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<td>Author(s)</td>
<td>Louboutin, Stephane R.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1384: 133-145</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-07</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25743">http://hdl.handle.net/2433/25743</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
On the use of explicit bounds on residues of Dedekind zeta functions taking into account the behavior of small primes (Abridged version)

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21 Jan, 2004

Abstract

Lately, we have obtained explicit upper bounds on \(|L(1, \chi)|\) (for primitive Dirichlet characters \(\chi\)) taking into account the behaviors of \(\chi\) on a given finite set of primes. This yields explicit upper bounds on residues of Dedekind zeta functions of abelian number fields taking into account the behavior of small primes, and we explained how such bounds yield improvements on lower bounds of relative class numbers of CM-fields whose maximal totally real subfields are abelian. We present here some other applications of such bounds together with new bounds for non-abelian number fields.

1 Introduction

\textbf{Theorem 1} (See [Lou04a] and [Lou05], and [Ram1] and [Ram2] for a slight improvement, and see [BHM], [Le], [MP], [Mos], [MR], [SSW] and [Ste] for various applications). Let \(S\) be a given finite set of pairwise distinct rational primes. Then, for any primitive Dirichlet character \(\chi\) of conductor \(q_\chi > 1\) we have

\[
|L(1, \chi)| \leq \frac{1}{2} \left\{ \prod_{p \in S} \frac{p-1}{|p-\chi(p)|} \right\} \left( \log q_\chi + \kappa_\chi + \omega \log 4 + 2 \sum_{p \in S} \frac{\log p}{p-1} \right) + R_S(q_\chi),
\]

\(0\)1991 Mathematics Subject Classification. Primary 11R29, 11M20.
Key words and phrases. \(L\)-functions, Dedekind zeta functions, number fields, class number.
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where

$$\kappa_{\chi} = \begin{cases} \kappa_{\text{even}} = 2 + \gamma - \log(4\pi) = 0.046191 \cdots & \text{if } \chi(-1) = +1 \\ \kappa_{\text{odd}} = 2 + \gamma - \log\pi = 1.432485 \cdots & \text{if } \chi(-1) = -1, \end{cases}$$

where $\omega \geq 0$ is the number of primes $p \in S$ which do not divide $q_{\chi}$, and where $R_{S}(q_{\chi})$ is an explicit error term which tends rapidly to zero when $q_{\chi}$ goes to infinity. Moreover, if $S = \emptyset$ or if $S = \{2\}$, then this error term $R_{S}(q_{\chi})$ is always less than or equal to zero, and if none of the prime in $S$ divides $q_{\chi}$ then this error term $R_{S}(q_{\chi})$ is less than or equal to zero for $q_{\chi}$ large enough.

**Lemma 2** Assume that none of the primes $p \in S$ divide $q_{\chi}$. If $\chi$ is even then $R_{S}(q_{\chi}) \leq 0$ for $S = \{2\}$ and $q_{\chi} \geq 3$, $R_{S}(q_{\chi}) \leq 0$ for $S = \{2, 3\}$ and $q_{\chi} \geq 3$, and $R_{S}(q_{\chi}) \leq 0$ for $S = \{2, 3, 5\}$ and $q_{\chi} \geq 3$. If $\chi$ is odd then $R_{S}(q_{\chi}) \leq 0$ for $S = \{2, 3\}$ and $q_{\chi} \geq 9$, and $R_{S}(q_{\chi}) \leq 0$ for $S = \{2, 3, 5\}$ and $q_{\chi} \geq 217$.

**Proof.** We assume that $S \neq \emptyset$ and that none of the primes $p \in S$ divide $q_{\chi}$. Hence, $\omega = \# S \geq 1$ and we set $d = \prod_{p \in S} p > 1$.

1. Assume that $\chi$ is even. Set

$$\theta(x) = \sum_{n \geq 1} e^{-\pi n^{2}x} \quad (x > 0).$$

According to [Lou04a, Theorem 5]¹, we may take

$$R_{S}(q_{\chi}) = -\frac{\phi(d)}{2d\sqrt{q_{\chi}}} (\kappa_{\text{even}} - 2 - \log(q_{\chi}/4^{\omega}) - 2 \sum_{p \mid d} \frac{\log p}{p - 1}) + \tilde{R}_{S}(q_{\chi})$$

where in setting $\delta_{\max} = \max\{\delta \geq 1; \delta \mid d \text{ and } \mu(\delta) = -1\}$ we have

$$\tilde{R}_{S}(q_{\chi}) = -\sum_{\delta \mid d} \frac{\mu(\delta)}{\delta} \int_{1}^{\infty} \theta(\delta q_{\chi}x) dx$$

$$\leq \frac{\pi e^{-\pi q_{\chi}/\delta_{\max}^{2}}}{6 \cdot 4^{\omega} q_{\chi}} \sum_{\delta \mid d} \delta = \frac{\pi e^{-\pi q_{\chi}/\delta_{\max}^{2}}}{3 \cdot 4^{\omega+1} q_{\chi}} \left( \prod_{p \mid d} (p + 1) - \prod_{p \mid d} (1 - p) \right),$$

in using

$$\int_{1}^{\infty} \frac{\theta(Ax) dx}{x} \leq \int_{1}^{\infty} \frac{\theta(Ax) dx}{\pi n^{2}A} \leq \frac{\pi}{6A} e^{-\pi A}.$$

