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ABSORPTION AT ELECTRON CYCLOTRON RESONANCE IN SLIGHTLY IONIZED GASES (I)*

BY

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ABSTRACT

An expression of the absorption at cyclotron resonance was derived by using the specific inductive capacity, for the microwave propagating in a slightly ionized gas along the static magnetic field. It coincided with that derived by Brown *et al.* from the conductivity. It was shown that the cyclotron resonance occurs at $\omega = \omega_c$, the absorption curve is Lorentzian, and the half-width gives a direct measure of the collision frequency.

Approximate expressions of ordinary refractive index and extinction coefficient for the right-handed and the left-handed circularly polarized waves were derived.

1. Introduction

There have been some reports concerning the absorption at electron cyclotron resonance in a slightly ionized gas, both theoretical and experimental (**1, 2, 3, 4**). Solving the Boltzmann equation for the distribution of electrons under the influence of both constant magnetic field and high-frequency electric field, Kelly, Margenau and Brown have shown that the absorption i.e. the real conductivity exhibits a Lorentzian resonance peak and the half-width of this curve gives a direct measure of the collision frequency for the constant mean free time. It was also found that for the constant mean free path, though the absorption curve is complicated, the half-width is practically equal to that for the former case of constant mean free time and the line shape is not very sensitive to the collision cross section depending upon electron velocity (**5**).

In the microwave diagnostics of plasma, however, it is commonly and easily done to measure the phase shift and the insertion loss of the microwave in the plasma, from which one can calculate the electron density and the collision frequency (**6, 7**). Therefore, it is necessary for the experimental study of plasmas to derive the expressions for the attenuation factor and the phase shift at electron cyclotron resonance.

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Drummond (8) has made detailed calculation for the very dense, hot magnetoplasma. However, since his results are extremely complicated, his analysis is not easily used for discussing the experimental results on a slightly ionized gas.

It is the purpose of the present paper to derive approximate expressions for the attenuation factor and the phase shift in the vicinity of the electron cyclotron resonance for the slightly ionized gas, which are useful for the analyses of experimental results.

The electron cyclotron resonance occurs when an electromagnetic wave has a transversal component to a static magnetic field $\boldsymbol{B}$, the behaviour depending on the direction of propagation $\boldsymbol{k}_0$. For the two interesting cases: (i) $\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$, (ii) $\boldsymbol{k}_0 \perp \boldsymbol{B}$, the calculations were carried out using the specific inductive capacity. The absorption curve which represents the attenuation factor near the electron cyclotron resonance is Lorentzian for the transversal wave ($\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$)*. On the contrary, for the hybrid wave ($\boldsymbol{k}_0 \perp \boldsymbol{B}$) the resonance curve not only appears non-Lorentzian, but also the resonance position shifts. The case of $\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$ is considered in the first part, that of $\boldsymbol{k}_0 \perp \boldsymbol{B}$ in the second and the experimental results are given in the third part of the present paper.

2. Dispersion of the right-handed circularly polarized wave

A linearly polarized wave incident on a slab of plasma in a parallel magnetic field is split up into two circularly polarized waves, the right-handed (or extraordinary) and the left-handed (or ordinary) waves propagating with different phase velocities (**11**).

For the right-handed wave in which the electric field vector rotates in the same sense as the electrons gyrate, the specific inductive capacity K_- is given, using Brown's notations, by (**6**, **12**):

$$K_- = 1 - \frac{\eta}{(1-\gamma)+j\beta}, \qquad (1)^{**}$$

where

$$\eta = (\omega_p/\omega)^2, \quad \gamma = \omega_c/\omega, \quad \beta = \nu_c/\omega,$$

with

$\omega_p = \sqrt{\dfrac{e^2N}{m\varepsilon_0}}$: plasma frequency,

e, m: charge and mass of electron,

ε_0 : permittivity of free space,

* Of course, the absorption curve comes to be very asymmetric with increasing electron density, as shown by Wharton's numerical results (**9**) and experimentally verified by Brown (**10**).

** The sign of the imaginary term is opposite to Brown's one, corresponding to the time factor $\exp(-j\omega t)$ here.

N : number density of electron,

ω : angular frequency of incident wave,

$\omega_c = \dfrac{eB}{m}$: cyclotron frequency,

ν_c : effective collision frequency of electron,

B : magnetic field strength.

