} \}.
\]

Now we can formulate the connection between SRS and greedy expansions.

**Theorem 1 (Hollander, 1996).** Let \( \beta > 1 \) be a Pisot number with minimal polynomial \( X^d - b_1 X^{d-1} - \cdots - b_{d-1} X - b_d \). Set
\[
\begin{align*}
\tau_1 & := 1, \\
\tau_j & := b_j \beta^{-1} + b_{j+1} \beta^{-2} + \cdots + b_d \beta^{-j-d+1},
\end{align*}
\]
where \( b_d = 1 \).

Then \( \beta \) has property (F) if and only if \( (r_1, \ldots, r_2) \in D_2^{\beta} \).

Let \( Z'[x] = \{ A(x) \in \mathbb{Z}[x] : \deg A < d \} \) and
\[
T(A) = \sum_{i=0}^{d-1} (A_{i+1} - qP_{i+1}) x^i,
\]
where \( A_d = 0 \) and \( q = [A_0/\beta_0] \).

Then \( T : Z'[x] \rightarrow Z'[x] \) and
\[
A = A_0 + zT(A), \quad \text{with } a_0 = A_0 - q\beta_0.
\]

This backward division process can become:
- divergent \( A(X) = -1 \) for \( P(X) = X^2 + 4X + 2 \)
  \( T_{X^2+4X+2}^{\beta}(-1) = -1, X + 4, -2X - 8, 4X + 16, \ldots \) or
- ultimately periodic \( A(X) = -1 \) for \( P(X) = X^2 - 2X + 2 \)
  \( T_{X^2-2X+2}^{\beta}(-1) = -1, X - 2, X - 1, X - 1, \ldots \) or
- can terminate after finitely many steps
  \( A(X) = -1 \) for \( P(X) = X^2 + 2X + 2 \)
  \(-1 = 1 + z^2 + z^3 + z^4. \)
Let \( \Pi(P) = \{ A : T^\ell_P(A) = A \text{ for some } \ell > 0 \} \)
denote the set of periodic points of the mapping \( T_P \).
We always have \( 0 \in \Pi(P) \). With help of this set we define

\[
C^0_d = \{(p_0, p_1, \ldots, p_{d-1}) \in \mathbb{Z}^d : \Pi(X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0) = \{0\}\}
\]

and

\[
C_d = \{(p_0, p_1, \ldots, p_{d-1}) \in \mathbb{Z}^d : \Pi(X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0) \text{ has only finite orbits}\}.
\]

Clearly, we have \( C^0_d \subset C_d \).
The elements of \( C^0_d \) will be called CNS polynomials.

It is convenient to replace \( T_P \) by the conjugate mapping
\( T_P : \mathbb{Z} \rightarrow \mathbb{Z} \) defined as

\[
T_P(A) = (A_1 - qP_1, \ldots, A_{d-1} - qP_{d-1}, -qP_d),
\]

where \( A = (A_0, \ldots, A_{d-1}) \) and \( q = \lfloor A_0/p_0 \rfloor \).

4.1. Affect of a new representation

H. Brunotte (2000) and K. Scheicher and J. Thuswaldner (2001) observed that the basis transformation

\[
\{1, x, \ldots, x^{d-1}\} \rightarrow \{w_1, \ldots, w_d\},
\]

where \( w_j = \sum_{i=d-j+1}^{d} p_ix^{i+j-d-1} \) of \( R \) implies a nicer and much better applicable transformation than \( T_P \) is. Indeed, if

\[
A(x) = \sum_{j=1}^{d} A_j w_j,
\]

then

\[
T_P(A) = -t w_d + \sum_{j=1}^{d-1} A_{j+1} w_j,
\]

where

\[
t = \lfloor \frac{p_1A_d + \cdots + p_dA_1}{p_0} \rfloor.
\]

Hence, \( T_P \) implies the mapping \( \tau_P : \mathbb{Z} \rightarrow \mathbb{Z} \)

\[
\tau_P(A) = \left( A_2, \ldots, A_d, -\left\lfloor \frac{p_1A_d + \cdots + p_dA_1}{p_0} \right\rfloor \right)
\]

where \( A = (A_1, \ldots, A_d) \). The mapping \( \tau_P \) will be called Brunotte's mapping.