2. Assume that $\chi$ is odd. Then (see [Lou0?]),

$$R_{S}(q_{\chi}) = -\frac{\pi}{2\sqrt{q_{\chi}}} \frac{\phi(d)}{d} + \frac{\pi}{12q_{\chi}} \left( 2^{\omega+1} \left( \frac{\phi(d)}{d} \right)^{2} + \frac{\mu(d)}{4^{\omega}} \phi(d) \right) + \frac{0.124}{q_{\chi}^{3/2}} R_{d}$$

¹Note the misprint in [Lou04a, page 128, line 2] where the factor $2^{\omega}$ should read $1/2^{\omega}$. 
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where

\[ R_d = \frac{4^\omega}{d^{3/2}} \left( \prod_{p|d} (p^2(p-2)+1) \right) + \frac{q^{3/2}}{8^\omega} \left( \prod_{p|d} (p^2+1) \right). \]

3. Notice that the error term is much more satisfactory for even characters. •

2 Upper bounds for relative class numbers

Corollary 3 Let \( q \equiv 5 \pmod{8} \), \( q \neq 5 \), be a prime, let \( \chi_q \) denote any one of the two conjugate odd quartic characters of conductor \( q \) and let \( h_q^- \) denote the relative class number of the imaginary cyclic quartic field \( N_q \) of conductor \( q \). Then,

\[ h_q^- = \frac{q}{2\pi^2} |L(1, \chi_q)|^2 \leq \frac{q}{A\chi_\pi^2} (\log q + 2 + \gamma - \log(\pi) + \log B_\chi)^2, \]

which implies \( h_q^- < q \) for \( q \leq C_\chi \), where \( A_\chi, B_\chi \) and \( C_\chi \) are as follows:

<table>
<thead>
<tr>
<th>Values of ((A_\chi, B_\chi, C_\chi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_q(3) = +1 )</td>
</tr>
<tr>
<td>( \chi_q(5) = +1 )</td>
</tr>
<tr>
<td>( \chi_q(5) = -1 )</td>
</tr>
<tr>
<td>( \chi_q(5) = \pm i )</td>
</tr>
</tbody>
</table>

Proof. Since \( q \equiv 5 \pmod{8} \), we have \( \chi_q(2)^2 = \left(\frac{2}{q}\right) = -1 \) and \( \chi_q(2) = \pm i \).

Set \( S = \{p \in \{2, 3, 5\}; \chi(p) \neq +1\} \). Then \( 2 \in S \) and according to Theorem 1 we may choose

\[ A_\chi = 8 \prod_{p \in S} \left| \frac{p - \chi(p)}{p - 1} \right|^2 = 40 \prod_{2 \neq p \in S} \left| \frac{p - \chi(p)}{p - 1} \right|^2 \]

and

\[ \log B_\chi = \omega \log 4 + 2 \sum_{p \in S} \frac{\log p}{p - 1} = (\omega + 1) \log 4 + 2 \sum_{2 \neq p \in S} \frac{\log p}{p - 1}, \]

for according to Lemma 2 we have \( R_S(q_\chi) \leq 0 \) for \( q_\chi \geq 217 \). •

Remarks 4 Using Corollary 3 to alleviate the amount of required relative class number computation, M. Jacobson and the author are now trying to solve the open problem hinted at in [Lou98]: determine the least (or at least one) prime \( q \equiv 5 \pmod{8} \) for which \( h_q^- > q \). Indeed, according to Corollary 3, for finding such a \( q \) in the range \( q < 5 \cdot 10^{10} \), we may assume that \( \chi_q(3) = +1 \), which amounts to eliminating three quarters of the primes \( q \) in this range. In the same way, in the range \( q < 3 \cdot 10^{13} \) we may assume that \( \chi_q(3) = +1 \) or \( \chi_q(5) = +1 \), which amounts to eliminating 9/16 of the primes \( q \) in this range.
3 Applications of bounds taking into account the behavior of small primes

3.1 Using simplest cubic fields

In [CW], G. Cornell and L. C. Washington used simplest cubic fields to produce real cyclotomic fields $\mathbb{Q}^+(\zeta_p)$ of class number $h^+_p > p$, where the simplest cubic fields are the real cyclic cubic number fields associated with the $\mathbb{Q}$-irreducible cubic polynomials $P_m(x) = x^3 - mx^2 - (m + 3)x - 1$ of discriminants

$$d_m = \Delta_m^2$$

where $\Delta_m := m^2 + 3m + 9$.