On the other hand, the specific inductive capacity K is related with the refractive index n (**13**) :

$$K = K_{\text{re}} + jK_{\text{im}} = n^2 = (n' + jn'')^2, \tag{2}$$

where n' : ordinary refractive index,

n'' : extinction coefficient.

Therefore,

and

$$\left.\begin{aligned} n' &= \left\{\frac{1}{2}(|K| + K_{\text{re}})\right\}^{1/2}, \\ n'' &= \left\{\frac{1}{2}(|K| - K_{\text{re}})\right\}^{1/2}. \end{aligned}\right\} \tag{3}$$

Substituting (1) into (3), the curves representing the dispersion $(n'-1)$ and the absorption (n'') can be plotted against γ, taking η and β as parameters. A few curves were given by Wharton (**9**), but it is actually rather troublesome to calculate these curves for each pair of η and β. Moreover, in our experimental cases, the following conditions are fulfilled :

$$\eta \ll 1, \quad \beta^2 \ll 1,$$

corresponding to the gaseous discharge plasmas at low pressure. In such cases, the expressions (3) become approximately :

$$n'_- = 1 - \frac{\eta}{2}\frac{1-\gamma}{(1-\gamma)^2+\beta^2} - \frac{\eta^2}{8}\frac{(1-\gamma)^2-\beta^2}{\{(1-\gamma)^2+\beta^2\}^2}, \tag{4}$$

and

$$n''_- = \frac{1}{2}\frac{\eta\beta}{(1-\gamma)^2+\beta^2}\left\{1 + \frac{\eta}{2}\frac{1-\gamma}{(1-\gamma)^2+\beta^2}\right\}. \tag{5}$$

Hence, the relative phase shift $\Delta\theta_-$ and the attenuation α_- per one free space wavelength (λ_0) are :

$$\Delta\theta_- = \pi\eta\frac{1-\gamma}{(1-\gamma)^2+\beta^2} + \frac{\pi\eta^2}{4}\frac{(1-\gamma)^2-\beta^2}{\{(1-\gamma)^2+\beta^2\}^2} \quad \text{(radian/wavelength)}, \tag{6}$$

and

$$\alpha_- = \frac{\pi\eta}{\beta}\frac{\beta^2}{(1-\gamma)^2+\beta^2}\left\{1 + \frac{\eta}{2}\frac{1-\gamma}{(1-\gamma)^2+\beta^2}\right\} \quad \text{(neper/wavelength)}. \tag{7}$$

If the second term in the bracket { } of (7) can be neglected, the attenuation α_- is approximated by:

$$\alpha_- = \alpha_m \frac{\beta^2}{(1-\gamma)^2+\beta^2} \quad \text{(neper/wavelength)}, \tag{8}$$

where

$$\alpha_m = \pi\eta/\beta. \tag{9}$$

In the above approximation, it follows from (8) that the cyclotron resonance occurs at $\gamma=1$, i. e. $\omega=\omega_c$ and the maximum attenuation at resonance is $\alpha_m(\text{nep}/\lambda_0)$. The absorption curve ($\alpha_-$) plotted against γ is symmetrical with respect to the line $\gamma=1$, and the half-width ($\Delta\gamma$) is determined by:

$$\Delta\gamma/2 = \beta. \tag{10}$$

If B_0 is the magnetic field intensity at which cyclotron resonance occurs and the half-width expressed in magnetic field intensity is ΔB, (10) is transformed into:

$$\Delta B/B_0 = 2\beta = 2\nu_c/\omega. \tag{11}$$

It is to be noted that the above approximation in which (8) has been derived is equivalent to $n' \simeq 1$ and thus the absorption curve given by (8) is equivalent to the one given by the real conductivity:

$$\sigma_R = \sigma_0 \frac{\beta^2}{(1-\gamma)^2+\beta^2}, \tag{12}$$

where $\sigma_0 = e^2N/m\nu_c$ is static conductivity (see appendix).