4.2. \( C^0_d \) for small \( d \)'s

- \( C^0_1 = \{p_0 : p_0 \geq 2\} \), V. Grünwald
- \( C^0_2 = \{(p_0, p_1) : -1 \leq p_1 \leq p_0, p_0 \geq 2\} \), Kátai, Szabó, B. Kovács, Gilbert.
- Conjecture of Gilbert, 1981: \((p_0, p_1, p_2) \in C^0_3\) if and only if \( r = (1, p_0, p_1, p_2, \frac{p_0}{p_0-1}, \frac{p_1}{p_0-1}, \frac{p_2}{p_0-1}) \in D^0_d \).

Visualization of Gilbert's conjecture, \( p_0 = 44 \).
5. Basic properties of SRS

For a matrix $M$ denote the spectral norm by $||M||$. For a vector $v, ||v||$ shall denote the Euclidean norm.

For $r = (r_1, \ldots, r_d) \in D_d$ let

$$R := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & -r_d \end{pmatrix}. \quad (6)$$

Lemma 1. Let $d \in \mathbb{N}$. If $r = (r_1, \ldots, r_d) \in D_d$ then the spectral radius of $R$ is less than or equal to 1.

In the opposite direction we get the following result.

Lemma 2. Let $r \in \mathbb{R}^d$ such that the spectral radius $\rho$ of the matrix $R$ given above is less than 1. Then $r \in D_d$.

It is not to hard to prove the following statement

Theorem 3. The sets $D_d$ and $D_0^d$ are Lebesgue measurable.

5.1. Convexity property of $\tau_T$

Theorem 4. Let $r_1, \ldots, r_k \in \mathbb{R}^d$ and $a \in \mathbb{Z}^d$ be such that $\tau_{r_1}(a) = \cdots = \tau_{r_k}(a)$. Let $s$ be any convex linear combination of $r_1, \ldots, r_k$. Then we have $\tau_s(a) = \tau_{r_1}(a) = \cdots = \tau_{r_k}(a)$.

Corollary 1. Let $r_1, \ldots, r_k \in \mathbb{R}$ have the same period, i.e. $\tau_{r_1}(a) = \cdots = \tau_{r_k}(a), l = 0, \ldots, q$ and $a = \tau_{r_1}(a)$. Then if $s$ is lying in the convex hull of $r_1, \ldots, r_k$ the mapping $\tau_s$ is periodic and has the same period as $\tau_{r_1}$.
For example, it is easy to check that for the plane vectors $r_1 = (251, 253), r_2 = (252, 253)$ and $r_3 = (344, 172)$ the corresponding mappings have the same period $(-2, 1); 3, -2, 1, 1, -2$, hence the corresponding mapping for any point lying in the rectangle $r_1, r_2, r_3$ have this period.

5.2. Brunotte's algorithm

To decide $r \in C^0_d$ Brunotte gave an algorithm, which was realized independently by Scheicher and Thuswaldner. We give here a generalization for $D^0_d$.

**Theorem 5.** Suppose that there exists a set $E \subset \mathbb{Z}^d$ satisfying

(i) $E$ contains $2d$ elements of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$.

(ii) $\tau(E) \cup \tau_*(E) \subseteq E$, where $\tau_*(x) = -\tau(-x)$.

(iii) For each $a \in E$ there is some $k > 0$ such that $\tau_k^a = 0$.

Then $r \in D^0_d$.

6. Lifting theorem

Let $d \in \mathbb{N}$ and

\[(a_1, \ldots, a_{d+j}) \in \mathbb{Z}^d, \quad (0 \leq j \leq L-1), \quad (7)\]

with $a_{d+1} = a_1, \ldots, a_{L+d} = a_d$.