Since $-x^3 P_m(1/x) = P_{-m-3}(x)$, we may assume that $m \geq -1$. We let

$$\rho_m = \frac{1}{3} \left( 2 \sqrt{\Delta_m} \cos \left( \frac{1}{3} \arctan \left( \frac{\sqrt{27}}{2m + 3} \right) \right) + m \right) = \sqrt{\Delta_m} - \frac{1}{2} + O \left( \frac{1}{\sqrt{\Delta_m}} \right) \ (1)$$

denote the only positive root of $P_m(x)$. Moreover, we will assume that the conductor of $K_m$ is equal to $\Delta_m$, which amounts to asking that (i) $m \not\equiv 0$ (mod 3) and $\Delta_m$ is squarefree, or (ii) $m \equiv 0, 6$ (mod 9) and $\Delta_m/9$ is squarefree (see [Wa, Prop. 1 and Corollary]). In that situation, $\{-1, \rho_m, -1/(\rho_m + 1)\}$ generate the full group of algebraic units of $K_m$ and the regulator of $K_m$ is

$$\text{Reg}_{K_m} = \log^2 \rho_m - (\log \rho_m)(\log(1 + \rho_m)) + \log^2(1 + \rho_m), \ (2)$$

which in using (1) yields

$$\text{Reg}_{K_m} = \frac{1}{4} \log^2 \Delta_m - \frac{\log \Delta_m}{\sqrt{\Delta_m}} + O \left( \frac{\log \Delta_m}{\Delta_m} \right) \leq \frac{1}{4} \log^2 \Delta_m. \ (3)$$

Lemma 5 The polynomial $P_m(x)$ has no root mod 2, has at least one root mod 3 if and only if $m \equiv 0$ (mod 3), and has at least one root mod 5 if and only if $m \equiv 1$ (mod 5). Hence, if $\Delta_m$ is square-free, then 2 and 3 are inert in $K_m$, and if $m \not\equiv 1$ (mod 5) then 5 is also inert in $K_m$.

As in [Lou02b, Section 5.1], we let $\chi_{K_m}$ be the primitive cubic Dirichlet characters modulo $\Delta_m$ associated with $K_m$ satisfying

$$\chi_{K_m}(2) = \begin{cases} \omega^2 & \text{if } m \equiv 0 \mod 2 \\ \omega & \text{if } m \equiv 1 \mod 2. \end{cases}$$

Since the regulators of these $K_m$'s are small, they should have large class numbers. In fact, we proved (see [Lou02d, (12)]):

$$h_{K_m} = \frac{\Delta_m}{4\text{Reg}_{K_m}} |L(1, \chi_{K_m})|^2 \geq \frac{\Delta_m}{e\log^3 \Delta_m} \ (4)$$

Corollary 6 Assume that $m \geq -1$ is such that $\Delta_m$ is squarefree. Then, $\chi_{K_m}(2) \neq +1, \chi_{K_m}(3) \neq +1$ and

$$h_{K_m} \leq \begin{cases} \Delta_m/60 & \text{if } m \geq 6, \\ \Delta_m/100 & \text{if } m \not\equiv 1 \mod 5 \text{ and } m \geq 9. \end{cases}$$
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Proof. If a prime \( l \geq 2 \) is inert in \( K_m \) then \( \chi_{K_m}(l) \in \{\exp(2i\pi/3), \exp(4i\pi/3)\} \).

According to the previous Lemma, to Theorem 1 (with \( S = \{2, 3\} \) and \( S = \{2, 3, 5\} \)) and to Lemma 2, we have

\[
|L(1, \chi_{K_m})|^2 \leq \begin{cases} 
(\log \Delta_m + \kappa \text{even} + \log(192))^2 / 91, \\
16(\log \Delta_m + \kappa \text{even} + \log(768\sqrt{5}))^2 / 2821 
\end{cases} \text{ if } m \neq 1 \pmod{5}.
\]

Now, according to (4) and (3), the desired results follow for \( mn \geq 95000 \). The numerical computation of the class numbers of the remaining \( K_m \) provides us with the desired bounds (see [Lou02b]).