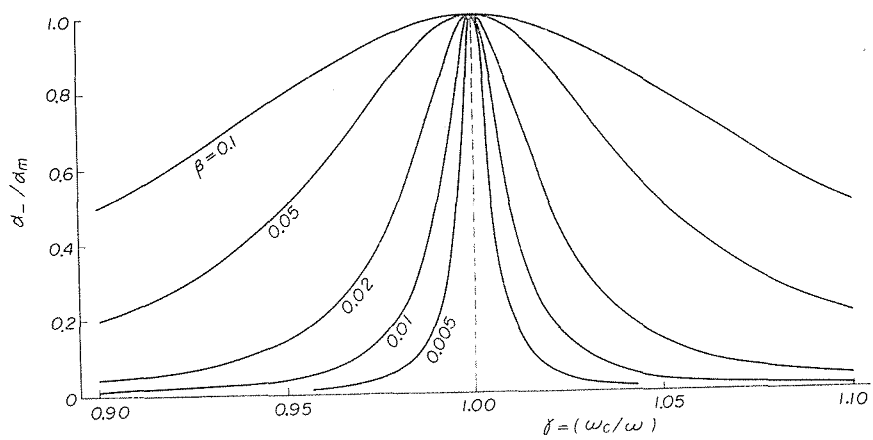

Fig. 1. Absorption curves.

The absorption curves (α_-/α_m) expressed by (8) are plotted against γ, taking β as parameter, in Fig. 1. For comparison, the absorption curves calculated from

(7) and the approximate form (8) are plotted in Fig. 2. In addition, we plot $(n'_{-}-1)$ against γ in Fig. 3, where n'_{-} is approximated by:

$$n'_{-} \simeq 1-\frac{\eta}{2}\frac{1-\gamma}{(1-\gamma)^2+\beta^2}. \tag{4a}$$

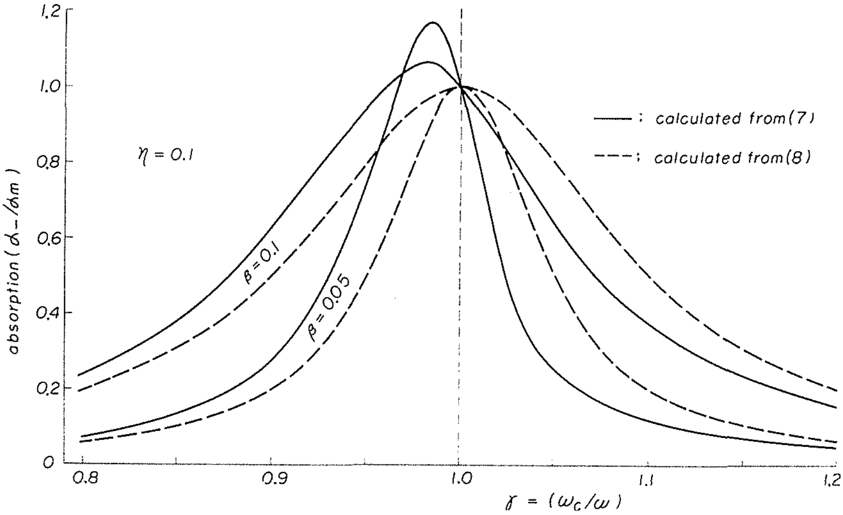

Fig. 2. Comparison between absorption curves calculated by (7) and (8).

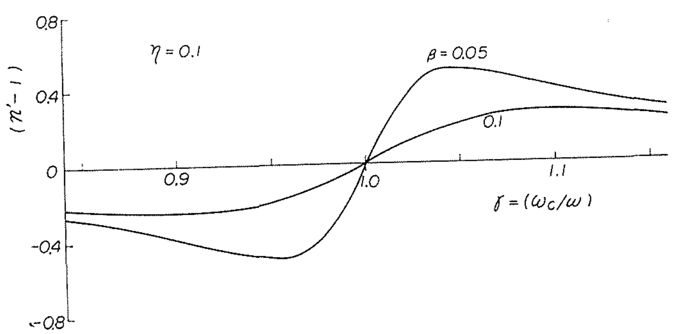

Fig. 3. Dispersion curves.

The maximum absorption α_m is proportional to η and inversely proportional to β. This is natural, since the power absorbed by plasma increases with increasing electron density (η) and with decreasing collision frequency (β). On the contrary, if the collision frequency is so high that the electrons collide before they finish respective orbits of gyrations (i. e. $\nu_c/\omega=\beta>1$), then no cyclotron resonance occurs. In Figs. 4 and 5 the maximum absorption (α_m) is plotted against η and β respectively.