For which $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$ these vectors form a period $\pi$ of $D_d$? By the definition of $\tau$ this is the case if and only if the inequalities

\[0 \leq r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (8)\]

hold simultaneously for all $0 \leq j \leq L-1$. They define a (possibly degenerated) polyhedron, which will be denoted by $P(\pi)$.

Since $r \in D^0_d$ if and only if $\tau$ has 0 as its only period we conclude that

\[D^0_d = D_d \setminus \bigcup_{\pi \neq 0} P(\pi)\]

where the union is extended over all families of vectors $\pi$ of the shape (7). We call the family of (non-empty) polyhedra corresponding to this choice the family of cutout polyhedra of $D^0_d$.

An example: Let $r = (\frac{2}{5}, -\frac{1}{2})$.

Starting from $E_0 = ((\pm 1, 0), (0, \pm 1))$ and using that $\tau_1(0, 0), \tau_{-1}(0, -1) = (0, 1), \tau(0, 1) = (1, 1)$, $\tau_0(-1, 0) = (-1, 0)$ we get

$E_1 = \tau(E_0) \cup \tau^*_1(E_0) = E_0 \cup \{(0, 0), (1, 1), (-1, -1)\}$.

Now $\tau_1(1, 1) = (1, 0), \tau_{-1}(1, -1) = (-1, 1)$, hence we may take

$E_2 = \tau(E_1) \cup \tau^*_1(E_1) = E_1 \cup \{(1, -1), (-1, 1)\}$.

Finally because $\tau(-1, 1) = (1, 1), \tau_1(1, -1) = (-1, 0)$, we get that

$E = E_2$ proves $r \in D^0_d$.

Let $\pi$ be a period of $C_d$ or $D_d$ which corresponds to a non-degenerate cutout polyhedron. Then we call $\pi$ a non-degenerate period. We will show that we can "lift" a non-degenerate period to higher dimensions.

**Definition 1.** Let

\[\pi : (a_1, \ldots, a_d); a_{d+1}, \ldots, a_L \quad (9)\]

be a non-degenerate period of length $L$ of $C_d$ or $D_d$. Then we call

\[l(\pi) : (a_1, a_2, \ldots, a_{d+1}); a_{d+2}, \ldots, a_L \quad (10)\]

the lift of $\pi$ to $d+1$.

Note that $\pi$ and $l(\pi)$ have the same period length $L$. 
Theorem 6 (Lifting Theorem). Let $d \geq 1$ be an integer.

(i) Let $p_0 \geq 2$ and let $\pi$ be a non-degenerate period for $C_d$. Then $\pi$ is also a non-degenerate period for $D_d$. More precisely, there exist $p_1, \ldots, p_d \in \mathbb{Z}$ such that $(p_0, \ldots, p_{d-1}) \in \text{int}(P(\pi))$ and

\[ \frac{1}{p_0}, \frac{p_1}{p_0}, \ldots, \frac{p_{d-1}}{p_0} \in \text{int}(P(\pi)). \]

(ii) Let $\pi$ be a non-degenerate period of $D_d$. Then its lift $\lambda := l(\pi)$ is a non-degenerate period of $C_{d+1}$ for each sufficiently large $p_0$. More precisely, for all $(r_1, \ldots, r_d) \in \text{int}(P(\pi))$ there exists $\epsilon > 0$ such that for all $(p_0, \ldots, p_d) \in \mathbb{Z}^{d+1}$ with

\[ \max_{1 \leq k \leq d} \left| \frac{p_{d+1-k}}{p_0} - r_k \right| < \epsilon \]

we have $(p_0, \ldots, p_d) \in \text{int}(P(\lambda))$.

7. Long periods

Consider the following family of edges.