From now on, we assume (i) that \( p = \Delta_m = m^2 + 3m + 9 \) is prime and (ii) that \( p \equiv 1 \pmod{12} \). In that case, both \( K_m \) and \( k_m := Q(\sqrt{\Delta_m}) \) are subfields of the real cyclotomic field \( Q^+(\zeta_p) \) and the product \( h_2h_3 \) of the class numbers \( h_2 := h_{k_m} \) and \( h_3 := h_{K_m} \) of \( k_m \) and \( K_m \) divides the class number \( h_p \) of \( Q^+(\zeta_p) \). Since \( h_3 \leq \Delta_m/60, h_2h_3 \geq \Delta_m \) implies \( h_2 \geq 60 \), hence \( h_2 \geq 61 \) (for \( h_2 \) is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime conductors with class numbers greater than or equal to 61 are few and far between. Hence, such simplest cubic fields \( K_m \) of prime conductors \( \Delta_m = m^2 + 3m + 9 \equiv 1 \pmod{4} \) with \( h_2h_3 > \Delta_m \) are few and far between. As we have at hand a very efficient method for computing class numbers of real quadratic fields (see [Lou02c] and [WB]), we used this explicit necessary condition \( h_2 \geq 61 \) to compute (using [Lou02b]) the class numbers of only few of the simplest cubic fields \( K_m \) of prime conductors \( \Delta_m \equiv 1 \pmod{12} \) with \(-1 \leq m \leq 1066285 \) to obtain the following Table. Notice that the authors of [CW] and [SWW] only came up with one such \( K_m \), the one for \( m = 106253 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \theta(\chi_{K_m}) )</th>
<th>( \arg W(\chi_{K_m}) )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th>( h_2h_3/\Delta_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>102496</td>
<td>20.268 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3} )</td>
<td>891</td>
<td>13152913</td>
<td>1.115 \ldots</td>
</tr>
<tr>
<td>106253</td>
<td>34.364 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>2685</td>
<td>6209212</td>
<td>1.476 \ldots</td>
</tr>
<tr>
<td>319760</td>
<td>202.162 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>1887</td>
<td>57772549</td>
<td>1.066 \ldots</td>
</tr>
<tr>
<td>554869</td>
<td>88.861 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3} )</td>
<td>7983</td>
<td>93739324</td>
<td>2.430 \ldots</td>
</tr>
<tr>
<td>726845</td>
<td>20.938 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>13553</td>
<td>76702419</td>
<td>4.526 \ldots</td>
</tr>
<tr>
<td>791021</td>
<td>129.812 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>1737</td>
<td>44512272</td>
<td>1.235 \ldots</td>
</tr>
<tr>
<td>796616</td>
<td>357.252 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>1155</td>
<td>696739264</td>
<td>1.268 \ldots</td>
</tr>
<tr>
<td>839401</td>
<td>293.373 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) + \pi )</td>
<td>1575</td>
<td>554491633</td>
<td>1.239 \ldots</td>
</tr>
<tr>
<td>906437</td>
<td>93.697 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) )</td>
<td>1955</td>
<td>469911916</td>
<td>1.118 \ldots</td>
</tr>
<tr>
<td>1066285</td>
<td>140.662 \ldots</td>
<td>( \frac{1}{3} \arctan(\frac{3\sqrt{3}}{2m+3}) + \frac{\pi}{3} )</td>
<td>5389</td>
<td>473034223</td>
<td>2.242 \ldots</td>
</tr>
</tbody>
</table>

Finally, it is much more efficient to use simplest sextic fields to produce real cyclotomic fields of prime conductors and class numbers greater than their conductors (see [Lou04c]).
3.2 Using simplest quartic fields

In [Laz1] and [Laz2], A. Lazarus dealt with various class number problems for the so-called simplest quartic fields, the real cyclic quartic number fields associated with the quartic polynomials $P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$ of discriminants

$$d_m = 4\Delta_m^3$$

where $\Delta_m := (m^2 + 16)^3$.

Since $P_m(-x) = P_{-m}(x)$, we may and will assume that $m \geq 0$. The reader will easily check (i) that $P_m(x)$ has no rational root, (ii) that $P_m(x)$ is Q-irreducible, except for $m = 0$ and $m = 3$, and (iii) that $P_m(x)$ has a only one root $\rho_m > 1$. Set $\beta_m = \rho_m - \rho_m^{-1} > 0$. Then, $\beta_m^2 - m\beta_m - 4 = 0$ and $\rho_m = (m + \sqrt{\Delta_m})/2$. In particular, $k_m = \mathbb{Q}(\sqrt{\Delta_m})$ is the quadratic subfield of the cyclic quartic field $K_m$. It is known that $h_{k_m}$ divides $h_{K_m}$, and we set $h_{K_m}^* = h_{K_m}/h_{k_m}$. Since $\rho_m > 1$ and $\rho_m^2 - \beta_m\rho_m - 1 = 0$, we obtain

$$\rho_m = \frac{1}{2} \left( \frac{m + \sqrt{\Delta_m}}{2} + \frac{\sqrt{\Delta_m + m\sqrt{\Delta_m}}}{2} \right) = \sqrt{\Delta_m} \left( 1 - \frac{3}{\Delta_m} + O\left( \frac{1}{\Delta_m^2} \right) \right)$$