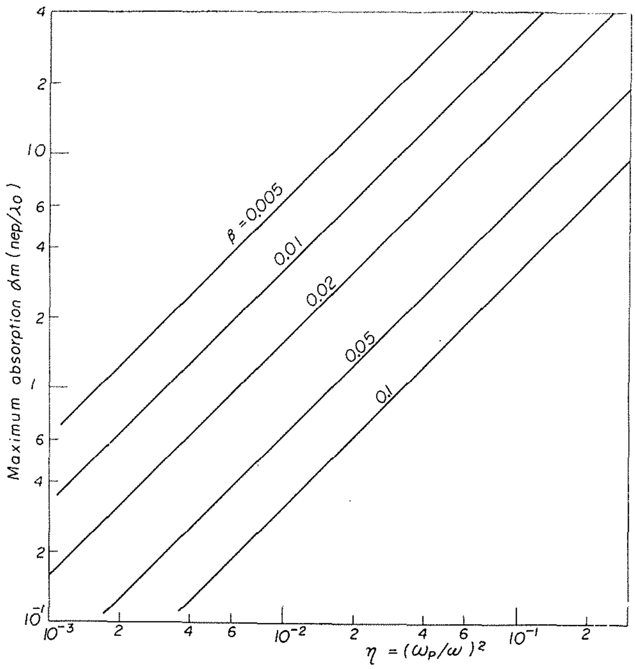

Fig. 4. α_m *versus* η.

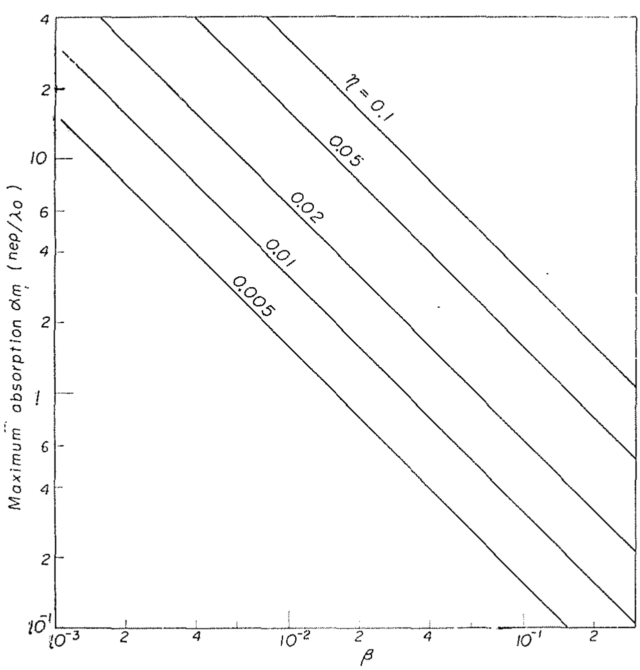

Fig. 5. α_m *versus* β.

3. Dispersion of the left-handed circularly polarized wave

We can obtain an approximate dispersion relation for the left-handed wave in the similar way to the right-handed one.

The specific inductive capacity K_+ for this wave is given by:

$$K_+ = 1 - \frac{\eta}{(1+\gamma)+j\beta}, \tag{13}$$

and thus n'_+ and n''_+ are given by:

$$n'_+ = 1 - \frac{\eta}{2}\frac{1+\gamma}{(1+\gamma)^2+\beta^2} - \frac{\eta^2}{8}\frac{(1+\gamma)^2-\beta^2}{\{(1+\gamma)^2+\beta^2\}^2}, \tag{14}$$

and

$$n''_+ = \frac{1}{2}\frac{\eta\beta}{(1+\gamma)^2+\beta^2}\left\{1+\frac{\eta}{2}\frac{1+\gamma}{(1+\gamma)^2+\beta^2}\right\}. \tag{15}$$

Hence, the relative phase shift $\Delta\theta_+$ and the absorption α_+ of the left-handed wave per one free space wavelength (λ_0) are respectively:

$$\Delta\theta_+ = \pi\eta\frac{1+\gamma}{(1+\gamma)^2+\beta^2} + \frac{\pi\eta^2}{4}\frac{(1+\gamma)^2-\beta^2}{\{(1+\gamma)^2+\beta^2\}^2} \quad \text{(radian/wavelength)}, \tag{16}$$

and

$$\alpha_+ = \alpha_m\frac{\beta^2}{(1+\gamma)^2+\beta^2} \quad \text{(neper/wavelength)}. \tag{17}$$

For the left-handed wave, naturally no resonance phenomena occur, since the electric vector rotates in the opposite sense to the electron gyration.