- $\alpha_k : (-k-1, -n+k) \rightarrow (-n+k, k+1)$
  \[ (0 \leq k \leq n-1), \]
- $\beta_k : (-n+k, k+1) \rightarrow (k+1, n+1-k)$
  \[ (0 \leq k \leq n-1), \]
- $\gamma_0 : (1, n+1) \rightarrow (n+1, 1)$
  \[ (1 \leq k \leq n-1), \]
- $\gamma_k : (k+1, n+1-k) \rightarrow (n+1-k, -k)$
  \[ (1 \leq k \leq n-1). \]

With these edges we form the cycle $\zeta_n : \alpha_0 \beta_0 \gamma_0 \alpha_{n-1} \beta_{n-1} \gamma_{n-1} \delta_{n-1} \alpha_{n-2} \ldots \alpha_1 \beta_1 \delta_1$.

Note that $\delta_1$ ends up in $(-1, -n)$. In this node $\alpha_0$ starts. Thus $\zeta_n$ is indeed a cycle. We wonder whether there exists $r := (x_n, y_n) \in D_2$ such that $r$ has $\zeta_n$ as a non-degenerate period. This is done in the following result.

By a direct application of the Lifting Theorem we obtain.

Theorem 8. Let $d \geq 2$ be an integer, fix $n \in \mathbb{N}$, $n > 3$. Then there exist some $r \in \mathbb{R}^d$ such that $l^{d-2}(\zeta_n)$ is a non-degenerate period of $r$.

Since we can select $n$ arbitrarily large and the length of the period $l^{d-2}(\zeta_n)$ is $4n+1$ this implies that there exist non-degenerate periods of arbitrarily large length for $D_2$ and $C_{d+1}$.

Corollary 2. Fix $n \in \mathbb{N}$, $n > 3$, and set $d \geq 2$ and $r = (0, \ldots, 0, x_n, y_n) \in \mathbb{R}^d$ with $x_n, y_n$ as in Theorem 7. Then $l^{d-2}(\zeta_n)$ is a period of $r$. 

\[ x_n := 1 - \frac{1}{2n^2} x_n \quad \text{and} \quad y_n := -\frac{2n+1}{2n(n+1)} + y_n, \]

where $|x_n|, |y_n| < 1/n^4$. Then $\zeta_n$ is a non-degenerate period of $r$.
8. Critical points

Definition 2. Let \( x \in D_d \).
- If there exists an open neighborhood of \( x \) which contains only finitely many cutout polyhedra then we call \( x \) a regular point.
- If each open neighborhood of \( x \) has nonempty intersection with infinitely many cutout polyhedra then we call \( x \) a weak critical point for \( D_d \).
- If for each open neighborhood \( U \) of \( x \) the set \( U \setminus D_d^0 \) can not be covered by finitely many cutout polyhedra then \( x \) is called a critical point.

We will show the existence of critical points for each \( d \geq 2 \). This shows that there is no way to characterize either of the sets \( D_d^0 \) by finitely many cutouts if \( d \geq 2 \).

Lemma 3. Let \( x \) be a weak critical point for \( D_d \). Then \( x \in \partial D_d \).

Lemma 4. Let \( \{x_n\}_{n \geq 1} \) and \( \{y_n\}_{n \geq 1} \) be sequences with \( x_n < 1 \), \( y_n < 0 \), \( \lim x_n = 1 \), \( \lim y_n = 0 \) and \( 1 - x_n = o(y_n) \). Let \( \{a_m\}_{m \geq 1} \) be a sequence of integers such that \( |a_i| < K \) for some constant \( K \). Then there exists \( N \in \mathbb{N} \) such that
\[
0 \leq a_{i-1}x_n + a_i y_n + a_{i+1} < 1 \quad (11)
\]
can not hold for all \( i \) if \( n \geq N \) unless \( a_i = 0 \) for all \( i \) large enough. Thus nonzero periods whose elements are bounded by \( K \) can not occur in \( D_d \) for \( n \) large enough.

Theorem 9. Let \( d \geq 2 \). Then \( K_d := (0, \ldots, 0, 1) \in \mathbb{R}^d \) is a critical point of \( D_d \).

Problem 1. Characterize the critical points of \( D_d \). Can one show that for a given \( d \) there exist only finitely many critical points?