(\text{use } m = \sqrt{\Delta_m - 16}),

$$\rho_m' = \frac{1}{2} \left( \frac{m - \sqrt{\Delta_m}}{2} + \frac{\sqrt{\Delta_m - m\sqrt{\Delta_m}}}{2} \right) = 1 - \frac{2}{\sqrt{\Delta_m}} + O\left( \frac{1}{\Delta_m} \right),$$

and

$$\text{Reg}_{K_m}^* = \log^2 \rho_m + \log^2 \rho_m' = \frac{1}{4} \log^2 \Delta_m - \frac{3\log \Delta_m}{\Delta_m} + O\left( \frac{1}{\Delta_m} \right) \leq \frac{1}{4} \log^2 \Delta_m$$

(5)

for $m \geq 1$. We will say that $K_m$ is a simplest quartic field if $m \geq 1$ is such that $\Delta_m$ is squarefree (which implies $m$ odd and $m \neq 3$).

**Proposition 7** Assume that $m \geq 1$ is odd and that $\Delta_m = m^2 + 16$ is prime. Then, the discriminant of the real quadratic subfield $k_m = \mathbb{Q}(\sqrt{\Delta_m})$ of $K_m$ is equal to $\Delta_m$, the discriminant of $K_m$ is equal to $\Delta_m^3$, its conductor is equal to $\Delta_m$, the class numbers of $K_m$ and $k_m$ are odd, and (see [Lou04b])

$$h_{K_m}^* = \frac{\Delta_m}{4\text{Reg}_{K_m}^*} |L(1, \chi_{K_m})|^2 \geq \frac{2\Delta_m}{3e(\log \Delta_m + 0.35)^4}$$

(6)

where $\chi_{K_m}$ is any one of the two conjugate primitive quartic Dirichlet characters modulo $\Delta_m$ associated with $K_m$. Moreover, $\chi_{K_m}(2) = -1$, and $m \geq 5$ implies

$$h_{K_m}^* < \Delta_m/26.$$

**Proof.** According to the class number formula (6), to Theorem 1 (with $S = \{2\}$) and Lemma 2 which yield

$$|L(1, \chi_{K_m})|^2 \leq (\log \Delta_m + \kappa_{\text{odd}} + \log(16))^2/36,$$
and to the asymptotic (5), we have $h_{K_{m}}^{*} \leq \Delta_{m}/(36 + o(1))$. Hence, $h_{K_{m}}^{*} < \Delta_{m}/24$. for $m \geq 148000$. The numerical computation of the class numbers of the remaining $K_{m}$ provides us with the desired bound (see [Lou02b]).

Since, $h_{K_{m}} = h_{k_{m}} h_{K_{m}}^{*} \geq \Delta_{m}$ and $h_{K_{m}}^{*} < \Delta_{m}/26$ imply $h_{k_{m}} \geq 27$ (for $h_{k_{m}}$ is odd), and Cohen-Lenstra heuristics predict that real quadratic number fields of prime discriminants with class numbers greater than or equal to 27 are few and far between. As we have at hand a very efficient method for computing rigorously class numbers of real quadratic fields (see [Lou02c] and [WB]), we used this explicit necessary condition $h_{k_{m}} \geq 27$ to compute only few of the class numbers of the simplest quartic fields $K_{m}$ of prime conductors $\Delta_{m} = m^{2} + 16 \equiv 1 \pmod{4}$ with $1 \leq m \leq 1680401$ to obtain the following Table. Notice that G. Cornell and L. C. Washington did not find any such $K_{m}$ (see [CW, bottom of page 268]).

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\Delta_{m}$</th>
<th>$h_{k_{m}}$</th>
<th>$h_{K_{m}}^{*}$</th>
<th>$h_{k_{m}}^{*}/\Delta_{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>524285</td>
<td>274874761241</td>
<td>1911</td>
<td>181442581</td>
<td>1.261 \ldots</td>
</tr>
<tr>
<td>1680401</td>
<td>282374752081</td>
<td>1537</td>
<td>1878644993</td>
<td>1.022 \ldots</td>
</tr>
</tbody>
</table>

### 4 The imaginary cyclic quartic fields with ideal class groups of exponent $\leq 2$

We explain how one could alleviate the determination in [Lou95] of all the non-quadratic imaginary cyclic fields of 2-power degrees $2n = 2^{r} \geq 4$ with ideal class groups of exponent $\leq 2$ (the time consuming part being the computation of the relative class numbers of the fields sieved by Proposition 9). To simplify, we will only deal with imaginary cyclic quartic fields of odd conductors, and we will prove Proposition 9 below. Recall the following result whose proof makes use of Theorem 1 (for even characters) with $S = \{2\}$:

**Theorem 8** (See [Lou03, Theorem 22]). Let $K$ be an imaginary cyclic quartic field of conductor $f_{K}$, Let $k$, $f_{k}$ and $\chi_{K}$ denote the real quadratic subfield of $K$, the conductor of $k$, and any one of the two conjugate primitive quartic Dirichlet characters modulo $f_{K}$ associated with $K$. Then,

$$h_{K}^{-} \geq \frac{C_{K} f_{K}}{e \pi^{2} (\log f_{k} + \kappa) \log (f_{k}f_{K}^{2})},$$

where

$$C_{K} = \frac{32}{|2 - \chi_{K}(2)|^{2}} = \begin{cases} 32 & \text{if } \chi_{K}(2) = +1, \\
32/9 & \text{if } \chi_{K}(2) = -1, \\
32/5 & \text{if } \chi_{K}(2) = \pm i, 
\end{cases}$$


Proposition 9 Assume that the exponent of the ideal class group of an imaginary cyclic quartic field $K$ of odd conductor $f_K$ is less than or equal to 2. Then, $f_k \leq 1889$ and $f_K \leq 10^7$ (where $k$ is the real quadratic subfield of $K$). Moreover, whereas there are 1 377 361 imaginary cyclic fields $K$ of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, only 400 out of them may have their ideal class groups of exponents $\leq 2$, the largest possible conductor being $f_K = 5619$ (for $f_k = 1873$ and $f_K/k := f_K/f_k = 3$).

Proof. It is known that if the exponent of the ideal class group of $K$ of odd conductor $f_K$ is $\leq 2$, then $f_k \equiv 1 \pmod{4}$ is prime and

$$h_K^- = 2^{t_{K/k} - 1},$$

where $t_{K/k}$ denotes the number of prime ideals of $k$ which are ramified in $K/k$ (see [Lou95, Theorems 1 and 2]). Conversely, for a given real quadratic field $k$ of prime conductor $f_k \equiv 1 \pmod{4}$, the conductors $f_K$ of the imaginary cyclic quartic fields $K$ of odd conductors and containing $k$ are of the form $f_K = f_k f_{K/k}$ for some positive square-free integer $f_{K/k} \geq 1$ relatively prime with $f_k$ and such that

$$(f_k - 1)/4 + (f_{K/k} - 1)/2 \text{ is odd}$$

(in order to have $\chi_K(-1) = -1$, i.e. in order to guarantee that $K$ is imaginary). Moreover, for such a given $k$ and such a given $f_{K/k}$, there exists only one imaginary cyclic quartic field $K$ containing $k$ and of conductor $f_K = f_k f_{K/k}$, and for this $K$ we have

$$t_{K/k} = 1 + \sum_{p | f_{K/k}} (3 + (\frac{p}{f_k}))/2,$$

where $(\frac{p}{f_k})$ denote the Legendre's symbol. Finally, if we let $\phi_k$ denote any one of the two conjugate quartic characters modulo a prime $f_k \equiv 1 \pmod{4}$, then $\chi_K(n) = \phi_k(n) (\frac{n}{f_{K/k}})$, where $(\frac{\cdot}{f_{K/k}})$ denote the Jacobi's symbol, and

$$\chi_K(2) = \begin{cases} 
\phi_k(2) = 1 & \text{if } f_k \equiv 1 \pmod{8} \text{ and } 2 f_k^{-1} \equiv 1 \pmod{f_k} \\
\phi_k(2) = -1 & \text{if } f_k \equiv 1 \pmod{8} \text{ but } 2 f_k^{-1} \not\equiv 1 \pmod{f_k} \\
-\phi_k(2) = \pm i & \text{if } f_k \equiv 5 \pmod{8}.
\end{cases}$$

Hence, we may easily compute $\kappa_k$, $c_K$ and $t_{K/k}$ from $f_k$ and $f_{K/k}$. In particular, we easily obtain that there are 1 377 361 imaginary cyclic fields $K$ of odd conductors $f_K \leq 10^7$ and such that $f_k \leq 1889$, and that $c_K = 32$ for 149 187 out of them, $c_K = 32/5$ for 938 253 out of them, and $c_K = 32/9$ for 289 921 out of them. Now, let $P_n$ denote the product of the first $n$ odd primes $3 = p_1 < 5 = p_2 < \cdots < p_n < \cdots$ (hence, $P_0 = 1$, $P_1 = 3$, $P_2 = 15$, $\cdots$). There are two cases to consider:
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1. If $\chi_k(2) = +1$. Then, $f_k \equiv 1 \pmod{8}$ is prime, $\kappa_k < 0.05$, $c_K \geq 32/9$, $f_K = f_k f_{k/k}$ where $f_{k/k}$ is a product of $n \geq 0$ distinct odd primes. Hence, $f_{k/k} \geq P_n$, $t_{k/k} \leq 1 + 2n$, $h_{k/k} = 2^{t_{k/k} - 1} \leq 4^n$ and using (7) we obtain

$$F_k(n) := \frac{32f_k P_n}{9e^n 4^n (\log f_k + 0.05) \log(f_k^3 P_n^2)} \leq 1.$$