4. Dispersion of the transversal wave ($k_0 /\!\!/ B$)

A transversal wave incident on a slab of plasma in a parallel magnetic field is considered. In passing through the plasma, the plane of polarization is rotated by an angle $\Delta\theta_{//}$ per one free space:

$$\Delta\theta_{//} = \pi\eta\left\{\frac{1-\gamma}{(1-\gamma)^2+\beta^2} - \frac{1+\gamma}{(1+\gamma)^2+\beta^2}\right\} \quad \text{(radian/wavelength)}, \tag{18}$$

analogously to the Faraday effect in crystal.

It is evident from the discussions in §§ 2 and 3, that the resonance absorption for the case $\eta \ll 1$ and $\beta^2 \ll 1$ can be approximated by:

$$\alpha_{//} = \frac{\alpha_m}{2}\left[\frac{\beta^2}{(1-\gamma)^2+\beta^2}\left\{1+\frac{\eta}{2}\frac{1-\gamma}{(1-\gamma)^2+\beta^2}\right\} + \frac{\beta^2}{(1+\gamma)^2+\beta^2}\right] \quad \text{(neper/wavelength)}. \tag{19}$$

Moreover, in the extreme approximation, the above equation becomes:

$$\alpha_{//} = \frac{\alpha_m}{2}\left\{\frac{\beta^2}{(1-\gamma)^2+\beta^2} + \frac{\beta^2}{(1+\gamma)^2+\beta^2}\right\} \quad \text{(neper/wavelength)}. \tag{20}$$

This expression agrees with that calculated by Brown *et al.* (5) from the conductivity. Under ordinary experimental conditions of the electron cyclotron resonance, $\beta^2 \ll 1$. Then, as the antiresonance term in (20) is almost constant and negligibly small compared with the resonance term, in the neighbourhood of the resonance, the absorption curve for the transversal wave actually coincides with that for the right-handed one.

Next, we consider the absorption curve including both resonance and antiresonance terms. Since $\beta^2 \ll 1$ and it is considered in the vicinity of the resonance, (20) can be approximated by:

$$\alpha_{//} = \frac{\alpha_m}{2}\frac{(2\beta)^2}{(\gamma_0^2-\gamma^2)^2+(2\beta)^2}, \tag{21}$$

where

$$\gamma_0^2 = 1-\beta^2. \tag{22}$$

Therfore, in the above approximation the cyclotron resonance occurs at $\gamma_0 = (1-\beta^2)^{1/2} \simeq 1-\beta^2/2$, and the maximum attenuation is $\alpha_m/2$. The absorption $(\alpha_{//})$ plotted against γ^2 is indeed symmetrical, but when plotted against γ it is not symmetical. The half-width of the absorption curve $(\Delta\gamma)$ is:

$$\Delta\gamma = (\Delta\gamma_-/2)+(\Delta\gamma_+/2) \simeq 2\beta, \tag{23}$$

and the measure of asymmetry is given by:

$$(\Delta\gamma_-/2)-(\Delta\gamma_+/2) \simeq \beta^2, \tag{24}$$

(see Fig. 6).

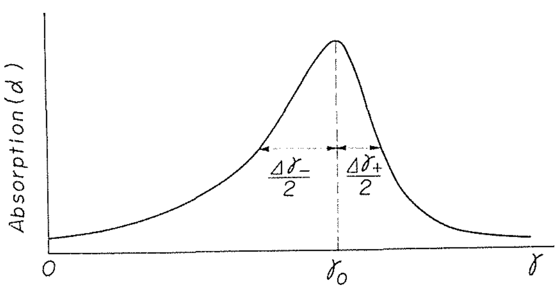

Fig. 6. Asymmetrical absorption curve.

Integrating $\alpha_{//}$ expressed by (20) with respect to γ, we obtain:

$$\int_0^\infty \alpha_{//}d\gamma = \frac{\pi^2}{2}\eta = \frac{\pi^2}{2}\frac{e^2}{m\varepsilon_0\omega^2}N, \tag{25}$$

which implies that the area surrounded by the absorption curve and the γ- axis is determined only by the electron density, being independent of the collision frequency.

5. Conclusion

Using the specific inductive capacity, the expressions for the absorption and the relative phase shift at the cyclotron resonance in a slightly ionized gas were derived for the transversal wave ($\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$). The resulting absorption curve, which strictly represents the attenuation factor, is somewhat complicated in comparison with that which represents the real conductivity σ_R, because the extinction coefficient n'' is more complicated than σ_R. As shown, in the extreme approximation, both absorption curves are coincident with each other.

In our present slightly ionized gas in which $\eta \ll 1$ and $\beta^2 \ll 1$, the expression (8) for the right-handed and the expression (20) for the transversal wave are appropriately used for the analyses of the experimental results. If this approximation is not permissible, we must use (4) or (3) which is rather complicated. Wharton calculated n' and n'' from (3) using tentative numerical values and it was shown that the absorption curve becomes more asymmetrical with increasing η (**9**), and experimentally verified by Brown (**10**).

The microwave propagating transversely to the magnetic field will be discussed in the next part.

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Appendix

The complex conductivity $\sigma_c = \sigma_R - j\sigma_i$ is related with the specific inductive capacity by:

$$K = 1 + j\frac{\sigma_c}{\omega\varepsilon_0} = \left(1 - \frac{\sigma_i}{\omega\varepsilon_0}\right) + j\frac{\sigma_R}{\omega\varepsilon_0}. \tag{26}$$

From this and (3) we have

$$n'^2 = \frac{1}{2}\left\{\sqrt{\left(1-\frac{\sigma_i}{\omega\varepsilon_0}\right)^2 + \left(\frac{\sigma_R}{\omega\varepsilon_0}\right)^2} + \left(1-\frac{\sigma_i}{\omega\varepsilon_0}\right)\right\}, \tag{27}$$

and

$$n''^2 = \frac{1}{2}\left\{\sqrt{\left(1-\frac{\sigma_i}{\omega\varepsilon_0}\right)^2 + \left(\frac{\sigma_R}{\omega\varepsilon_0}\right)^2} - \left(1-\frac{\sigma_i}{\omega\varepsilon_0}\right)\right\}. \tag{28}$$

If

$$\frac{\sigma_i}{\omega\varepsilon_0} \ll 1, \quad \frac{\sigma_R}{\omega\varepsilon_0} \ll 1,$$

(27) and (28) can be approximated by:

$$n' = 1-\frac{1}{2}\frac{\sigma_i}{\omega\varepsilon_0}-\frac{1}{8}\left\{\left(\frac{\sigma_i}{\omega\varepsilon_0}\right)^2-\left(\frac{\sigma_R}{\omega\varepsilon_0}\right)^2\right\}, \tag{29}$$

and

$$n'' = \frac{1}{2}\frac{\sigma_R}{\omega\varepsilon_0}\left\{1+\frac{1}{2}\frac{\sigma_i}{\omega\varepsilon_0}\right\}. \tag{30}$$

On the other hand, equating the imaginary part of (26) to that of (2), we have

$$n'' = \frac{1}{2}\frac{\sigma_R}{\omega\varepsilon_0}\frac{1}{n'}. \tag{31}$$

Accordingly,

(i) if $n' \simeq 1$,

$$n'' \simeq \frac{1}{2}\frac{\sigma_R}{\omega\varepsilon_0}, \tag{32}$$

which implies that the variation of n'' *versus* γ is equivalent to that of σ_R; and

(ii) if $n'^2 \simeq 1-\sigma_i/(\omega\varepsilon_0)$,

$$n'' \simeq \frac{1}{2}\frac{\sigma_R}{\omega\varepsilon_0}\left(1-\frac{\sigma_i}{\omega\varepsilon_0}\right)^{-1/2} \simeq \frac{1}{2}\frac{\sigma_R}{\omega\varepsilon_0}\left(1+\frac{1}{2}\frac{\sigma_i}{\omega\varepsilon_0}\right), \tag{33}$$

which agrees with (30).

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