Assume that $f_k \geq 36$. Then $3f_k^{3/2} \geq 5^4$ and for $n \geq 1$ we have $p_{n+1} \geq p_2 = 5, P_n \geq P_1 = 3$ and

$$\frac{F_k(n+1)}{F_k(n)} = \frac{p_{n+1} \log(f_k^{3/2} P_n)}{4 \log(p_{n+1} f_k^{3/2} P_n)} \geq \frac{5 \log(f_k^{3/2} P_n)}{4 \log(5 f_k^{3/2} P_n)} \geq \frac{5 \log(3 f_k^{3/2})}{4 \log(15 f_k^{3/2})} \geq 1.$$

Since we clearly have $F_k(1) \leq F_k(0)$, we obtain $\min_{n \geq 0} F_k(n) = F_k(1)$ and

$$\frac{8f_k}{3e^n (\log f_k + \kappa_k) \log(9f_k^3)} = F_k(1) \leq F_k(n) \leq 1,$$

which implies $f_k \leq 1899$, hence $f_k \leq 1889$ (for $f_k \equiv 1 \pmod{8}$ must be prime). Hence, using (7), we obtain

$$h_{k/k} \geq \frac{32f_K}{9 \pi^2 (\log(1889) + 0.05) \log(1889 P_n^2)}.$$

Let now $n$ denote the number of distinct prime divisors of $f_K$. Then $f_K \geq P_n$, $t_{k/k} \leq 2(n - 1) + 1$ and $h_{k/k} = 2^{t_{k/k} - 1} \leq 4^{n-1}$. Hence, using (7), we obtain

$$4^{n-1} \geq \frac{32P_n}{9 \pi^2 (\log(1889) + 0.05) \log(1889 P_n^2)},$$

which implies $n \leq 7$, $h_{k/k} \leq 4^6$,

$$4^6 \geq \frac{32f_K}{9 \pi^2 (\log(1889) + 0.05) \log(1889 f_K^2)}$$

and yields $f_K \leq 10^7$.

2. If $\chi_k(2) = -1$. Then $f_k \equiv 5 \pmod{8}$ is prime, $\kappa_k \leq 2.82$, $c_K \geq 32/5$ and we follow the previous case. We obtain $f_k \leq 1329$, hence $f_k \leq 1301$ (for $f_k \equiv 5 \pmod{8}$ must be prime), $n \leq 7$, $h_{k/k} \leq 4^6$ and $f_K \leq 7 \cdot 10^6$.

Hence, the first assertion Proposition 9 is proved. Now, for a given odd prime $f_k \leq 1889$ equal to 1 modulo 4, and for a given odd square-free integer $f_{k/k} \leq 10^7/f_k$ relatively prime with $f_k$, we compute $\kappa_k$, $t_{k/k}$ (using (10)), $c_K$ (using (11)) and use (7) and (8) to deduce that if the exponent of the ideal class group of $K$ is less than or equal to 2, then

$$2^{t_{k/k} - 1} \geq \frac{c_K f_k f_{k/k}}{e^{\pi^2 (\log f_k + \kappa_k) \log(f_k^3 f_{k/k}^2)}}.$$ (12)
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Now, an easy calculation yields that only 400 out of 1 377 361 imaginary cyclic fields $K$ of odd conductors and such that $f_k \leq 1889$ and $f_K \leq 10^7$ satisfy (12), and the second assertion of the Proposition is proved. $
$

5 The non-abelian case

We showed in [Lou03] how taking into account the behavior of the prime 2 in CM-fields can greatly improve upon the upper bounds on the root numbers of the normal CM-fields with abelian maximal totally real subfields of a given (relative) class number. We now explain how we can improve upon previously known upper bounds for residues of Dedekind zeta functions of non-necessarily abelian number fields by taking into account the behavior of the prime 2. Let $K$ be a number field of degree $m \geq 1$. We set

$$
\Pi_{K}(2, s) := \prod_{\mathfrak{p}|2} (1 - (N(\mathfrak{p}))^{-s})^{-1}
$$

(which is $\geq 1$ for $s > 0$) and $\Pi_{K}(2) := \Pi_{K}(2, 1)$. In particular, $\Pi_{K}(2)/\Pi_{Q}^{m}(2) \leq 1$. However, if 2 is inert in $K$, then $\Pi_{K}(2)/\Pi_{Q}^{m}(2) = 1/(2^m - 1)$ is small.

**Theorem 10** Let $K$ be a number field of degree $m \geq 3$ and root discriminant $\rho_K = d_K^{1/m}$. Set $v_m = (m/(m - 1))^{m-1} \in [9/4, e)$, and $E(x) := (e^x - 1)/x = 1 + O(x)$ for $x > 0$. Then,

$$
\text{Res}_{s=1}(\zeta_{K}(s)) \leq \left(\frac{e}{2}\right)^{m-1} v_m \frac{\Pi_{K}(2)}{\Pi_{Q}^{m}(2)} \left(\log \rho_K + \frac{\log 4}{\log \rho_K}\right)^{m-1}.
$$

Moreover, $0 < \beta < 1$ and $\zeta_{K}(\beta) = 0$ imply

$$
\text{Res}_{s=1}(\zeta_{K}(s)) \leq (1 - \beta)\left(\frac{e}{2}\right)^m \left(\log \rho_K + \frac{\log 4}{\log \rho_K}\right)^m.
$$

**Proof.** We only prove (13), the proof of (14) being similar. According to [Lou01, Section 6.1] but using the bound

$$
\zeta_{K}(s) \leq \frac{\Pi_{K}(2, s)}{\Pi_{Q}^{m}(2, s)} \zeta^{m}(s)
$$

instead of the bound $\zeta_{K}(s) \leq \zeta^{m}(s)$, we have

$$
\text{Res}_{s=1}(\zeta_{K}(s)) \leq \left(\frac{e \log d_K}{2(m - 1)}\right)^{m-1} g(s_K)
$$

$$
= \left(\frac{e}{2}\right)^{m-1} v_m \frac{\Pi_{K}(2)}{\Pi_{Q}^{m}(2)} (\log \rho_K)^{m-1} g(s_K),
$$

for $x > 0$.
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where $s_K = 1 + 2(m - 1)/\log d_K \in [1,6]$ and

$$g(s) := \frac{\Pi_K(2,s)/\Pi_K(2)}{\Pi_Q(2,s)/\Pi_Q(2)} \leq h(s) := \frac{\Pi_Q^m(2)/\Pi_Q^m(2)}{\Pi_Q(2,s)/\Pi_Q(2)}$$

(for $\Pi_K(2,s) \leq \Pi_K(2,1) = \Pi_K(2)$ for $s \geq 1$). Now, $\log h(1) = 0$ and $(h'/h)(s) = \frac{m \log 2}{2^s - 1} \leq m \log 2$ for $s \geq 1$. Hence,

$$\log h(s_K) \leq (s_K - 1)m \log 2 = \frac{(m - 1)\log 4}{\log \rho_K},$$

$$g(s_K) \leq h(s_K) \leq \left(\exp\left(\frac{\log 4}{\log \rho_K}\right)\right)^{m-1},$$

and (13) follows. •

**Corollary 11** (Compare with [Lou01, Theorems 12 and 14] and [Lou03, Theorems 9 and 22]). Set $c = 2(\sqrt{3} - 1)^2 = 1.07 \cdots$ and $v_m := (m/(m - 1))^{m-1} \in [2,e]$. Let $N$ be a normal CM-field of degree $2m > 2$, relative class number $h_N^-$ and root discriminant $\rho_N = d_N^{1/2m} \geq 650$. Assume that $N$ contains no imaginary quadratic subfield (or that the Dedekind zeta functions of the imaginary quadratic subfields of $N$ have no real zero in the range $1 - (c/\log d_N) \leq s < 1$).

Then,

$$h_N^- \geq \frac{c}{2mv_me^{c/2-1}} \left(\frac{4\sqrt{\rho_N}}{3\pi e(\log \rho_N + (\log 4)E(\log \rho_N))}\right)^m. \quad (15)$$

Hence, $h_N^- > 1$ for $m \geq 5$ and $\rho_N \geq 14610$, and for $m \geq 10$ and $\rho_N \geq 9150$. Moreover, $h_N^- \to \infty$ as $[N : Q] = 2m \to \infty$ for such normal CM-fields $N$ of root discriminants $\rho_N \geq 3928$.

**Proof.** To prove (15), follow the proof of [Lou01, Theorems 12 and 14] and [Lou03, Theorems 9 and 22], but now make use of Theorem 10 instead of [Lou01, Theorem 1] and finally notice that

$$\frac{\Pi_N(2)}{\Pi_K(2)/\Pi_Q^m(2)} = 2^m \Pi_N(2)/\Pi_K(2) = 2^m \prod_{\mathfrak{p}|(2)} (1 - \chi(\mathfrak{p})/N(\mathfrak{p}))^{-1} \geq (4/3)^m$$

($\chi$ is the quadratic character associated with the quadratic extension $N/K$, and $\mathfrak{p}$ ranges over the primes ideals of $K$ lying above the rational prime 2). •

**References**


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